Incompressible transformations

by

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1. Introduction. This paper is concerned with the study of a 1-1 measurable non-singular transformation T of a σ-finite measure space (X, μ) onto itself, and of the sequence $\{\omega_n\}$ of Radon-Nikodym derivatives of the powers of T. Our starting point is the observation that one can imbed X in a larger measure space (X^*, μ^*) and (in a sense) extend T to a 1-1 transformation T^* of X^* onto itself in such a way that T^* is *measure-preserving* (Theorem 1, 3.2). Thus, in a sense, the theory of 1-1 measurable non-singular transformations can be reduced to the theory of measure-preserving transformations.

For this to be useful, one must be able to interpret further properties of T in terms of T^*; and our main theorem (Theorem 2, 5.1) is that the T^* which we construct will be incompressible (on X^*) if and only if T is incompressible (on X). This leads fairly immediately to ergodic theorems for incompressible T; by applying an ergodic theorem of Halmos to T^*, we obtain variants of the theorems of Halmos, Hopf and Hurewicz (Theorem 3, 7.4). Another application of Theorem 2 (or rather of one of the lemmas leading to it) gives the following result (a special case of Theorem 4, 8.2): if $\omega_n(x) \to 0$ on a set of positive measure then T is compressible. As a corollary we have (8.3) that if μ is finite, then $\sum \omega_n(x)$ converges almost everywhere where $\omega_n(x) \to 0$. We also obtain some results related to these, and study the case in which $\mu X = \infty$ (Theorem 5, 8.4). The method can also be applied to give relative density properties of $\{\omega_n\}$, and the author hopes to deal with this in a subsequent paper.

2. Preliminaries; notation.

2.1. We make the standing hypothesis, throughout this paper, that (X, μ) denotes a σ-finite complete measure space, and that T is a 1-1 mapping of X onto itself such that, for all $A \subset X$, (i) if A is measurable, so are TA and $T^{-1}A$, (ii) if A is null, so are TA and $T^{-1}A$ [1]. We some-

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[1] We often omit brackets to simplify the notation, writing TA for $T(A)$, etc.

times consider other measures on X; these are always assumed to be σ-finite and equivalent to μ.

Throughout what follows, all subsets of X referred to are understood to be measurable. A *function* on X is a measurable extended-real function (the values $\pm\infty$ being allowed); letters like f, g, h, H are understood to refer to functions. The letters m, n, i, j are understood to run over *all* integers $(0, \pm 1, \pm 2, \ldots)$ in the absence of qualifying phrases.

The iterates T^m of T also have the properties (i) and (ii) above; hence each null set E is contained in the invariant null set $\bigcup T^m E$. We shall often discard invariant null sets from X without further warning.

2.2. Each T^n has a Radon-Nikodym derivative $\omega_n(\ ;T)$, unique (modulo null sets), such that

(1) $$\mu T^n A = \int_A \omega_n(x; T)\, d\mu(x) \quad \text{for all} \quad A \subset X;$$

usually the transformation T intended is clear, and we omit it, writing $\omega_n(\ ;T)$ and $\omega_n(x;T)$ simply as ω_n and $\omega_n(x)$. We usually write ω_1 as ω; of course, $\omega_0(x) = 1$. The following known properties ([2], p. 750) follow easily; in the first instance they are true only "almost everywhere", but by discarding an invariant null set from X we ensure that they hold exactly, for all $x \in X$.

(2) $$0 < \omega_n(x) < \infty.$$

(3) $$\omega_{i+j}(x) = \omega_j(x)\,\omega_i(T^j x);$$

in particular,

(4) $$\omega_{n+1}(x) = \omega(x)\,\omega_n(Tx) = \omega_n(x)\,\omega(T^n x),$$

(5) $$\omega_{-n}(x)\,\omega_n(T^{-n}x) = 1.$$

From (4) we obtain:

(6) $$\textit{If } n > 0, \quad \omega_n(x) = \omega(x)\,\omega(Tx)\ldots\omega(T^{n-1}x).$$

Finally we note:

(7) $$\textit{If } f \geqslant 0, \quad \int_X f(x)\,d\mu(x) = \int_X f(T^n x)\,\omega_n(x)\,d\mu(x) = \int_X \{f(T^{-n}x)/\omega_n(T^{-n}x)\}\,d\mu(x).$$

2.3. A set $A \subset X$ such that $T^m A \cap T^n A = \emptyset$ if $m \neq n$ is *wandering* (under T), and the set $\bigcup T^n A$ is then called *dissipative*. The transformation T is *compressible* if there exists $B \subset X$ such that $TB \subset B$ and $\mu(B - TB) > 0$. Clearly $B - TB$ is then wandering, whence:

(1) The following statements are equivalent [2]:

(i) T is compressible,

(ii) There exists a non-null wandering set,

(iii) There exists a non-null dissipative set,

(iv) T^{-1} is compressible.

2.4. If $A \subset X$ is invariant under T, then the restriction $T|A$ of T to A is a transformation of the measure space (A, μ) satisfying our assumptions. It is easy to see that, if $x \in A$, $\omega_n(x; T|A) = \omega_n(x; T)$. Accordingly we shall usually not bother to distinguish between T and $T|A$.

3. The space X^*.

3.1. Let (Y, ν) denote the measure space consisting of the half-line $y \geqslant 0$, with Lebesgue measure ν. We take (X^*, μ^*) to be the measure-theoretic product $(X, \mu)\times(Y, \nu)$, and define T^* on X^* by: for each $(x, y) \in X^*$,

(1) $$T^*(x, y) = \big(Tx,\ y/\omega(x)\big).$$

Because of 2.2 (2), T^* is a well-defined 1-1 map of X^* onto itself; and T^* is an extension of T in the sense that, for each $x \in X$, T^* maps the "x-fiber" $x \times Y$ onto the "Tx-fiber" $Tx \times Y$ (scaling it down by the factor $\omega(x)$). We easily verify (using 2.2 (4) and (5)) that

(2) $$T^{*n}(x, y) = \big(T^n x,\ y/\omega_n(x)\big) = \big(T^n x,\ y\,\omega_{-n}(T^n x)\big).$$

We shall later verify that T^* is measurable (and in fact measure-preserving); but before doing so it is convenient to introduce some further notation. Let h be any non-negative function on X; that is, $0 \leqslant h(x) \leqslant \infty$ for all $x \in X$ (and h is measurable). We define its "upper" and "lower" ordinate sets by:

(3) $$S^0 h = \{(x, y) \mid x \in X,\ 0 \leqslant y \leqslant hx\},$$ $$S_0 h = \{(x, y) \mid x \in X,\ 0 \leqslant y < hx\},$$

and say that an *ordinate set* of h is any set Sh between them (that is, $S_0 h \subset Sh \subset S^0 h$). From Fubini's theorem, $S^0 h - S_0 h$ is null; thus each Sh is measurable, two different ordinate sets of h differ by at most a null set, and further

(4) $$\mu^*(Sh) = \int_X h(x)\,d\mu(x).$$

For each n we define a function h_n on X by

(5) $$h_n(x) = h(T^n x)/\omega_{-n}(T^n x) = h(T^n x)\,\omega_n(x) \quad (x \in X).$$

[2] By a theorem of Halmos ([2], p. 738) these properties are also equivalent to saying that T^n is compressible for some n, or for all $n \neq 0$.

Note that $h_0 = h$, that

(6) $$h_{n+1}(x) = h_n(Tx)\,\omega(x)\,,$$

and that (from 2.2 (7))

(7) $$\int_X h_n(x)\,d\mu(x) = \int_X h(x)\,d\mu(x)\,.$$

It is also easy to verify that $T^{*n}(S^0 h) = S^0 h_{-n}$ and $T^{*n}(S_0 h) = S_0 h_{-n}$, so that, in an obvious sense, we may say

(8) $$T^{*n}(Sh) = S(h_{-n})\,.$$

3.2. THEOREM 1. *T^* is a measure-preserving transformation of (X^*, μ^*).*

We first observe that, from 3.1 (8), (4) and (7), the transforms T^*Sh and $T^{*-1}Sh$ of an arbitrary ordinate set Sh are always measurable and satisfy

(1) $$\mu^*(T^*Sh) = \mu^*(Sh) = \mu^*(T^{*-1}Sh)\,.$$

Since the family of all ordinate sets generates the family of all measurable subsets of X^* (modulo null sets), it follows by standard techniques ([3]) that, for each (measurable) $B^* \subset X^*$, T^*B^* and $T^{-1}B^*$ are measurable and $\mu^*(T^*B^*) = \mu^*B^* = \mu^*(T^{-1}B^*)$, proving the theorem.

3.3. For later use we derive some further properties of T^*. First, for (measurable) $A^* \subset X^*$ and $x \in X$, write

(1) $$A^*(x) = \{y \mid (x, y) \in A^*\}\,;$$

this is measurable for almost all $x \in X$, and we define a measurable function $\nu(A^*;\ )$ on X by setting

(2) $$\nu(A^*; x) = \nu\big(A^*(x)\big) \qquad (\text{a.e. in } X)\,.$$

We have $(T^{*-1}A^*)(x) = \{y\omega(x) \mid y \in A^*(Tx)\}$, whence

(3) $$\nu(T^{*-1}A^*; x) = \omega(x)\,\nu(A^*; Tx)\,,$$

and therefore

(4) $$\nu(T^*A^*; Tx)\,\omega(x) = \nu(A^*; x)\,.$$

Next we derive an extension of 3.1 (8). As before, let h be any non-negative (possibly infinite) function on X, and let n be any integer. We define the (measurable) non-negative function H_n on X by:

(5) $$H_n(x) = \sup_{i \geqslant n} h_i(x) = \sup_{i \geqslant n} h(T^i x)\,\omega_i(x)\,,$$

([3]) See, for example, [6], p. 235, for a treatment of a similar but more general situation.

from 3.1 (5). A routine verification shows that

(6) $$H_n(x) \geqslant H_{n+1}(x) = H_n(Tx)\,\omega(x)$$

and that

(7) $$H_n(x) = \max\{h_n(x), H_n(Tx)\,\omega(x)\}\,.$$

From 3.1 (8) (applied to T^{*-1} and H_n) we now obtain

(8) $$T^{*-1}S(H_n) = S(H_{n+1}) \subset S(H_n)\,.$$

Further, we have

(9) $$S(H_n) = \bigcup_{i \geqslant n} T^{*-i}(Sh) \text{ modulo null sets}\,,$$

since when $S = S_0$ the two are easily verified to be equal.

One final remark will be useful in the next section. If h satisfies $h(x) = h(Tx)\,\omega(x)$ $(x \in X)$, one easily sees that $h(x) = h(T^n x)\,\omega_n(x)$, and so $H_n(x) = h(x)$ in this case.

4. T-invariant measures and ergodicity of T^*.

4.1. Let λ be another measure on X (σ-finite and equivalent to μ), with Radon-Nikodym derivative φ with respect to μ. By altering φ on a null set, if necessary, we may assume φ everywhere positive and finite. The measure λ_1 on X defined by $\lambda_1(A) = \lambda(TA)$ $(A \subset X)$ is also σ-finite and equivalent to μ; an elementary calculation shows that its Radon-Nikodym derivative φ_1 with respect to μ is given by

(1) $$\varphi_1(x) = \varphi(Tx)\,\omega(x) \qquad (x \in X)\,;$$

this is of course also finite and strictly positive. Thus

(2) $$S\varphi_1 = T^{*-1}(S\varphi)\,.$$

The measure λ will clearly be *invariant* under T if and only if $\varphi = \varphi_1$ (almost everywhere), or equivalently $T^{*-1}S\varphi = S\varphi$. The following well-known result of Halmos ([2], p. 751), which we state as a lemma, is an immediate consequence.

4.2. LEMMA. *A necessary and sufficient condition for T to admit an invariant measure (σ-finite and equivalent to μ) is that there exist a function φ on X such that (almost everywhere)*

(1) $$0 < \varphi(x) < \infty\,,$$

(2) $$\varphi(x) = \varphi(Tx)\,\omega(x)\,, \quad \textit{or equivalently} \quad S\varphi \textit{ is invariant under } T^* \text{ ([4])}.$$

([4]) Here φ is $1/f$ in Halmos's notation.

We note that, together, conditions (1) and (2) here are also equivalent to each of the following:

(3) *There exists a T^*-invariant $A^* \subset X^*$ such that, for all $x \in X$, $0 < \nu(A^*; x) < \infty$.*

(If (3) holds, 3.3 (4) shows that $\nu(A^*; \ )$ fulfils the above requirements on φ; conversely, take $A^* = S\varphi$.)

(4) *There exists a function h on X such that $0 < h(x) < \infty$ and $H_0(x) = H_1(x) < \infty$ $(x \in X)$.*

(If (4) holds, 3.3 (6) shows that H_1 fulfils the requirements on φ; conversely, take $h = \varphi$, which satisfies (4) in view of the remark at the end of 3.3.)

Remark. It follows from a later result that in (2) the equality can be replaced by either inequality. In fact, if $\varphi(x) \leqslant \varphi(Tx)\omega(x)$ $(x \in X)$, Lemma 5.3 below shows that on the non-dissipative part of X we have equality, and so have an invariant measure (on this part and thence on all X); while if instead $\varphi(x) \geqslant \varphi(Tx)\omega(x)$ $(x \in X)$, we apply the previous reasoning to the function $1/\varphi$.

We shall also see (Lemma 5.4) that a weakened form of (4) is sufficient for X to have a non-null invariant subset admitting a T-invariant measure. Though apparently irrelevant, these considerations are needed for our study of the incompressibility of T^*.

The following elementary remark will also be useful:

(5) *If a is a positive real number, and if the ordinate set Sh is invariant under T^*, then so is $S(ah)$.*

4.3. It should be noted that it will happen only rather rarely that T^* will be ergodic on X^*. Of course, T^* cannot be ergodic on X^* unless T is ergodic on X; but moreover, whenever T admits an invariant measure (σ-finite and equivalent to μ), T^* cannot be ergodic. To see this, apply Lemma 4.2; the set $S\varphi$ is invariant under T^*, and both $S\varphi$ and $X^* - S\varphi$ are non-null. It would be interesting to know whether, conversely, if T is ergodic on X and admits no invariant measure then T^* is ergodic on X^*. One can obtain a further necessary condition for ergodicity of T^*, in terms of the sequence $\{\omega_n\}$, but this condition may perhaps be a consequence of the other two; the author hopes to deal with this elsewhere.

5. Compressibility.

5.1. Theorem 2. *A necessary and sufficient condition for T^* to be compressible (on X^*) is that T be compressible (on X).*

The sufficiency is trivial; for if $A \subset X$ is a non-null wandering set under T, then $A \times Y$ is a non-null wandering set under T^*. We prove

the converse in the next section, after deriving a sequence of lemmas, the first of which is a restatement of a result of Halmos and Ornstein ([3], pp. 89, 90).

5.2. Lemma. *If $\mu TE \leqslant \mu E$ for all $E \subset X$, and if $\mu TF < \mu F$ for some $F \subset X$, then T is compressible.*

In other words, if $\omega(x) \leqslant 1$ for almost all $x \in X$, and if $\mu\{x|\ \omega(x) < 1\} > 0$, then T is compressible.

5.3. Lemma. *Suppose that there exists a non-negative function g on X such that (i) $g(Tx)\omega(x) \leqslant g(x)$ $(x \in X)$, (ii) there exists a non-null $Z \subset X$ such that $g(Tz)\omega(z) < g(z)$ for all $z \in Z$; then T is compressible* (⁵).

Remark. When $g = 1$, this reduces to the previous lemma.

Proof. Suppose not, and put $M = \{x|\ g(x) = 0\}$; then (i) gives $TM \subset M$, and since T is incompressible, it follows that $M - TM$ is null. Put $\bar{M} = \bigcup\{T^{-n}M|\ n \geqslant 0\}$; then $\bar{M}$ is an invariant set containing M, and $\bar{M} - M$ is null. Now, from (ii), $Z \cap M = \emptyset$ and therefore $\mu(Z - \bar{M}) \neq 0$, whence $\mu(TZ - \bar{M}) = \mu\big(T(Z - \bar{M})\big) > 0$. Let $A = \{x|\ x \in X - \bar{M}, 0 < g(x) < \infty\}$; then $A \supset TZ - \bar{M}$, from (ii), and so is also not null. Also, from (i), $TA \subset A$, so (from incompressibility) $\mu(A - TA) = 0$. Let $B = \bigcap\{T^n A|\ n \geqslant 1\}$; B is invariant, $B \subset A$, and $\mu(A - B) = 0$. Thus B likewise is not null.

Since A contains TZ except for a null set, B does the same; but B is invariant, and therefore B contains Z except for a null set. Note that, on restricting g and T to B, we still have conditions (i) and (ii) satisfied (with Z replaced by $Z \cap B$), together with the further condition

(iii) $\quad 0 < g(x) < \infty \quad (x \in B)$.

It follows that for some $a, b > 0$ the set $D = \{x|\ x \in Z \cap B,\ a < g(x) < b\}$ has positive μ-measure.

We define a measure λ (equivalent to μ) on B by setting, for $C \subset B$, $\lambda C = \int_C g(x)\,d\mu(x)$. Then, by (i):

$$\lambda(TC) = \int_{TC} g(x)\,d\mu(x) = \int_C g(Tx)\,\omega(x)\,d\mu(x) \leqslant \lambda C;$$

and a similar calculation shows that $\lambda(TD) < \lambda D$. By Lemma 5.2 applied to the measure space (B, λ), $T|B$ is compressible (with respect to λ

(⁵) Conversely, if T is compressible it is easy to construct such a g, so that 5.3 gives a necessary and sufficient condition. However, 5.2 (as it stands) does not, as when $\mu X < \infty$ the hypotheses in 5.2 can never be satisfied. A further sufficient condition for compressibility, which neither includes nor is included in 5.2, will be given later (Theorem 4, 8.2).

and so with respect to μ); *a fortiori* so is T, giving the desired contradiction.

5.4. LEMMA. *Suppose that the only T-invariant sets $D \subset X$ admitting T-invariant measures are null, and let h be a function on X such that $0 < h(x) < \infty$ $(x \in X)$. Then, for (almost) all $x \in X$, $H_0(x) = \infty$; that is* (3.3 (9)) $\bigcup\{T^{*-n}Sh \mid n \geqslant 0\} = X^*$ *(modulo null sets).*

Proof. Put $A = \{x \mid H_0(x) = \infty\}$; we first show that A is invariant. In fact (from 3.3 (7) and 3.1 (5)), $H_0(x) = \max\{h(x), H_0(Tx)\omega(x)\}$; since both h and ω are positive and finite, it follows that $H_0(x) = \infty \Leftrightarrow H_0(Tx) = \infty$; that is, A is invariant. Thus $B = X - A$ is an invariant set on which $0 < h(x) \leqslant H_0(x) < \infty$; and in any case $H_0(x) \geqslant H_0(Tx)\omega(x)$.

Put $Z = \{x \mid x \in B,\ H_0(x) > H_0(Tx)\omega(x)\}$; we show that Z is null. If not, Lemma 5.3 (applied to $T|B$, with $g = H_0$) would prove that T is compressible; but then X contains a non-null dissipative set D, necessarily invariant, on which T trivially admits an invariant measure, contrary to hypothesis. Thus Z is null, and we have $H_0(x) = H_0(Tx)\omega(x)$ almost everywhere on B. By Lemma 4.2, $T|B$ admits an invariant measure; thus by hypothesis, B is null, q.e.d.

5.5. LEMMA. *Suppose that T is measure-preserving, and that there exists a subset A of X satisfying* (i) $0 < \mu A < \infty$, (ii) $\mu\big(X - \bigcup\{T^iA \mid i \geqslant 0\}\big) = 0$. *Then T is incompressible.*

Proof. We may assume $X = \bigcup\{T^iA \mid i \geqslant 0\}$ (by removing an invariant null set). In the present argument, we restrict i to run over the *non-negative* integers only; n runs over all integers, as usual.

Suppose that the lemma is false; then, from 2.3 (1), there exist disjoint non-null sets B_n such that $TB_n = B_{n+1}$. The set $C = \bigcup B_n$ is invariant, and can be written $C = C \cap \bigcup T^iA = \bigcup(T^iC \cap T^iA)$ (since $C = T^iC$) $= \bigcup T^iD$ where $D = A \cap C = \bigcup(D \cap B_n)$. Note that D is not null (else C and B_n are). Thus $0 < \mu D = \sum \mu(D \cap B_n) \leqslant \mu A < \infty$, and hence there exists (large negative) n_0 such that

$$\mu\Big(D \cap \bigcup_{j \leqslant n_0} B_j\Big) = \sum_{j \leqslant n_0} \mu(D \cap B_j) < \mu(B_1). \tag{1}$$

On the other hand,

$$B_{n_0} \subset C = \bigcup T^iD = \bigcup_{i,n} T^i(D \cap B_n),$$

where $T^i(D \cap B_n) \subset T^iB_n = B_{n+i}$ which is disjoint from B_{n_0} unless $n+i = n_0$. Thus $B_{n_0} \subset \bigcup T^i(D \cap B_{n_0-i})$, giving $\mu(B_{n_0}) \leqslant \sum \mu T^i(D \cap B_{n_0-i}) = \sum \mu(D \cap B_{n_0-i})$ (because T is measure-preserving) $= \sum_{j \leqslant n_0} \mu(D \cap B_j) < \mu(B_1)$, from (1). But T is measure-preserving, so $\mu(B_{n_0}) = \mu(B_1)$, giving a contradiction.

6. Proof of Theorem 2.

6.1. We assume that T^* is compressible (on X^*), and have to show that T is compressible (on X). We first consider two special cases separately (6.2, 6.3), and then combine them to cover the general case in 6.4.

6.2. Assume first that *X admits a T-invariant measure.*

From Lemma 4.2, there exists a function φ on X such that $0 < \varphi(x) < \infty$ and $\varphi(x) = \varphi(Tx)\omega(x)$ $(x \in X)$. Thus $S\varphi$ is invariant under T^*. By 4.2 (5), so is $S(n\varphi)$ for each $n > 0$.

By hypothesis, there is a non-null $D^* \subset X^*$ which is dissipative under T^*. Since clearly $\bigcup\{S(i\varphi) \mid i > 0\} = X^*$, there is a positive j such that $D^* \cap S(j\varphi) = E^*$ say, is not null. Being the intersection of a dissipative set and an invariant set, E^* is also dissipative; thus we may write E^* as the union of disjoint sets A_n $(n = 0, \pm 1, \ldots)$ such that $T^*A_n = A_{n+1}$ and $\mu^*(A_n) \neq 0$. Write $B = \bigcup\{A_{-n} \mid n \geqslant 0\}$; then $T^{*-1}B = \bigcup\{A_{-n} \mid n \geqslant 1\} \subset B \subset E^* \subset Sh$, where $h = j\varphi$. Thus, for (almost) all $x \in X$, $\nu(T^{*-1}B; x) \leqslant \nu(B; x) \leqslant h(x) < \infty$. From 3.3 (3) we therefore have $\omega(x)\nu(B; Tx) = \nu(T^{*-1}B; x) \leqslant \nu(B; x) < \infty$.

Now $B - T^{*-1}B = A_0$; and since $\mu^*A_0 > 0$, there exists $Z \subset X$ such hat $\mu Z > 0$ and all the "sections" $A_0 \cap (z \times Y)$, $z \in Z$, are of positive ν-measure. For all $z \in Z$ we have $\nu(T^{*-1}B; z) + \nu(A_0; z) = \nu(B; z) < \infty$, giving $\nu(T^{*-1}B; z) < \nu(B; z)$ $(z \in Z)$. Thus the hypotheses of Lemma 5.3 hold (with $g = \nu(T^{*-1}B;\ )$), and T is therefore compressible.

6.3. Now assume instead that *there is no T-invariant measure on any non-null invariant subset of X.*

We take a function h on X such that (i) $0 < h(x) < \infty$ $(x \in X)$, (ii) $\int_X h(x)\,d\mu(x) < \infty$; such an h is easily constructed. By Lemma 5.4, we have $\bigcup\{T^{*-n}Sh \mid n \geqslant 0\} = X^*$. But $\mu^*(Sh) < \infty$; thus, from Lemma 5.5 (applied to X^* and T^{*-1}, with $A = Sh$), T^* is compressible, and the present case cannot arise.

6.4. In the general case, consider the family $\mathcal{F}$ of all (measurable) invariant subsets F of X which admit T-invariant measures (σ-finite and equivalent to $\mu|F$). The algebra of measurable sets modulo null sets being complete, there exists $Z \subset X$ which is (modulo null sets) a least upper bound for $\mathcal{F}$; further, we may suppose $Z = \bigcup\{F_n \mid n \geqslant 1\}$ where $F_n \in \mathcal{F}$. Clearly Z is T-invariant; and a T-invariant measure on Z is easily constructed by combining such measures on the sets $F_n - (F_1 \cup \ldots \cup F_{n-1})$. Of course, $Z \times Y$ is a T^*-invariant subspace of X^*; and the construction (of X^* from X) in § 3, if applied to (Z, μ), yields just the subspace $(Z \times Y, \mu^*)$ of X^*. In other words, we may consistently write $Z^* = Z \times Y$; and similarly we have $(X-Z)^* = (X-Z) \times Y = X^* - Z^*$.

The argument in 6.3 now shows that T^* is incompressible on X^*-Z^*; and the argument in 6.2 shows that if T is incompressible (on X and hence on Z) then T^* is incompressible on Z^*. It follows at once that T^* is incompressible on X^*.

7. Ergodic theorems.

7.1. Assuming that T is incompressible on X, we have seen (Theorems 1 and 2) that T^* will be both measure-preserving and incompressible on X^*. By applying standard ergodic theorems to T^* we can readily obtain information about T. We illustrate the method by giving 3 closely related results (Theorem 3 below). The first two of these are similar to Hopf's theorem ([4], p. 49), and constitute generalizations of it to the case of transformations which need not preserve measure. The third resembles a theorem of Hurewicz ([5]; see also [1]), but does not seem to contain or be contained in it. Our starting point is a slight sharpening of a sharpened form, due to Halmos ([1], Th. 5), of the Hopf theorem. Before stating it, we develop the necessary notation.

Let h be a (measurable) function on X; we write

$$(1)\qquad Ph = \{x|\ h(x) \neq 0\}, \qquad Qh = \bigcup T^n(Ph) \qquad (n = 0, \pm 1, \ldots).$$

Thus Qh is invariant, and $X-Qh$ is the largest invariant set on which h vanishes identically. Following Halmos, we call h *invariantly positive* if $h \geqslant 0$ and $\mu(X-Qh) = 0$.

If f, g are (almost everywhere) *finite* functions on X, if n is a *positive* integer, and if $x \in X$, we write

$$(2)\qquad f_n(T; x) = \sum \{f(T^i x)|\ 0 \leqslant i \leqslant n-1\},$$

$$(3)\qquad L_n(T; f, g; x) = f_n(T; x)/g_n(T; x),$$

with the conventions (here and in similar expressions) that $a/0 = \infty$ if a is positive, $-\infty$ if a is negative, and that $0/0 = 0$. We simplify the notation by omitting the symbols T, f, g when the intention is clear, writing for example f_n for $f_n(T;\ )$, and $L_n(x)$ for $L_n(T; f, g; x)$. From these conventions it results that

$$(4)\qquad L_n(x) = 0 \quad \text{if and only if} \quad f_n(x) = 0.$$

It is clear that if $x \in X-Qf$ then $f_n(x) = 0$ $(n > 0)$, and consequently

$$(5)\qquad L_n(x) = 0 \qquad (x \in X-Qf).$$

Conversely, if T is incompressible and $f \geqslant 0$, one sees that for almost all $x \in Qf$ we have $f_n(x) > 0$ for all large enough n.

We can now state Halmos's extension of Hopf's theorem in the following form.

7.2. Lemma. *Suppose that T is measure-preserving and incompressible, that f is finite and summable, that $0 \leqslant g(x) < \infty$ $(x \in X)$, and that $Qf \subset Qg$. Then, for almost all $x \in X$, $L_n(x)$ converges to a finite limit as $n \to \infty$.*

Remark. This differs from Halmos's formulation in that he requires g to be invariantly positive—that is, in effect, $Qg = X$. The extra generality in 7.2 is trivial, from 7.1 (5), but is convenient for our applications. In applying the Lemma we shall assume $Pf \subset Pg$, which of course implies $Qf \subset Qg$. Hopf himself had required $g(x) > 0$ almost everywhere—that is, in effect, $Pg = X$.

7.3. For our modified ergodic theorems we need still further notation. As before, let f, g be finite functions on X, and n a positive integer; further, let J be any set of real numbers (usually an interval). We define, for each $x \in X$ and each $t > 0$,

$$(1)\qquad K_n(x, t) = K_n(T; f, g; x, t) = \frac{\sum \{f(T^i x)|\ 0 \leqslant i < n,\ \omega_i(x) \geqslant t\}}{\sum \{g(T^i x)|\ 0 \leqslant i < n,\ \omega_i(x) \geqslant t\}},$$

$$(2)\qquad M_n(x, J) = M_n(T; f, g; x, J) = \frac{\sum \{f(T^i x)\omega_i(x)|\ 0 \leqslant i < n,\ \omega_i(x) \in J\}}{\sum \{g(T^i x)\omega_i(x)|\ 0 \leqslant i < n,\ \omega_i(x) \in J\}},$$

with the convention $0/0 = 0$ as before.

7.4. Theorem 3. *Let T be incompressible on X, let f and g be finite (measurable) functions on X, and suppose that f is summable, that g is non-negative, and that* (i) $g(x) = 0$ *implies* $f(x) = 0$ $(x \in X)$. *Then*

(a) *for almost all $x \in X$, and for almost all $t > 0$ (depending on x) $K_n(x, t)$ tends to a finite limit as $n \to \infty$;*

(b) *for almost all $x \in X$, and for all t in a fixed set D of positive real numbers, dense in $[0, \infty)$ (and not depending on x) $K_n(x, t)$ tends to a finite limit as $n \to \infty$;*

(c) *there exists a sequence $\{J_m\}$ $(m = 1, 2, \ldots)$ of closed intervals such that $J_1 \subset J_2 \subset \ldots$ and $\bigcup J_m = (0, \infty)$, such that, for each m and for almost all $x \in X$, $M_n(x, J_m)$ tends to a finite limit as $n \to \infty$.*

Proof of (a) and (b). Let $C^* = X \times [0, 1] \subset X^*$, and let χ^* denote the characteristic function of C^*. We define finite measurable functions f^*, g^* on X^* by: $f^*(x, y) = f(x)\chi^*(x, y)$, $g^*(x, y) = g(x)\chi^*(x, y)$. Clearly f^* is summable (on X^*). In analogy with 7.1 (1) we use the notation $P^*h^* = \{(x, y)|\ h^*(x, y) \neq 0\}$ (where h^* is a function on X^*), and similarly $Q^*h^* = \bigcup T^{*n}P^*h$. Then clearly $P^*f^* = (Pf \times Y) \cap C^* \subset (Pg \times Y) \cap C^* = P^*g^*$, and therefore $Q^*f^* \subset Q^*g^*$. By Theorems 1 and 2, T^* is measure-preserving and incompressible on X^*. Hence, from Lemma 7.2, we have that $L_n(T^*; f^*, g^*; (x, y))$ tends, for all $(x, y) \in X^*-N^*$ where $\mu^*N^* = 0$, to a finite limit as $n \to \infty$. On applying 3.1 (2) to the definitions of K_n and L_n, we readily find $L_n(T^*; f^*, g^*; (x, y)) = K_n(T; f, g; x, y)$; and (a) follows immediately by an application of Fubini's theorem to N^*.

To derive (b), we note that there is a null set $Y_0 \subset Y$ such that, for all $y \in Y - Y_0$, we have $\mu\{x | (x, y) \in N^*\} = 0$. We merely pick a dense sequence $D = \{d_m | m = 1, 2, \ldots\} \subset Y - Y_0$.

7.5. The proof of Theorem 3 (c) uses a different but similar construction. For each $a > 1$, put $C^*(a) = X \frown [a^{-1}, a] \subset X^*$, and let χ_a^* be the characteristic function of $C^*(a)$. Define f_a^* by: $f_a^*(x, y) = y^{-1}f(x)\chi_a^*(x, y)$, and similarly for g_a^*; we see that f_a^*, g_a^* are finite and measurable, that f_a^* is summable and g_a^* non-negative, and that $P^*f^* \subset P^*g^*$, so that $Q^*f^* \subset Q^*g^*$. Thus, for each $a > 1$, Lemma 7.2 gives a null set $N_a^* \subset X^*$ such that, for all $(x, y) \in X^* - N_a^*$, $L_n(T^*; f_a^*, g_a^*; (x, y))$ converges to a finite limit as $n \to \infty$. We let a run over the integers $2, 3, \ldots$, and put $N^* = \bigcup N_a^*$. Since N^* is null, we can choose $y_0 > 0$ such that, for almost all $x \in X$, $(x, y_0) \notin N^*$; in fact y_0 is "almost arbitrary". We take $J_m = [y_0 m^{-1}, y_0 m]$ and easily verify that, for $m = 2, 3, \ldots$,

$$L_n(T^*; f_m^*, g_m^*; (x, y_0)) = M_n(T; f, g; x, J_m),$$

whence (c) follows.

7.6. Remarks. When T is measure-preserving, we have $\omega_i(x) = 1$ for all i and x; thus $K_n(x, t) = K_n(x, 0) = L_n(x)$ whenever $t \leqslant 1$. That is, we may in this case simplify parts (a) and (b) of Theorem 3 by putting $t = 0$, and we then recover Hopf's theorem. We do not quite, however, recover Halmos's extension of Hopf's theorem, our starting point (7.2), because of the assumption labelled (i) in Theorem 3—that $Pf \subset Pg$. It would be desirable to weaken this to $Qf \subset Qg$ (compare 7.2); but I do not know whether this is possible. If it can be done, Theorem 3 (a) will include Halmos's theorem.

Again, as was remarked in 7.1, part (c) of Theorem 3 is similar to the Hurewicz theorem. In fact, if we could replace the intervals J_m by their union, the whole half-line $(0, \infty)$, (c) would then assert that the sequence

$$\sum_{i=0}^{n-1} f(T^i x)\omega_i(x) \Big/ \sum_{i=0}^{n-1} g(T^i x)\omega_i(x)$$

converges (as $n \to \infty$) to a finite limit almost everywhere. This is just the assertion of the theorem of Hurewicz [6], though of course this is not a proof of that theorem (nor does that theorem appear to imply Theorem 3 (c) immediately). We remark further that, as the proof shows,

[6] As improved by Halmos ([1], Th. 4) — and except for our restriction $Pf \subset Pg$. Halmos requires $Qg = X$, which trivially generalizes to $Qf \subset Qg$.

the intervals J_m can be taken to be of the form $[y_0/a_m, y_0 a_m]$, where $\{a_m\}$ is any sequence we please tending to ∞, and y_0 is an almost arbitrary positive real number. It would be easy to enlarge the family of possible J_m's still further.

8. Convergence of $\sum \omega_n$.

8.1. We shall say that a sequence $\{f_n\}$ $(n = 0, \pm 1, \ldots)$ of functions on X "*steps down*" at $x \in X$ if, for some n, we have $f_n(x) > \sup\{f_i(x) | i > n\}$.

Clearly if the sequence $f_n(x)$ is strictly decreasing then $\{f_n\}$ steps down at x. Again, if $f_n(x) > 0$ for all n and $\lim_{n\to\infty} f_n(x) = 0$, then $\{f_n\}$ steps down at x.

We write

$$X_0 = \{x | \omega_n(x) \to 0 \text{ as } n \to \infty\} \subset X,$$

$$X_1 = \{x | \{\omega_n\} \text{ steps down at } x\} \subset X.$$

Thus $X_0 \subset X_1$; from 2.2 (4), both sets are invariant under T.

8.2. THEOREM 4. *If $\mu X_1 > 0$, then T is compressible, and X_1 is dissipative (modulo null sets). If further $\mu X_1 < \infty$, then $X_1 = X_0$ (modulo null sets), and moreover $\sum_{-\infty}^{\infty} \omega_n(x)$ is convergent almost everywhere on X_1* [7]

Before giving the proof, we remark that the theorem will apply to any invariant subset of X (merely consider the restriction of T to this invariant subset). In particular, if $\mu X_0 < \infty$, then $\sum \omega_n(x)$ converges almost everywhere on X_0. We shall show later (8.6) that the finiteness restrictions on μ here cannot be removed.

Proof of Theorem 4. Since X_1 is invariant, it is enough to consider the restriction of T to the measure space (X_1, μ); that is, we may assume $X = X_1$ throughout, and have therefore

(1) $\{\omega_n\}$ steps down at x for all $x \in X$.

To prove that T is compressible, we apply 3.3 (5) to the constant function $h = 1$; then

$$H_n(x) = \sup\{h(T^i x)\omega_i(x) | i \geqslant n\} = \sup\{\omega_i(x) | i \geqslant n\},$$

and consequently, from (1), there exists for each x an $n(x)$ such that $H_{n(x)+1}(x) < H_{n(x)}(x)$. For some integer m, the set $E_m = \{x | H_{m+1}(x) < H_m(x)\}$ is thus of positive measure. We note that, from 3.3 (6), $H_m(Tx)\omega(x) = H_{m+1}(x) \leqslant H_m(x)$ $(x \in X)$, and moreover $H_m(Tx)\omega(x)$

[7] This is a sharpening of the known result that the set where $\sum\{\omega_n(x) | n > 0\}$ converges is dissipative. The last part of the theorem is a restatement of another known result: if $\mu X < \infty$ and X is dissipative then $\sum \omega_n(x)$ converges almost everywhere.

$< H_m(x)$ if $x \in E_m$. The hypotheses of Lemma 5.3 are therefore satisfied (with $Z = E_m$ and $g = H_m$), and T is therefore compressible.

To prove $X(= X_1)$ dissipative, we note that, by the same argument as in 6.4, X has a maximal dissipative subset A (modulo null sets), necessarily invariant. If $\mu(X-A) > 0$, the preceding argument applies to $X-A$ to prove that $T|X-A$ is compressible, so that $X-A$ contains a non-null dissipative set B. But then $A \cup B$ is dissipative, contradicting the maximality of A. Thus, modulo null sets, $X = A$ and is dissipative.

We may therefore write $X = \bigcup T^n W$ $(n = 0, \pm 1, \ldots)$ where the sets $T^n W$ are pairwise disjoint. Assume now that $\mu X < \infty$; then for each integer k we have

$$\int\limits_{T^kW} \sum_{-\infty}^{\infty} \omega_n(x)\, d\mu(x) = \sum_n \int\limits_{T^kW} \omega_n(x)\, d\mu(x) = \sum_n \mu(T^{k+n}W) = \mu X < \infty,$$

so that the series $\sum \omega_n(x)$ has a finite sum almost everywhere on each $T^k W$, and therefore on X. *A fortiori*, $\omega_n(x) \to 0$, so $x \in X_0$, for almost all $x \in X$ $(=X_1)$, completing the proof.

8.3. COROLLARY. *If $\mu X < \infty$, then for almost all $x \in X$ all five of the following assertions are equivalent:*

$(1)^+$ $\{\omega_n\}$ *steps down at* x, $(1)^-$ $\{\omega_{-n}\}$ *steps down at* x,

$(2)^+$ $\lim\limits_{n\to\infty} \omega_n(x) = 0$, $(2)^-$ $\lim\limits_{n\to-\infty} \omega_n(x) = 0$,

(3) $\sum\limits_{-\infty}^{\infty} \omega_n(x)$ *is convergent.*

This follows from the preceding plus the remark that (from 2.1 (1)) $\omega_n(x; T) = \omega_{-n}(x; T^{-1})$, so that Theorem 4 applied to T^{-1} gives $(1)^- \Rightarrow (3)$.

8.4. Let C denote the set of all $x \in X$ for which $\sum\limits_{-\infty}^{\infty} \omega_n(x)$ converges; C is, of course, a subset of X_0, and we have just seen that if $\mu X_1 < \infty$ then $C = X_0 = X_1$ (modulo null sets). We now investigate the relation between these sets when $\mu X_1 = \infty$ [8].

A familiar argument (compare 6.4 and 8.2) shows that X has a (possibly null) subset V which is (modulo null sets) the least upper bound of the family of all invariant sets of finite measure. We write $X - V = U$, and have:

(1) U, V are disjoint invariant sets, $U \cup V = X$, V is a countable union of invariant sets of finite measure; every non-null invariant subset of U has infinite measure.

[8] Actually Theorem 5 below applies even if $\mu X_1 < \infty$, but then gives only $C = X_0 = X_1$ again.

These properties are easily seen to characterize U, V uniquely (modulo null sets). Further, they have a "hereditary" character, and therefore, for every invariant set $Z \subset X$, the sets $Z \cap U$, $Z \cap V$ constitute the corresponding decomposition of Z.

8.5. We shall prove

THEOREM 5. *Modulo null sets,* $C = X_0 \cap V = X_1 \cap V$.

We disregard null sets freely in what follows.

As was just remarked, the decomposition of X_1 corresponding to 8.4 (1) is $\{X_1 \cap U, X_1 \cap V\}$; thus we may continue to restrict attention to X_1. That is, we may assume $X = X_1$, and have to prove $C = V$.

Let Z be any invariant subset of X $(= X_1)$ of positive finite measure. Then, by 8.3 applied to the subspace Z, we see that $\sum \omega_n(x)$ converges on Z; that is, $Z \subset C$. Hence $V \subset C$.

To prove $C \subset V$, we note that $C \subset X$ $(= X_1) = \bigcup T^n W$ as at the end of 8.2; thus $C = \bigcup \{C_{km} \mid k = 0, \pm 1, \ldots, m = 1, 2, \ldots\}$ where $C_{km} = T^k W \cap \{x \mid \sum \omega_n(x) \leqslant m\}$. Further, each C_{km} can be written as the union of sets $D(k, m, r)$ $(r = 1, 2, \ldots)$ of finite measure. Write $E(k, m, r) = \bigcup \{T^n D(k, m, r) \mid n = 0, \pm 1, \ldots\}$, an invariant subset of X, and note that each $D(k, m, r)$ is a wandering set (being contained in $T^k W$). Thus, by essentially the same argument as in 8.2, we have

$$\mu E(k, m, r) = \sum_n \mu\big(T^n(D(k, m, r))\big) = \int\limits_{D(k,m,r)} \sum_n \omega_n(x)\, d\mu(x)$$
$$\leqslant m\mu D(k, m, r) < \infty.$$

This proves that $E(k, m, r) \subset V$. Thus each $D(k, m, r) \subset V$, and so finally $C \subset V$.

8.6. One might expect, by analogy with 8.3, that the series $\sum\limits_{n>0} \omega_n(x)$ and $\sum\limits_{n<0} \omega_n(x)$ might have the same convergence set C as $\sum\limits_{-\infty}^{\infty} \omega_n(x)$ even when $\mu X_1 = \infty$; but this is not the case in general. It will be clear from the following considerations that all three series can have different convergence sets. The same reasoning also shows that the restriction $\mu X_1 < \infty$, in the last part of Theorem 4, is essential.

Suppose that (X, λ) is any normal (and non-atomic) measure space, and A any non-null subset of X such that $\lambda(X-A) \neq 0$. Then, given any (measurable) functions ψ_n on A $(n = 0, \pm 1, \ldots)$ such that $0 < \psi_n < \infty$, *we can always find a compressible* 1-1 *measurable transformation T of X, and a measure μ on X* (σ-finite and equivalent to λ), *agreeing with λ on A, such that $\omega_n(x) = \psi_n(x)$ for almost all $x \in A$ (and for all n).* For we can easily arrange that the sets $T^m A$ are disjoint, of positive measure, and cover X; and we define μ on $T^m A$ by

$$\mu B = \int_{T^{-n}B} \psi_n(x)\, d\lambda(x) \qquad (B \subset T^n A)\,.$$

In other words, the functions ω_n can be prescribed arbitrarily on any proper subset A of X, provided the transformation T is allowed to be compressible and the total measure μX is not required to be finite.

Thus, for example, we can arrange that $\omega_n \to 0$ but $\sum \omega_n = \infty$ throughout any such A; and so on.

As another application of this construction, suppose we start with an *incompressible* transformation T_1 on (X, λ) as above; then we can find a *compressible* transformation T_2 on (X, μ) such that μ and λ agree on A, and further $\omega_n(x; T_2) = \omega_n(x; T_1)$ on A for all n (these ω's being calculated in terms of μ, λ, respectively). Thus, it is impossible to tell, solely from the behavior of the functions ω_n on a proper subset A of X, whether or not the transformation T is compressible on X.

8.7. We conclude with one more corollary to Theorem 4 (8.2):

COROLLARY. *If T is incompressible, then for almost all $x \in X$*

$$\limsup_{n\to\infty} \omega_n(x) = \limsup_{n\to-\infty} \omega_n(x) = \sup_{-\infty<n<\infty} \omega_n(x) > 0\,.$$

For the set where $\limsup_{n\to\infty} \omega_n(x) < \sup_n \omega_n(x)$ is just X_1, and so is null from Theorem 4. This proves one equality; the other follows from the first applied to T^{-1}.

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On products of sets in a locally compact group

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Introduction. Let G be a locally compact group, μ a left Haar measure on G, μ_* the corresponding inner measure. The group G is said to be *unimodular* if the left invariant measure μ is also right invariant; this happens, for instance, when G is either compact, or abelian, or discrete, or a semi-simple Lie group.

Let further A and B be given non-empty subsets of G. Then AB will denote the set of all elements $x \in G$ which admit at least one representation as a product $x = ab$ with $a \in A$ and $b \in B$.

THEOREM 1.1. *Suppose that G is unimodular and connected. Then*

$$\mu_*(AB) \geqslant \mu_*(A) + \mu_*(B)\,, \tag{1.1}$$

unless $\mu(G) < \mu_*(A) + \mu_*(B)$, in which case G is compact and $AB = G$.

The special case, where G is abelian, is due to Kneser [6]. The further special case, that G is also compact and second countable, is due to Shields [9]. It remains to determine the class of pairs (A, B) such that (1.1) holds with the equality sign. For an abelian connected group, this problem was solved by Kneser [6].

THEOREM 1.2. *Suppose that G is unimodular, and further that there exists a pair of non-empty subsets A and B of G such that*

$$\mu_*(AB) < \mu_*(A) + \mu_*(B)\,. \tag{1.2}$$

Assertion: *G contains at least one open and compact subgroup F of size $\mu(F) \leqslant \mu_*(AB)$.*

More precisely, the set AB is both open and compact, and the open and compact subgroup F can be chosen in such a way that

$$aFb \subset AB \quad \textit{whenever} \quad a \in A \textit{ and } b \in B\,. \tag{1.3}$$

Finally, if a subgroup F satisfies (1.3) *then $AB = A_1B_1$ as soon as*

$$A_1 \subset AF,\quad B_1 \subset FB,\quad \mu_*(A_1) + \mu_*(B_1) > \mu(AB)\,. \tag{1.4}$$

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