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On the differentiability of functions

by

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Dedicated to E. Hille
on the occasion of his 70-th birthday

Chapter I

1. In this paper* we extend and generalize the main results of our paper [9]. The knowledge of [9], however, is not indispensable here.

In this chapter we formulate the main results of the paper. Their proofs are given in Chapter II. Chapter III contains some additional results.

Let $F(x)$ be a function defined in the neighborhood of the point x_0 (in what follows we consider only measurable sets and functions). The two functions

$$\varphi_{x_0}(t) = \varphi_{x_0}(t; F) = \tfrac{1}{2}[F(x_0+t)+F(x_0-t)],$$

$$\psi_{x_0}(t) = \psi_{x_0}(t; F) = \tfrac{1}{2}[F(x_0+t)-F(x_0-t)],$$

whose sum is equal to $F(x_0+t)$, will be called respectively the *even* and *odd* part of $F(x_0+t)$; we shall also use the expression the *even* and *odd part of* F *at* x_0.

These parts are of importance in certain problems of the Theory of Functions and, in particular, in Fourier series. Let $S[F]$ denote the Fourier series of a periodic function F (by "periodic" we shall always mean "of period 2π") and $\tilde{S}[F]$ the conjugate series. By $S^{(k)}[F]$ and $\tilde{S}^{(k)}[F]$ we shall mean the series $S[F]$ and $\tilde{S}[F]$ differentiated termwise k times. It is a familiar fact that for the summability (and, in particular, convergence) of $S[F]$ at a given point x_0 decisive is the behavior of the even part $\varphi_{x_0}(t; F)$ near $t=0$. The same holds for the summability of $S^{(k)}[F]$ if k is even and the summability of $\tilde{S}^{(k)}[F]$ if k is odd. Similarly, the behavior of $\psi_{x_0}(t; F)$ near $t=0$ is decisive for the summability of $S^{(k)}[F]$ if k is odd and of $\tilde{S}^{(k)}[F]$ if k is even.

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In the present paper we are primarily interested in the problems of the differentiability of the even and odd parts of a function. The problems belong essentially to Real Variable, but the methods we use lean heavily on the theory of Fourier series and integrals and Complex Variable; in view of the remarks just made about Fourier series this is rather natural.

2. To make the picture more clear we begin with the case of derivatives of order 1.

The differentiability of the odd part $\psi_{x_0}(t; F)$ at $t=0$ is the same thing as the existence of the first symmetric derivative

$$\lim_{t\to 0}\frac{F(x_0+t)-F(x_0-t)}{2t}$$

of F at x_0. The differentiability of the even part $\varphi_{x_0}(t; F)$ at $t=0$ is clearly equivalent to the relation

$$F(x_0+t)+F(x_0-t)-2F(x_0) = o(t). \tag{1}$$

The latter relation is usually called the *smoothness* of the function F at the point x_0 and was first considered by Riemann in his memoir on trigonometric series. Functions which are continuous and smooth at each point have a number of interesting properties (see [12] or [11_{I}], p. 42 sqq.). If (1) holds we shall also say that F *satisfies condition* λ *at* x_0. If we merely have

$$F(x_0+t)+F(x_0-t)-2F(x_0) = O(t),$$

we shall say that F *satisfies condition* Λ at x_0 (¹).

(¹) The following reflexion upon the significance of smooth functions may be not totally out of place. In view of the fact that smooth functions play important role in certain problems of the Theory of Functions one may ask about the origin of this importance. The answer is not immediately obvious and one may be easily led to irrelevant notions and generalizations. For example, it may appear that the expression

$$\omega_2(h) = \operatorname*{Max}_{x,\,|t|\leqslant h}|F(x+t)+F(x-t)-2F(x)|$$

is merely an analogue of the modulus of continuity

$$\omega_1(h) = \operatorname*{Max}_{x,\,|t|\leqslant h}|F(x+t)-F(x)|$$

of F, and one is naturally led to considering expressions $\omega_k(h)$ defined in a similar way but using the k-th differences. Such expressions are interesting and useful, but after $k=1$ only the case $k=2$ seems to be of real importance, the reason being that the behavior of $\omega_2(h)$ expresses a property of the *even part* of the function. Here, it seems, lies the source of significance of smooth functions. A good illustration are applications of smooth functions to elliptic differential equations discussed in [1].

It is obvious that the differentiability of F at x_0 implies the differentiability of both $\varphi_{x_0}(t)$ and $\psi_{x_0}(t)$ at $t=0$. It is equally clear that neither the differentiability of $\varphi_{x_0}(t)$ at $t=0$, nor that of $\psi_{x_0}(t)$ implies the existence of $F'(x_0)$. It is natural, however, to ask about theorems of the "almost everywhere" type. It turns out that the roles played here by the even and odd parts of the function are completely different. We list a few known results.

THEOREM A. *If F has a first symmetric derivative at each point of a set E, then F is differentiable almost everywhere in E.*

THEOREM B. *There exist continuous functions which satisfy condition* (1) *everywhere, even uniformly in x, and which are differentiable in sets of measure* 0 *only* (²).

If, however, we strengthen condition (1) somewhat, the function becomes differentiable. The precise result is as follows:

THEOREM C. *Let $\varepsilon(t)$ be a function defined in some interval $0<t\leqslant\eta$, monotonically decreasing to* 0 *with t and such that the integral*

$$\int_0^\eta \frac{\varepsilon^2(t)}{t}\,dt \tag{2}$$

is finite. If for each x belonging to a set E we have

$$F(x+t)+F(x-t)-2F(x) = O\{t\varepsilon(t)\} \quad (t\to 0), \tag{3}$$

not necessarily uniformly in x, then F is differentiable almost everywhere in E.

(²) Theorems A and B clearly show the difference between $\varphi_{x_0}(t)$ and $\psi_{x_0}(t)$ as regards differentiability. The picture is a little different for the continuity of the even and odd part of F. The problems here are easier and we state a few facts.

The continuity at $t=0$ of the even and odd parts of $F(x_0+t)$ means respectively

(a) $F(x_0+t)+F(x_0-t)-2F(x_0)\to 0$,

(b) $F(x_0+t)-F(x_0-t)\to 0$

for $t\to 0$. It is not difficult to show (see Lemma 9 in Chapter II) that if we have either (a) or (b) at each point of a set E, then F is continuous at almost all points of E. Thus there is no difference between the continuity of the even and odd part of F. (In particular, if F satisfies condition Λ at each point of E, then F is continuous almost everywhere in E).

The result just stated can be generalized as follows. Let $a_1, a_2, \ldots, a_k$ be a sequence of real numbers all different, and let $\beta_1, \beta_2, \ldots, \beta_k$ be another sequence such that $\sum\beta_i=0$. Suppose that at some point x_0 we have

(c) $\sum\beta_i F(x_0+a_i t)\to 0$

or, what is the same thing, (c′) $\sum\beta_i\{F(x_0+a_i t)-F(x_0)\}\to 0$, as $t\to 0$. We may then say that F is *conditionally* continuous at x_0 (relative to the sequences $\{a_i\}$ and $\{\beta_i\}$). It can be shown that if (c) holds for $t\to +0$ (or $t\to -0$) at each $x_0\in E$, then F is continuous almost everywhere in E. This stems from the fact that F is anyway approximately continuous almost everywhere in E and this coupled with condition (c) gives the desired result. Similarly, if at each $x_0\in E$ the left side of (c) is ultimately bounded as $t\to +0$, then F is bounded in the neighborhood of almost all points of E.

That this result is, in a sense, best possible is shown by the following

THEOREM D. *Let $\varepsilon(t)$, $0 < t \leqslant \eta$, be a function monotonically decreasing with t, satisfying the condition*

$$\varepsilon(2t)/\varepsilon(t) \to 1$$

for $t \to 0$ and such that the integral (2) diverges. Then there is a continuous function $F(x)$ satisfying for all x the condition

$$|F(x+t)+F(x-t)-2F(x)| \leqslant t\varepsilon(t) \quad (0 < t \leqslant \eta) \tag{4}$$

and differentiable in a set of measure 0 only.

Theorem A is an old result of Khintchin [3]. The function

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin 2^n x}{2^n n^{1/2}} \tag{5}$$

can be taken for the function of Theorem B (see e. g., [11_{I}], p. 47-48). It is known that continuous functions which are smooth at each point must necessarily have points of differentiability so that the exceptional set of measure 0 in Theorem B cannot be empty (see [11_{I}], p. 43). Theorem C is the main result of paper [1], and it is indicated there that if $\varepsilon(t)$ satisfies the hypotheses of Theorem D, then the continuous function

$$F(x) = \sum \varepsilon\left(\frac{1}{2^n}\right) \frac{\sin 2^n x}{2^n} \tag{6}$$

is differentiable in a set of measure 0 only and satisfies condition (4) if multiplied by a suitable positive constant.

That the function (6) is differentiable in a set of measure 0 only follows from the fact that the divergence of the integral (2) is equivalent to the divergence of $\sum \varepsilon^2(2^{-n})$ so that the lacunary series $\sum \varepsilon(2^{-n}) \cos 2^n x$ obtained by the termwise differentiation of (6) is not in L^2 and therefore, as is well known (see [11_{I}], p. 203) cannot be summable by any given linear method of summation except, perhaps, in a set of measure 0 only; in particular, F' can exist in a set of measure 0 only.

The remaining part of the conclusion of Theorem D is easy to verify by a familiar argument. We have

$$F(x+t)+F(x-t)-2F(x) = -4\sum_{1}^{\infty} \varepsilon(2^{-n})2^{-n} \sin 2^n x \sin^2 \tfrac{1}{2} 2^n t$$
$$= -4\sum_{1}^{N} - 4\sum_{N+1}^{\infty} = P+Q,$$

say, where N is determined by the condition $2^{-N-1} \leqslant t < 2^{-N}$, $N = 1, 2, \ldots$ Clearly,

$$|Q| \leqslant 4\sum_{N+1}^{\infty} \varepsilon(2^{-n})2^{-n} \leqslant 4\cdot 2^{-N}\varepsilon(2^{-N-1}) \leqslant 8t\varepsilon(t),$$
$$|P| \leqslant t^2 \sum_{1}^{N} \varepsilon(2^{-n})2^n = t^2 \sum_{1}^{N} \varepsilon(2^{-n})(\tfrac{3}{2})^n(\tfrac{4}{3})^n,$$

say. Since $(\frac{3}{2})^n \varepsilon(2^{-n}) \leqslant (\frac{3}{2})^{n+1}\varepsilon(2^{-n-1})$ for n large enough, there is a constant A independent of N such that $(\frac{3}{2})^n\varepsilon(2^{-n}) \leqslant A(\frac{3}{2})^N\varepsilon(2^{-N})$ for $n = 1, 2, \ldots, N$. It follows that $|P| \leqslant O(t^2)\varepsilon(2^{-N})2^N = O(t\varepsilon(t))$. Hence, collecting results we see that $P+Q = O(t\varepsilon(t))$ and Theorem D follows. (We easily see from the proof that the condition $\varepsilon(2t)/\varepsilon(t) \to 1$ can be replaced by $\limsup_{t\to 0}\{\varepsilon(2t)/\varepsilon(t)\} = \gamma < 2$; for $\gamma = 2$ the result is false.)

It may be observed that (5) is essentially a special case of (6) with $\varepsilon(t) = \left(\log \frac{1}{t}\right)^{-(1/2)}$, and that in this case the integral (2) diverges.

3. In what follows we shall sometimes say that the function $\varepsilon(t)$ defined in a right-hand side neighborhood of $t = 0$ *satisfies condition* N if $\varepsilon^2(t)/t$ is integrable over some interval $(0, \eta)$.

If we have (3) and the integral (2) is finite, then the function

$$\frac{[F(x_0+t)+F(x_0-t)-2F(x_0)]^2}{t^3} \tag{7}$$

is integrable near $t = 0$. Conversely, the integrability of the function (7) near $t = 0$ implies that we have a relation (3) with $\varepsilon(t)$ satisfying condition N. But it is important to observe that in the latter case the function $\varepsilon(t)$ may, first, depend on x_0 (that is, $\varepsilon(t) = \varepsilon_{x_0}(t)$) and, second, that it does not necessarily tend to 0 with t; and even if it does, it need not tend to 0 monotonically.

Theorem E which follows clearly generalizes Theorem D; it is one of the main results of the paper.

THEOREM E. *If $F(x)$ satisfies condition Λ at each point x_0 of a set E, and if for each $x_0 \in E$ the function (7) is integrable near $t = 0$, then $F'(x)$ exists almost everywhere in E.*

The integrability of the function (7) near $t = 0$ was first considered by Marcinkiewicz [4] who proved the following theorem:

THEOREM F. *Suppose that F is differentiable at each point of a set E. Then at almost all $x_0 \in E$ the function (7) is integrable near $t = 0$.*

Since the differentiability of F at a point implies that F satisfies condition Λ (even, λ) at that point, Theorems E and F can be combined in the following single theorem:

THEOREM G. *Suppose that $F(x)$, defined in an interval, satisfies condition Λ at each point of a set E. Then the necessary and sufficient condition for F to be differentiable almost everywhere in E is that for almost all $x_0 \in E$ the function*

$$\varepsilon_{x_0}(t) = \frac{F(x_0+t)+F(x_0-t)-2F(x_0)}{2t} \tag{8}$$

satisfies condition N.

The following result is merely a variant of Theorem G:

THEOREM G′. *Suppose that $F \in L^2(-\infty, +\infty)$ and satisfies condition Λ at each point of a set E. Then F is differentiable almost everywhere in E if and only if for almost all $x \in E$ the expression*

$$\mu(x) = \mu(x, F) = \left\{\int_0^\infty \frac{[F(x+t)+F(x-t)-2F(x)]^2}{t^3}\,dt\right\}^{1/2} \tag{9}$$

if finite.

This is clear since, if $F \in L^2(-\infty, +\infty)$, the part of the integral in (9) which extends over any interval $\eta \leqslant t < \infty$ $(\eta > 0)$ is always finite.

The integral (9) is sometimes called *the integral of Marcinkiewicz.*

4. The question naturally arises what are the necessary and sufficient conditions for the existence of the integral (9) almost everywhere in E if we no longer assume that F satisfies condition Λ in E. To answer this question we must generalize the notion of derivative.

Let $1 \leqslant p \leqslant \infty$ and suppose that F belongs to L^p in the neighborhood of the point x_0. Suppose further that there is a polynomial

$$P(t) = \sum_{j=0}^{k} \frac{a_j}{j!} t^j \tag{10}$$

of degree k such that

$$\left\{\frac{1}{2h}\int_{-h}^{h} |F(x_0+t)-P(t)|^r\,dt\right\}^{1/r} = o(h^k) \tag{11}$$

as $h \to +0$; the polynomial P, if it exists, is unique. We shall then say that F is *differentiable of order* k at x_0 in L^p. The polynomial $P(t)$ may be called the *k-th differential* of F at x_0, and the number a_k the *k-th derivative* of F at x_0, both in the metric L^p (these notions were introduced in [1]). It is clear that the existence of the k-th differential implies that of the $(k-1)$-th differential. If $p = \infty$ the left side of (11) means, of course, ess sup $|F(x_0+t)-F(x_0)|$ for $|t| \leqslant h$, and modifying F in a set of measure 0 we then have

$$F(x_0+t) = P(t)+o(t^k) \qquad (t \to 0),$$

so that F has at x_0 a k-th derivative, or k-th differential, in the classical sense (of Peano). In this case, the coefficient a_j of P we shall occasionally denote by $F_{(j)}(x_0)$.

We can now answer the question raised above.

THEOREM H. *Suppose that $F(x)$ is defined in an interval. The necessary and sufficient condition for*

$$\varepsilon_{x_0}(t) = \frac{F(x_0+t)+F(x_0-t)-2F(x_0)}{2t}$$

to satisfy condition N *almost everywhere in the set E is that F has a derivative in L^2 at almost all points of E.*

THEOREM H′. *Let $F \in L^2(-\infty, +\infty)$. The necessary and sufficient condition for $\mu(x, F)$ to be finite for almost all $x \in E$ is that F be differentiable in L^2 almost everywhere in E.*

These two results do not differ essentially. Observe that the integrability of $\varepsilon_{x_0}^2(t)/t$ near $t = 0$ implies the integrability of $[F(x+t)+F(x_0-t)-2F(x_0)]^2$ near $t = 0$, and it can be shown without much difficulty that the latter implies the integrability of F^2 near almost every point $x_0 \in E$. The version H′ is useful in some cases.

5. Theorems G, G′, H, H′ can be extended to higher derivatives. Suppose that k is *even* and the *even* part of F has a k-th Peano derivative at x_0, or that k is *odd* and the *odd* part of F has a k-th derivative at x_0. In the former case,

$$\varphi_{x_0}(t) = a_0 + \frac{1}{2!}a_2t^2 + \ldots + \frac{1}{k!}a_kt^k + o(t^k),$$

and in the latter,

$$\psi_{x_0}(t) = a_1t + \frac{1}{3!}a_3t^3 + \ldots + \frac{1}{k!}a_kt^k + o(t^k),$$

as $t \to 0$. In either case F has a symmetric k-th derivative at x_0 equal to a_k, that is

$$\lim_{t \to 0}\left\{t^{-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}F\big(x_0+(j-\tfrac{1}{2}k)t\big)\right\} = a_k. \tag{13}$$

For if, e. g., k is even, the sum $\sum$ here is equal to

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\varphi_{x_0}\big((j-\tfrac{1}{2}k)t\big)$$

and (13) is a consequence of the formula for $\varphi_{x_0}(t)$ and the formulas for the k-th differences of the function t^j. On the other hand, it is known (see [7]) that if a function has a k-th symmetric derivative in a set E, then it has a k-th differential almost everywhere in E. Hence we have the following analogue of Theorem A: *If for each x_0 in a set E the even part of F has a k-th differential at $t=0$ of even order k, or the odd part has a k-th differential of odd order k, then almost everywhere in E the function F itself has a differential of order k.*

The question is what happens if we interchange the roles of the even and odd parts. We shall say that F is *smooth of order* k at x_0 if k is even and $\psi_{x_0}(t;F)$ has a k-th differential at $t=0$, or if k is odd and $\varphi_{x_0}(t;F)$ has a k-th differential at $t=0$. In the former case,

$$\psi_{x_0}(t)=a_1t+\frac{1}{3!}a_3t^3+\ldots+\frac{1}{(k-1)!}a_{k-1}t^{k-1}+o(t^k), \tag{14}$$

and in the latter,

$$\varphi_{x_0}(t)=a_0+\frac{1}{2!}a_2t^2+\ldots+\frac{1}{(k-1)!}a_{k-1}t^{k-1}+o(t^k). \tag{15}$$

Either of these conditions will also be called λ_k, and condition Λ_k will be defined by replacing here o by O. Clearly, conditions λ and Λ introduced previously correspond to the case $k=1$.

A function may satisfy condition λ_k, even uniformly in x, and have a k-th differential in a set of measure 0 only. A simple example is obtained by integrating the series (5) term-by-term $k-1$ times. Since the function (5) satisfies condition λ uniformly in x, the sum of the integrated series satisfies, as one can easily verify, condition λ_k uniformly in x. At each point where the sum of the integrated series has a k-th differential, the series obtained by termwise differentiation $k-1$ times, that is, the series $\sum n^{-(1/2)}\cos 2^n x$, is summable by a linear method of summation, and this can occur only in a set of measure 0.

If at each point of a set E the function F satisfies condition λ_k, or even only Λ_k, then the last term in (14) or (15) (as the case may be) is $o(t^{k-1})$, and since the parity of $k-1$ is opposite to that of k the function F has, by the result stated above, a $(k-1)$-th differential, that is,

$$F(x_0+t)=\sum_{j=0}^{k-1}\frac{1}{j!}a_j(x_0)t^j+o(t^{k-1}) \tag{16}$$

at almost all points of E, and the problem is to find when F has a k-th differential almost everywhere in E. The theorem which follows is an extension of Theorem E.

THEOREM 1. *Suppose that at each point $x_0\in E$ the function F satisfies condition Λ_k, i. e., we have either*

(17) $\psi_{x_0}(t)$

$$=a_1(x_0)t+\frac{1}{3!}a_3(x_0)t^3+\ldots+\frac{1}{(k-1)!}a_{k-1}(x_0)t^{k-1}+\frac{\varepsilon_{x_0}(t)}{k!}t^k \qquad (k\text{–}even)$$

or

(18) $\varphi_{x_0}(t)$

$$=a_0(x_0)+\frac{1}{2!}a_2(x_0)t^2+\ldots+\frac{1}{(k-1)!}a_{k-1}(x_0)t^{k-1}+\frac{\varepsilon_{x_0}(t)}{k!}t^k \qquad (k\text{–}odd)$$

where $\varepsilon_{x_0}(t)$ is bounded near $t=0$. Suppose, moreover, that $\varepsilon_{x_0}(t)$ satisfies condition N *at each point of E. Then F has a k-th differential almost everywhere in E.*

That the result is best possible can be proved by means of the function obtained by integrating the series (6) termwise $k-1$ times, where $\varepsilon(t)$ satisfies the hypotheses of Theorem D. The resulting function will satisfy conditions (17) or (18) uniformly in x_0, with $|\varepsilon_{x_0}(t)|\leqslant A\varepsilon(t)$, and will have a k-th differential in a set of measure 0 only. We shall not dwell on this point.

6. The next theorem is an analogue of Theorem F.

THEOREM 2. *Suppose that $F(x)$ is defined in an interval and at each point of a set E has a k-th differential (and so, in particular, is smooth of order k). Then at almost all points $x_0\in E$ the function $\varepsilon_{x_0}(t)$ in* (17) *or* (18) *satisfies condition* N.

This theorem is proved in [13]. It is included in Theorem 4 below which asserts that it is enough to assume that at each point of E the function F has a k-th differential in L^2.

A corollary of Theorem 1 and 2 is the following Theorem 3:

THEOREM 3. *If $F(x)$ is defined in an interval and at each point of E satisfies condition Λ_k (and, in particular, has a $(k-1)$-th differential almost everywhere in E) then F has a k-th differential almost everywhere in E if and only if the function $\varepsilon_{x_0}(t)$ in* (17) *or* (18) *satisfies condition* N *almost everywhere in E*

or

THEOREM 3′. *If $F\in L^2(-\infty,+\infty)$ and satisfies condition Λ_k at each point $x\in E$, then F has a k-th differential almost everywhere in E if and only if the expression*

$$\mu_k(x_0)=\mu_k(x_0,F)=\left\{\int_0^\infty\frac{\varepsilon_{x_0}^2(t)}{t}\,dt\right\}^{1/2},$$

where $\varepsilon_{x_0}(t)$ is given by (17) *or* (18), *is finite almost everywhere in E.*

We now pass to the case when F is no longer supposed to satisfy condition Λ_k in E, so that $\varepsilon_{x_0}(t)$ in (17) or (18) need no longer be bounded as $t\to 0$, $x_0\in E$. We want, of course, the developments (17) and (18) to be unique, and conditions like, e. g.,

$$\varepsilon_{x_0}(t)=o(1/t) \qquad (t\to 0)$$

are certainly sufficient. Another condition guaranteeing the same result would be that F has at x_0 a $(k-1)$-differential in some sense, for example, in the metric L^p, $1 \leqslant p < \infty$. On the other hand, theorems can be formulated in such a way that we do not need the assumption of the existence of the $(k-1)$-th differential of F in any sense, and we will follow this approach ([3]).

THEOREM 4. *Suppose that at each point $x_0 \in E$ we have* (17) *or* (18). *Then* a) *if $\varepsilon_{x_0}(t)$ satisfies condition* N *everywhere in E, the function F has a k-th differential in L^2 almost everywhere in E. Conversely,* b) *if at each point $x_0 \in E$ the function F has a k-th differential $\sum_0^k a_j(x_0)t^j/j!$ in L^2, then the function $\varepsilon_{x_0}(t)$ defined by* (17) *or* (18), *as the case may be, satisfies condition* N *almost everywhere in E.*

THEOREM 4′. *Suppose that $F \in L^2(-\infty, +\infty)$ and satisfies* (17) *or* (18) *at each point $x_0 \in E$. Then,* a) *if $\mu_k(x, F) < \infty$ everywhere in E, the function F has a k-th differential in L^2 almost everywhere in L^2, and, conversely,* b) *if F has a k-th differential in L^2 in E, then $\mu_k(x, F) < \infty$ almost everywhere in E.*

Chapter II

1. In this chapter we prove the theorems stated in Chapter I. All the theorems are essentially of local character and in their proofs we may assume that the functions under consideration are periodic of period 2π and use the properties of their Fourier series and Poisson integrals.

Given a periodic and integrable function $F(\theta)$ we denote by $F(\varrho, \theta)$ its Poisson integral

$$F(\varrho, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)P(\varrho, \theta-t)\,dt,$$

where

$$P(\varrho, \theta) = \frac{1}{2} + \sum_1^\infty \varrho^\nu \cos\nu\theta = \frac{1}{2}\,\frac{1-\varrho^2}{1-2\varrho\cos\theta+\varrho^2}$$

is the Poisson kernel.

([3]) Consider, e. g., the case of Theorem 4′ below, and denote by P_{k-1} the polynomial on the right of (17) or (18). The hypothesis that $\varepsilon_{x_0}(t)$ satisfies condition N implies that

$$\Big\{h^{-1}\int_0^h [\varphi_{x_0}(t)-P_{k-1}(t)]^2\,dt\Big\}^{1/2} \quad \text{or} \quad \Big\{h^{-1}\int_0^h [\varphi_{x_0}(t)-P_{k-1}(t)]^2\,dt\Big\}^{1/2}$$

is $o(h^{k-1})$, as the case may be. It is known that this implies the existence of the $(k-1)$-th differential of F, in L^2 and at almost all points of E (see also Chapter II, Section 11) but we prefer not to use this fact.

Given any number $0 < \sigma < 1$, we shall denote by $\Omega_\sigma(\theta)$ the convex domain limited by the two tangents from $\zeta = e^{i\theta}$ to the circumference $|\zeta| = \sigma$ and the arc of this circumference between the points of contact; if σ is fixed we shall write $\Omega(\theta)$ for $\Omega_\sigma(\theta)$.

LEMMA 1. *Let k be a positive integer and suppose that a periodic and quadratically integrable $F(\theta)$ satisfies at the point θ_0 a relation*

$$\varphi_{\theta_0}(t; F) = a_1 t + \frac{a_3}{3!}t^3 + \ldots + \frac{a_{k-1}}{(k-1)!}t^{k-1} + \frac{\varepsilon(t)}{k!}t^k \tag{1}$$

if k is even, or

$$\varphi_{\theta_0}(t; F) = a_0 + \frac{a_2}{2!}t^2 + \ldots + \frac{a_{k-1}}{(k-1)!}t^{k-1} + \frac{\varepsilon(t)}{k!}t^k \tag{2}$$

if k is odd, where, in either case, $\varepsilon(t)$ satisfies condition N. *Then for any $0 < \sigma < 1$ we have*

$$\iint\limits_{\Omega_\sigma(0)} \Big[\frac{\partial^{k+1}}{\partial\theta^{k+1}}F(\varrho, \theta_0+\xi) + \frac{\partial^{k+1}}{\partial\theta^{k+1}}F(\varrho, \theta_0-\xi)\Big]^2 \varrho\, d\varrho\, d\xi < \infty. \tag{3}$$

We shall denote, for brevity, the l-th derivative of $P(\varrho, \theta)$ with respect to θ by $P_l(\varrho, \theta)$. We shall also systematically write δ for $1-\varrho$. We need the following two inequalities, valid for $l = 0, 1, \ldots$:

$$|P_l(\varrho, \theta)| \leqslant A_l\,\delta^{-l-1}, \tag{4}$$

$$|P_l(\varrho, \theta)| \leqslant A_l\,\delta\,|\theta|^{-l-2}. \tag{5}$$

Here and in what follows (except when otherwise stated), A with various subscripts will mean constants (not always the same) depending only on parameters displayed in subscripts; A without subscripts will mean an absolute constant.

The inequality (4) follows by differentiating the series $\frac{1}{2}+\sum\varrho^\nu\cos\nu\theta$ termwise l times and observing that $\sum\varrho^\nu\nu^l = O(\delta^{-l-1})$. To prove (5), we use the formula

$$f_\theta = i(\zeta f_\zeta - \bar\zeta f_{\bar\zeta}) \qquad (\zeta = \varrho e^{i\theta}),$$

where $f_\zeta = \frac{1}{2}(f_\xi - if_\eta)$, $f_{\bar\zeta} = \frac{1}{2}(f_\xi + if_\eta)$, if $\zeta = \xi + i\eta$. If f is, say, a rational function of ζ and $\bar\zeta$, then f_ζ is obtained by formal differentiation with respect to ζ, treating $\bar\zeta$ as constant, and similarly for $f_{\bar\zeta}$. Hence, by observing that

$$2P(\varrho, \theta)(1-\varrho^2)^{-1} = \frac{1}{\zeta-\bar\zeta}\Big(\frac{1}{1-\zeta} - \frac{1}{1-\bar\zeta}\Big),$$

and that in the neighborhood of $\zeta = 1$ both $|\zeta-\bar\zeta|$ and $|1-\zeta|$ majorize $A|\theta|$, we obtain (5).

The proof of (3) is the same for k even and odd; for the sake of definiteness we assume that k is even. Then

$$(6)\qquad \frac{\partial^{k+1}}{\partial\theta^{k+1}}F(\varrho,\theta_0+\xi)+\frac{\partial^{k+1}}{\partial\theta^{k+1}}F(\varrho,\theta_0-\xi)$$
$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}F(\theta_0+t)[P_{k+1}(\varrho,t-\xi)+P_{k+1}(\varrho,t+\xi)]dt$$
$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}F(\theta_0+t)[P_{k+1}(\varrho,\xi+t)-P_{k+1}(\varrho,\xi-t)]dt$$
$$=-\frac{2}{\pi}\int_{0}^{\pi}\varphi_{\theta_0}(t)[P_{k+1}(\varrho,\xi+t)-P_{k+1}(\varrho,\xi-t)]dt.$$

By hypothesis, $\varphi_{\theta_0}(t)$ consists of an odd polynomial of degree $\leqslant k-1$ in t and a remainder $\varepsilon(t)t^k/k!$. Integration by parts shows (see (5)) that if l is odd and $l\leqslant k$, then

$$\int_0^\pi t^l\{P_{k+1}(\varrho,\xi+t)-P_{k+1}(\varrho,\xi-t)\}dt=-\int_{-\pi}^{\pi}t^lP_{k+1}(\varrho,\xi+t)dt$$
$$=O(\delta)+(-1)^l l!\int_{-\pi}^{\pi}P_{k+1-l}(\varrho,\xi+t)dt=O(\delta).$$

Hence the last term of (6) is

$$(7)\qquad -\frac{2}{\pi k!}\int_0^\pi\varepsilon(t)t^k\{P_{k+1}(\varrho,\xi+t)-P_{k+1}(\varrho,\xi-t)\}dt+O(\delta).$$

In showing that the integral in (3) is finite, we may restrict our considerations to that part of $\Omega_\sigma(0)$ which is in the neighborhood of the point $z=1$. If $\varrho e^{i\xi}\in\Omega_\sigma(0)$ and $\delta=1-\varrho$ is small enough, we have $|\xi|\leqslant\varkappa\delta$, where $\varkappa=\varkappa(\sigma)$. Let Ω' be the part of $\Omega_\sigma(0)$ where $|\xi|\leqslant\varkappa\delta$, $\delta\leqslant\delta_0$ and δ_0 is so small that $2\varkappa\delta_0\leqslant 1-\sigma$.

In estimating the integral of the square of the expression (7) over Ω' we may omit the term $O(\delta)$ whose contribution is finite. We split the integral in (7) into two parts extended over $0\leqslant t\leqslant 2\varkappa\delta$ and $2\varkappa\delta\leqslant t\leqslant\pi$, and denote the resulting expressions respectively by $S(\delta,\xi)$ and $T(\delta,\xi)$. We have, by (4),

$$|S|\leqslant A_k\delta^{-k-2}\int_0^{2\varkappa\delta}|\varepsilon(t)|t^k dt\leqslant A_{k\sigma}\delta^{-2}\int_0^{2\varkappa\delta}|\varepsilon(t)|dt,$$

and, by (5),

$$|T|\leqslant A_k\delta\int_{2\varkappa\delta}^{\pi}|\varepsilon(t)|\frac{t^k}{|t-\xi|^{k+3}}dt\leqslant A_k\delta\int_{2\varkappa\delta}^{\pi}|\varepsilon(t)|\frac{t^k}{(\frac{1}{2}t)^{k+3}}\leqslant A_k\delta\int_{2\varkappa\delta}^{\pi}|\varepsilon(t)|t^{-3}dt.$$

It is enough to show that both S^2 and T^2 are integrable over Ω'. Schwarz's inequality gives

$$S^2(\delta,\xi)\leqslant A_{k\sigma}\delta^{-3}\int_0^{2\varkappa\delta}\varepsilon^2(t)dt,$$

$$T^2(\delta,\xi)\leqslant A_k\delta^2\int_{2\varkappa\delta}^{\pi}\frac{\varepsilon^2(t)dt}{t^3}\int_{2\varkappa\delta}^{\infty}\frac{dt}{t^3}\leqslant A_{k\sigma}\int_{2\varkappa\delta}^{\pi}\varepsilon^2(t)t^{-3}dt,$$

so that

$$\iint_{\Omega}S^2(\delta,\xi)\varrho d\theta d\xi\leqslant\int_0^{\delta_0}d\delta\int_{-\varkappa\delta}^{\varkappa\delta}S^2(\delta,\xi)d\xi\leqslant\int_0^{\delta_0}d\delta\cdot 2\varkappa\delta\cdot A_{k\sigma}\delta^{-3}\int_0^{2\varkappa\delta}\varepsilon^2(t)dt$$
$$\leqslant A_{\varkappa\sigma}\int_0^{2\varkappa\delta_0}\varepsilon^2(t)dt\int_{t/2\varkappa}^{\delta_0}\delta^{-2}d\delta\leqslant A_{\varkappa\sigma}\int_0^{2\varkappa\delta_0}t^{-1}\varepsilon^2(t)dt<\infty,$$

and

$$\iint_{\Omega'}T^2(\delta,\xi)\varrho d\varrho d\theta\leqslant A_{\varkappa\sigma}\int_0^{\delta_0}\delta d\delta\int_{2\varkappa\delta}^{\pi}t^{-3}\varepsilon^2(t)dt\leqslant A_{\varkappa\sigma}\int_0^{\delta_0}\delta d\delta\int_{2\varkappa\delta}^{2\varkappa\delta_0}t^{-3}\varepsilon^2(t)dt+O(1)$$
$$=A_{\varkappa\sigma}\int_0^{2\varkappa\delta_0}t^{-3}\varepsilon^2(t)dt\int_0^{t/2\varkappa}\delta d\delta+O(1)\leqslant A_{\varkappa\sigma}\int_0^{2\varkappa\delta_0}t^{-1}\varepsilon^2(t)dt+O(1)<\infty.$$

This completes the proof of Lemma 1.

2. LEMMA 2. *If $U(x,y)+iV(x,y)$ is holomorphic in the interior of a finite rectilinear triangle D, and U^2 is integrable over D, so is V^2.*

This is a special case of a general theorem of Friedrichs [2] valid for a much wider class of domains D. If V, which contains an arbitrary additive constant, is suitably normalized we even have an inequality

$$\iint_D V^2dxdy\leqslant A_D\iint_D U^2dxdy,$$

but the weaker statement of the lemma is sufficient for our purposes.

LEMMA 3. *Let $\Phi(\zeta)=\Phi(\varrho e^{i\theta})$ be regular in $|\zeta|<1$, and suppose that for each θ_0 belonging to a set E of positive measure there exists a $\sigma=\sigma(\theta_0)$, $0<\sigma<1$, such that the integral*

$$\iint_{\Omega_\sigma(\theta_0)}|\Phi'(\zeta)|^2\varrho d\varrho d\theta$$

is finite. Then $\Phi(\zeta)$ has a finite non-tangential limit almost everywhere in E.

For the proof see, e. g., [11_{II}], p. 207.

LEMMA 4. *If the integral $\int_0^\eta t^{-1}\varepsilon^2(t)dt$ is finite for some $\eta>0$, then $\int_0^u|\varepsilon(t)|dt=o(u)$.*

The hypothesis implies that $\int_0^u \varepsilon^2(t)\,dt = o(u)$ and it is enough to apply Schwarz's inequality.

LEMMA 5. *Suppose that a periodic and integrable $F(\theta)$ satisfies at $\theta=\theta_0$ the relation* (1) *or* (2), *where*

$$\int_0^u |\varepsilon(t)|\,dt = o(u). \tag{8}$$

Then $\tilde{S}^{(k)}[F]$ is summable $(C, k+1)$ at θ_0 if and only if it is summable A.

Let $\tilde{\sigma}_n^a(\theta)$ be the (C, a) means of $\tilde{S}[F]$. It is well known that if F satisfies condition λ_k at θ_0, that is, if $\varepsilon(t)$ in (1) or (2), as the case may be, tends to 0 with t, then

$$\frac{d^k}{d\theta^k}\tilde{\sigma}_n^{k+1}(\theta_0) - \left(-\frac{2}{\pi}\int_{1/n}^{\infty}\frac{\varepsilon(t)}{t}\,dt\right) \to 0 \qquad (n\to\infty) \tag{9}$$

(see [11_{II}], p. 63), but a glance at the proof shows that the conclusion holds under the weaker condition (8). The proof of (9) is exclusively based on the following estimates for the conjugate $(C, k+1)$ kernel $\tilde{K}_n^{k+1}(\theta)$:

$$\left|\frac{d^k}{d\theta^k}\tilde{K}_n^{k+1}(\theta)\right| \leqslant A_k n^{k+1},$$

$$\left|\frac{d^k}{d\theta^k}\left\{\frac{1}{2}\cot\frac{1}{2}\theta - \tilde{K}_n^{k+1}(\theta)\right\}\right| \leqslant A_k n^{-1}\theta^{-k-2} \qquad (n^{-1}\leqslant\theta\leqslant\pi).$$

But the conjugate Poisson kernel

$$Q(\varrho,\theta) = \sum \varrho^\nu \sin\nu\theta = \varrho\sin\theta/(1-2\varrho\cos\theta+\varrho^2)$$

satisfies analogous inequalities, with n replaced by $1/\delta$. For, clearly $|d^kQ/d\theta^k| \leqslant \sum\varrho^\nu\nu^k \leqslant A_k\delta^{-k-1}$, and from the formula

$$\frac{\sin\theta}{2(1-\cos\theta)} - Q(\varrho,\theta) = \frac{1-\varrho}{1+\varrho}P(\varrho,\theta)\cot\frac{1}{2}\theta,$$

using (5), we obtain

$$\left|\frac{d^k}{d\theta^k}\left\{\frac{1}{2}\cot\frac{1}{2}\theta - Q(\varrho,\theta)\right\}\right| \leqslant A_k\delta^2\theta^{-k-3} \leqslant A_k\delta\theta^{-k-2},$$

where $\delta\leqslant\theta\leqslant\pi$. Hence, if $\tilde{F}(\varrho,\theta)$ is the harmonic function conjugate $F(\varrho,\theta)$, we have, under the hypotheses of the lemma, the following analogue of (9):

$$\frac{d^k}{d\theta^k}\tilde{F}(\varrho,\theta_0) - \left(-\frac{2}{\pi}\right)\int_\delta^\infty\frac{\varepsilon(t)}{t}\,dt \to 0 \qquad (\delta = 1-\varrho\to 0).$$

Observing that (8) implies the relation $\int_{1/(n+1)}^{1/n} t^{-1}|\varepsilon(t)|\,dt\to 0$, we obtain from (9) and (10)

$$\frac{d^k}{d\theta^k}\tilde{F}(\varrho,\theta_0) - \frac{d^k}{d\theta^k}\tilde{\sigma}_n^{k+1}(\theta_0)\to 0 \qquad \left(n = \left[\frac{1}{1-\varrho}\right]\right),$$

which completes the proof of the lemma.

LEMMA 6. *If a trigonometric series S with coefficients $o(n^l)$ $(l = 0,1,2,\ldots)$ is summable (C, l) at a point θ_0 to sum s, and if $F(\theta)$ is the sum of the series obtained by integrating S termwise $l+2$ times, then $\frac{1}{2}\{F(\theta_0+t)+(-1)^lF(\theta_0-t)\}$ has at $t=0$ an $(l+2)$-nd Peano derivative equal to s.*

For a proof, see [3_{II}], p. 66.

LEMMA 7. *If a trigonometric series S is summable (C, a), $a>-1$, in a set E, then the conjugate series $\tilde{S}$ is summable (C, a) almost everywhere in E.*

The proof may be found in [8]. Only the case $a = 0,1,2,\ldots$ is needed here.

3. In the preceding sections of this chapter we gave a number of lemmas pertaining to trigonometric series and harmonic and analytic functions. We are now going to give lemmas about functions of real variables. While most of these lemmas are known, and we shall be satisfied with giving references, the lemma which follows is essentially new and is basic for the proofs of theorems of Chapter I.

LEMMA 8. *Let $f(x,y)$ be defined and real-valued in the open half-plane $y>0$, and be in L^2 near each point of this open half-plane. Suppose also, for simplicity, that $f=0$ for $y>\omega>0$. Then, if there is a set P on the axis $y=0$ such that, for each $x\in P$,*

$$\int_0^\infty yf^2(x,y)\,dy < \infty \tag{11}$$

and, with some $a=a_x>0$,

$$\iint_{y\geqslant a|t|}[f(x+t,y)+f(x-t,y)]^2\,dy\,dt<\infty, \tag{12}$$

then for almost all $x\in P$ we have

$$\iint_{y\geqslant\beta|t|} f^2(x+t,y)\,dt\,dy<\infty \tag{13}$$

no matter how small is $\beta>0$.

It is easy to see that for each fixed a the integral in (12) is a lower-continuous, possibly infinite, function of x, and so is certainly measurable.

Also measurable, as a function of x, is the integral in (11). The domain of integration in (12) decreases as a increases, and so, by considering the sequence of values $a = 1, 2, 3, \ldots$ and the corresponding subsets of P we may suppose that 1° a is fixed troughout P; 2° the integrals in (11) and (12) are bounded on P:

$$(14)\qquad \int_0^\infty y f^2(x, y)\,dy \leqslant M; \quad \iint_{y\geqslant a|t|} \{f(x+t, y)+f(x-t, y)\}^2\,dy\,dt \leqslant M \quad (x\in P);$$

3° P is closed, bounded and of positive measure. It is enough to prove that (13) holds almost everywhere in P.

Integrating the second inequality (14) over P we have

$$\int_P dx \iint_{y\geqslant a|t|} \{f(x+t, y)+f(x-t, y)\}^2\,dy\,dt \leqslant M|P|,$$

or, making the change of variables $x+t = u$, $x-t = v$,

$$\tfrac{1}{2} \iint_{\frac{1}{2}(u+v)\in P} du\,dv \int_{y\geqslant \frac{1}{2}a|u-v|} \{f(u, y)+f(v, y)\}^2\,dy \leqslant M|P|.$$

If we reduce the domain of integration by restricting the variable v to P we have, a fortiori,

$$(15)\qquad \iint_{\frac{1}{2}(u+v)\in P,\ v\in P} du\,dv \int_{y\geqslant \frac{1}{2}a|u-v|} \{f(u, y)+f(v, y)\}^2\,dy \leqslant 2M|P|.$$

The main idea of the proof of the lemma consists a) in showing that, with our hypotheses, the integral

$$(16)\qquad I = \iint_{\frac{1}{2}(u+v)\in P,\ v\in P} du\,dv \int_{y\geqslant \frac{1}{2}a|u-v|} f^2(v, y)\,dy$$

is finite, so that, in view of (15), the integral

$$(17)\qquad J = \iint_{\frac{1}{2}(u+v)\in P,\ v\in P} du\,dv \int_{y\geqslant \frac{1}{2}a|u-v|} f^2(u, y)\,dy$$

is finite and then, b) deducing from the latter fact the inequality (13) for almost all $x\in P$.

Since $f(x, y) = 0$ for $y > \omega$, we may, if need be, restrict our integration in (16) or (17) to subdomains of the strip $0 < y \leqslant \omega$. Hence the values of u in (16) or (17) are actually confined to a finite interval. Dropping the condition $\frac{1}{2}(u+v)\in P$ we obtain from (16) that

$$I \leqslant \int_{-\infty}^{+\infty} du \int_P dv \int_{y\geqslant \frac{1}{2}a|u-v|} f^2(v, y)\,dy = \int_P dv \int_0^\infty f^2(v, y)\Big\{\int_{y\geqslant \frac{1}{2}a|u-v|} du\Big\}\,dy$$
$$\leqslant \int_P dv \int_0^\infty f^2(v, y)\frac{4y}{a}\,dy \leqslant \frac{4M}{a}|P|,$$

the last inequality being a cosequence of the first condition (14). From this and (15), (16), (17) we deduce that

$$(18)\qquad J = \iint_{\frac{1}{2}(u+v)\in P,\ v\in P} du\,dv \int_{y\geqslant \frac{1}{2}a|u-v|} f^2(u, y)\,dy \leqslant 4\left(1+\frac{2}{a}\right)M|P|.$$

Clearly,

$$(19)\qquad J = \int_{-\infty}^{+\infty} du \int_0^\infty f^2(u, y)\,dy\Big\{\int_{\substack{\frac{1}{2}(u+v)\in P,\ v\in P\\ y\geqslant \frac{1}{2}a|u-v|}} dv\Big\} = \int_{-\infty}^{+\infty}\int_0^\infty f^2(u, y)\mu(u, y)\,du\,dy,$$

where $\mu(u, y)$ denotes the integral in curly brackets. For fixed $y > 0$ and u, $\mu(u, y)$ is the measure of the set of points v on the real axis which lie in the interval

$$(20)\qquad u-\frac{2y}{a} \leqslant v \leqslant u+\frac{2y}{a}$$

and which, in addition, satisfy the conditions

$$(21)\qquad v\in P, \quad \tfrac{1}{2}(u+v)\in P.$$

We claim that if u_0 is any point of density of P, then, as the point (u, y) approaches $(u_0, 0)$ non-tangentially from the upper half-plane, $\mu(u, y)$ is asymptotically equal to $4y/a$, that is the length of the interval (20).

Suppose, e. g., that $u_0 = 0$ is a point of density of P. The non-tangential approach in this case means that

$$(22)\qquad y \geqslant \varepsilon|u|$$

for some $\varepsilon > 0$. Let $\varphi(v)$ be the characteristic function of the set P and $\psi(v) = 1-\varphi(v)$ that of the complementary set. Then

$$\mu(u, y) = \int_{u-2y/a}^{u+2y/a} \varphi(v)\varphi\big(\tfrac{1}{2}(u+v)\big)\,dv = \int_{u-2y/a}^{u+2y/a} \{1-\psi(v)\}\big\{\big(1-\psi\big(\tfrac{1}{2}(u+v)\big)\big)\big\}\,dv$$
$$= \frac{4y}{a} - \int_{u-2y/a}^{u+2y/a} \psi(v)\,dv - \int_{u-2y/a}^{u+2y/a} \psi\left(\frac{u+v}{2}\right)dv + \int_{u-2y/a}^{u+2y/a} \psi(v)\psi\left(\frac{u+v}{2}\right)dv,$$

and it is enough to show that each of the last three integrals is $o(y)$.

The first integral is, in view of (22), dominated by

$$(23)\qquad \int_{-(2a^{-1}+\varepsilon^{-1})y}^{(2a^{-1}+\varepsilon^{-1})y} \psi(v)\,dv = o(y).$$

The second integral can be written

$$2\int_{u-y/a}^{u+y/a} \psi(w)\,dw$$

and so is, likewise, $o(y)$. Finally, the third integral, being dominated by the first, is $o(y)$. Hence, actually, $\mu(u,y)\simeq 4a^{-1}y$ as (u,y) tends non-tangentially to any point of density of P.

It follows that there is a closed subset P_0 of P, with $|P-P_0|$ arbitrarily small, and a $\delta>0$ such that if $u_0\epsilon P_0$, then

$$\mu(u,y)>\frac{2y}{a} \tag{24}$$

provided $0<y\leqslant\delta$, $y\geqslant\beta|u-u_0|$, where β is any fixed positive number.

In particular, denoting by $\Delta_\beta(u_0)$ the set of points (u,y) satisfying these two conditions, and by U_β the union of the $\Delta_\beta(u_0)$ for $u_0\epsilon P_0$, we obtain

$$\frac{2}{a}\iint\limits_{U_\beta} f^2(u,y)\,y\,du\,dy\leqslant J. \tag{25}$$

Let now $g(u,y)$ be equal to $f^2(u,y)$ in U_β and to 0 elsewhere. Then

$$\int\limits_{-\infty}^{+\infty}dx\int\limits_{\Delta_\beta(x)}g(u,y)\,du\,dy$$
$$=\int\limits_{-\infty}^{+\infty}dx\iint\limits_{0<y\leqslant\delta,\ y\geqslant\beta|x-u|}g(u,y)\,du\,dy=\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{\delta}g(u,y)\,du\,dy\Big\{\int\limits_{\beta|x-u|\leqslant y}1\cdot dx\Big\}$$
$$=2\beta^{-1}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{\delta}g(u,y)\,y\,du\,dy=2\beta^{-1}\iint\limits_{U_\beta}f^2(u,y)\,y\,du\,dy\leqslant a\beta^{-1}J,$$

by (25). Since, by (18), J is finite, the integral $\iint\limits_{\Delta_\beta(x)}g(u,y)\,du\,dy$ is finite for almost all x. It follows that the integral

$$\iint\limits_{\Delta_\beta(x_0)}f^2(u,y)\,du\,dy$$

is finite for almost all $x_0\epsilon P_0$. Since $f^2(u,y)$ is locally integrable in the interior of the upper half-plane, and since δ and $|P-P_0|$ can be arbitrarily small, the lemma follows.

4. We add a few remarks about the lemma.

(i) In certain cases important for applications, condition (11) is, essentially, a corollary of condition (12). Suppose, for example, that $f(x,y)$ is harmonic in the closed triangle

$$a|x|\leqslant y,\quad 0\leqslant y\leqslant\omega, \tag{Δ}$$

except, possibly, at the vertex $(0,0)$. Let $\beta=(1+a^2)^{1/2}$. The circle with center $(0,y)$ and radius $r=\beta y$ is tangent to two sides of Δ and is contained in Δ provided y is sufficiently small, $0<y<\omega'<\omega$. Let $\chi_y(\xi,\eta)$

be the characteristic function of the disc limited by the circle. If $y\leqslant\omega'$, we have, by the familiar property of harmonic functions,

$$f(0,y)=\frac{1}{\pi r^2}\iint\limits_{\xi^2+(\eta-y)^2\leqslant r^2}f(\xi,\eta)\,d\xi\,d\eta$$
$$=\frac{1}{\pi r^2}\iint\limits_{\Delta}\frac{1}{2}\big(f(\xi,\eta)+f(-\xi,\eta)\big)\chi_y(\xi,\eta)\,d\xi\,d\eta,$$

$$f^2(0,y)\leqslant\frac{1}{4}\,\frac{1}{\pi r^2}\iint\limits_{\Delta}\{f(\xi,\eta)+f(-\xi,\eta)\}^2\chi_y(\xi,\eta)\,d\xi\,d\eta$$
$$=\frac{A_a}{y^2}\iint\limits_{\Delta}\{f(\xi,\eta)+f(-\xi,\eta)\}^2\chi_y\,d\xi\,d\eta.$$

Hence, multiplying by y, integrating over $0<y\leqslant\omega'$ and changing the order of integration,

$$\int\limits_0^{\omega'}yf^2(0,y)\,dy\leqslant A_a\iint\limits_{\Delta}\{f(\xi,\eta)+f(-\xi,\eta)\}^2\Big\{\int\limits_0^{\omega'}y^{-1}\chi_y(\xi,\eta)\,dy\Big\}d\xi\,d\eta.$$

But, for a fixed (ξ,η), we can have $\chi_y(\xi,\eta)=1$ at most if $y-r\leqslant\eta\leqslant\leqslant y+r$, that is to say, if y is contained between two fixed multiples (depending only on a) of η. Hence the integral in brackets is majorized by A_a and

$$\int\limits_0^{\omega'}yf^2(0,y)\,dy\leqslant A_a\iint\limits_{\Delta}\{f(\xi,\eta)+f(-\xi,\eta)\}^2\,d\xi\,d\eta.$$

Since the hypotheses of Lemma 8 imply that for almost all x the integral $\int\limits_\varepsilon^\infty yf^2(0,y)\,dy$ is finite, no matter how small $\varepsilon>0$, we immediately see that if $f(x,y)$ is harmonic in some strip $0<y<\eta$, condition (11) may be dropped without affecting the conclusions of the lemma.

(ii) It is clear that the quadratic integrabilities in Lemma 8 are not essential. If $1\leqslant p<\infty$, f is locally in L^p in the interior of the upper half-plane, and if the integrands in (11) and (12) are replaced respectively by $y|f(x,y)|^p$ and $|f(x+t,y)+f(x-t,y)|^p$ we still have the conclusion (13) with f^2 replaced by $|f|^p$ (this holds even for $0<p<\infty$). We may also assume that f is complex-valued. If f is harmonic in some strip $0<y<\eta$ we may again omit the analogue of condition (11), etc.

Certain problems lead to integrals analogous to (12) with sum in the integrand replaced by the difference, but the conclusion of the lemma still holds; instead of the sum in (12) we could take a linear combination with constant coefficients, and even some more general expressions.

(iii) This case corresponding to $p=\infty$ in (ii) is of interest, though it will not be needed in this paper. *Suppose that* $f(x,y)$ *is defined for* $y>0$ *and that for each* $x\epsilon P$, $|P|>0$, *there is an* $a=a_x$ *such that* $\frac{1}{2}\{f(x+t,y)+$ $+f(x-t,y)\}$ *tends to a finite limit* $g(x)$ *as* $y\to 0$ *and* $0<a|t|\leqslant y$. *Then at almost all points* $x\epsilon P$ *we have* $f(x+t,y)\to g(x)$ *as* $y\to 0$ *and* $\beta|t|\leqslant y$, *no matter what* $\beta>0$. The proof resembles that of Lemma 8 but the details are simpler. The hypotheses imply, in particular, that $f(x,y)\to$ $\to g(x)$ for each $x\epsilon P$ as $y\to 0$. We give a sketch of the proof.

We may suppose that a is fixed, that the convergence of the semi-sum $\frac{1}{2}\{f(x+t,y)+f(x-t,y)\}$ is uniform on P, that P is closed, and that $g(x)$ (being measurable on P) is continuous on P. We set, as before, $x+t=u$, $x-t=v$ and we deduce that

$$\frac{1}{2}\{f(u,y)+f(v,y)\}-g\left(\frac{u+v}{2}\right) \tag{26}$$

tends uniformly to 0 if $\frac{1}{2}(u+v)\epsilon P$, $y\geqslant\frac{1}{2}a|u-v|$, $y\to 0$. Let us augment these conditions by the requirement that $v\epsilon P$. We then have conditions (21) and (22) satisfied and, as we have shown, if u_0 is any point of density of P and the point (u,y) tends non-tangentially to $(u_0,0)$, the set of points v such that $v\epsilon P$, $\frac{1}{2}(u+v)\epsilon P$, $\frac{1}{2}a|u-v|\leqslant y$ has, asymptotically, measure $4y/a$ and so is (this is the only thing we need here) non-empty if y is small enough. To sum up, if (u,y) tends non-tangentially to $(u_0,0)$ and y is small enough, then there is a $v\epsilon P$ such that $\frac{1}{2}(u+v)\epsilon P$, $\frac{1}{2}a|u-v|$ $\leqslant y$. Applying this information to the expression (26) and using the facts that $g\left(\frac{u+v}{2}\right)\to g(u_0)$ and $f(v,y)=g(v)+\{f(v,y)-g(v)\}\to g(u_0)$, we see that $f(u,y)\to g(u_0)$ and the assertion is established.

5. Lemma 9 [4]. (i) *Suppose that a function* $F(x)$ *is defined in an interval and that at each point* x_0 *of a set* E *we have*

$$\lim_{t\to 0}\{F(x_0+t)-F(x_0-t)\}=0, \tag{27}$$

then F *is continuous at almost all points of* E. (ii) *The conclusion holds if* (27) *is replaced by*

$$\lim_{t\to 0}\{F(x_0+t)+F(x_0-t)-2F(x_0)\}=0. \tag{28}$$

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[4] This lemma is certainly known but since we cannot give an adequate quotation we give the proof.

Added in proof. The conclusion for assumption (28) is an immediate consequence of Lemma 8, Remark (iii), this page: it is enough to take $f(x,y)=F(x)$, $g(x)=F(x)$. Similarly for assumption (27).

Results analogous to Lemma 9 are discussed in a paper of C. J. Neugebauer which is to appear in the Duke Journal.

re (i). Let E_n be the set of $x_0\epsilon E$ such that $|F(x_0+t)-F(x_0-t)|\leqslant n^{-1}$ for $0<t\leqslant 1/n$. We have $E_1\subset E_2\subset\ldots\subset E_n\ldots$, $E=\sum E_n$, and it is enough to show that at almost all points of each set E_n we have $\limsup_{t\to 0}|F(x_0+t)-F(x_0)|\leqslant n^{-1}$. In accordance with our general assumptions, F and E are supposed to be measurable but we do not assume the measurability of the E_n. We fix $n=n_0$. Since F is measurable, there is a closed set P, with $|E-P|$ arbitrarily small, such that F is continuous on, and with respect to, P. If we show that at each point x_0 which is a point of external density for E_{n_0} and at the same time a point of density of P we have $\limsup_{t\to 0}|F(x_0+t)-F(x_0)|\leqslant n_0^{-1}$, part (i) of the lemma will be established. Suppose for the sake of simplicity that $x_0=0$, $F(x_0)=0$, and that $x\to +0$.

The set of points $y\epsilon P$ such that $0<y<x$ has, asymptotically, measure x as $x\to 0$. The set of points y, $0<y<x$, such that $z=\frac{1}{2}(x+$ $+y)\epsilon E_{n_0}$ has, as one can easily see, external measure asymptotically equal to x as $x\to 0$. Hence, if x is small enough, there is a $y\epsilon P$ such that $\frac{1}{2}(x+y)\epsilon E_{n_0}$. It follows that, with $h=\frac{1}{2}(x-y)$,

$$F(x)=F(y)+\{F(x)-F(y)\}=F(y)+\{F(z+h)-F(z-h)\}.$$

Since $|F(z+h)-F(z-h)|\leqslant n_0^{-1}$ for x sufficiently small and $F(y)\to$ $\to F(0)=0$, we see that $\limsup_{x\to 0}|F(x)|\leqslant n_0^{-1}$ and (i) is established.

re (ii). Let E_n be the set of x such that $|F(x+t)+F(x-t)-2F(x)|\leqslant$ $\leqslant n^{-1}$ for $0<t\leqslant n^{-1}$, and let P have the same meaning as before. It is enough to show that at every point x_0 which is a point of external density for E_n and a point of density for P we have $\limsup_{t\to 0}|F(x_0+t)-$ $-F(x_0)|\leqslant n^{-1}$. We fix $n=n_0$ and assume that $x_0=0$, $F(x_0)=0$, $x\to +0$. For a fixed x, the external measure of the set of the y's in $(0,x)$ such that $\frac{1}{2}(x+y)\epsilon E_{n_0}$ is, asymptotically, x. The measure of the set of y's in $(0,x)$ such that $y\epsilon P$, and $z=\frac{1}{2}(x+y)\epsilon P$ is also, asymptotically, x. Hence, if x is small enough, there is in $(0,x)$ a point y such that y is in P and $z=\frac{1}{2}(x+y)$ is both in P and E_{n_0}. Since, with $h=$ $=\frac{1}{2}(x-y)$,

$$F(x)=F(z+h)+F(z-h)-2F(z)+2F(z)-F(y),$$

and since the last two terms tend to 0 as $x\to 0$, it follows that $\limsup_{x\to +0}|F(x)|\leqslant n_0^{-1}$, and the proof of (ii) is complete.

6. Lemma 10. *If for each* x_0 *in a set* E *the even part* $\varphi_{x_0}(t)$ *of a function* F *has a* k*-th differential at* $t=0$ *of even order* k, *or the odd part* $\psi_{x_0}(t)$ *has at* $t=0$ *a* k*-th differential of odd order* k, *then almost everywhere in* E *the function* F *itself has a* k*-th differential.*

This result was already stated and used in Section 6 of Chapter I. As indicated above, a proof may be found in [7].

LEMMA 11. *Let $F(x)$ be a function defined in an interval and equal to 0 in a set E. Suppose, moreover, that at each point of E we have, with $k = 1, 2, \ldots$, independent of x,*

$$F(x+t)-F(x-t) = O(t^k) \tag{29}$$

or

$$F(x+t)+F(x-t) = O(t^k). \tag{30}$$

Then $F_{(k)}(x)$ exists almost everywhere in E.

This result is essentially contained in [10], but since now the assumptions are somewhat different we give the proof here. We note the basic difference between Lemmas 10 and 11: a possible reversal of the roles of k even and odd and, correspondingly, additional assumptions in Lemma 11 about the behavior of F on E (these assumptions could be considerably relaxed).

Consider first (29). Let E_n be the subset of E consisting of points x such that $|F(x+t)-F(x-t)| \leqslant nt^k$ for $0 \leqslant t \leqslant 1/n$. It is enough to show that $F(x_0+t) = o(t^k)$ at each point $x_0 \epsilon E_n$ which is a point of external density for E_n (and so also a point of density for E). Suppose, for simplicity, that $x_0 = 0$, $t > 0$, and let ε be an arbitrarily small but fixed positive number. If t is small enough, in particular $t \leqslant 1/n$, we can find in the interval $\big((1-\varepsilon)t, t\big)$ a point $\xi \epsilon E$ such that $u = \frac{1}{2}(\xi+t) \epsilon E_n$. Then, with $h = \frac{1}{2}(t-\xi)$, we have

$$|F(t)| = |F(u+h)-F(x-h)| \leqslant nh^k \leqslant n(\tfrac{1}{2}\varepsilon t)^k,$$

and since ε can be as small as we please, $F(t) = o(t^k)$. For assumption (30) the proof is the same.

7. LEMMA 12. *Let $f(x)$ be defined in a finite interval and suppose that f has a k-th differential at each point of a set E, $|E| > 0$. Then, for any $\varepsilon > 0$ we can find a closed subset P of E, with $|E-P| < \varepsilon$, and a decomposition*

$$f(x) = g(x)+h(x),$$

where $g \epsilon C^k$, $g = f$ on P, and, except possibly for a finite number of intervals contiguous to P,

$$|h(x)| \leqslant C\{\delta(x)\}^k,$$

$\delta(x)$ denoting the distance of x from the set P, and C a constant independent of x.

For a proof see [5] or [11_{II}], p. 73-77.

LEMMA 13. *Suppose that $F(x)$ is defined in an interval and that for each x_0 in a set E there exists a number $h = h_{x_0} > 0$ such that the integral $\int_0^h [F(x_0+t)+F(x_0-t)]^2 dt$ is finite. Then F^2 is integrable near almost all points of E. The same conclusion holds if in the assumption we replace $F(x_0+t)+F(x_0-t)$ by $F(x_0+t)-F(x_0-t)$.*

The proof follows the usual pattern. We may assume that the interval (a, b) of definition of F is finite and denote by E_n the set of points $x_0 \epsilon E$ such that $|F(x_0)| \leqslant n$ and $\int_0^{1/n} [F(x_0+t)+F(x_0-t)]^2 dt \leqslant n$ (hence the distance of x_0 from both a and b is $\geqslant 1/n$). Thus $E_1 \subset E_2 \subset E_3 \subset, \ldots$, $E = \sum E_n$. We fix n and integrate the last inequality over E_n which we denote by $\mathcal{E}$. Setting $x_0+t = u$, $x_0-t = v$, we have

$$\iint\limits_{\substack{v\epsilon\mathcal{E},\ \frac{1}{2}(u+v)\epsilon\mathcal{E}\\ 0\leqslant\frac{1}{2}(u-v)\leqslant 1/n}} [F(u)+F(v)]^2 du\, dv \leqslant 2n|\mathcal{E}|,$$

and in particular, since $|F(v)| \leqslant n$, the integral

$$\int_a^b F^2(u)\Big\{\int\limits_{\substack{v\epsilon\mathcal{E},\ \frac{1}{2}(u+v)\epsilon\mathcal{E}\\ 0\leqslant\frac{1}{2}(u-v)\leqslant 1/n}} dv\Big\} du = \int_a^b F^2(u)\xi(u)du, \tag{31}$$

say, is finite. If u is situated in $(a+2/n, b)$ if $0 < \eta \leqslant 2/n$, and if φ is the characteristic function of $\mathcal{E}$, then

$$\xi(u) = \int_{u-2/n}^{u} \varphi(v)\varphi\{\tfrac{1}{2}(u+v)\}dv \geqslant \int_{u-\eta}^{u} \varphi(v)\varphi\{\tfrac{1}{2}(u+v)\}dv.$$

But if u_0 is any point of density of $\mathcal{E}$ and if $|u-u_0| \leqslant \eta$, the last integral is asymptotically equal to η as $\eta \to 0$. Hence $\xi(u)$ is bounded below by a positive number in the neighborhood of any point of density of $\mathcal{E}$ that is situated in the interior of the interval $(a+2/n, b)$, and the finiteness of the integral (31) implies that F^2 is integrable in the neighbourhood of every such point. From this we deduce that F^2 is integrable in the neighbourhood of almost all points of the set E. This is the first part of the lemma and the proof of the second part is similar.

8. LEMMA 14 ([5]). *Let a be a positive number. Suppose that a function $F(x)$ is defined in an interval, vanishes on a set E, $|E| > 0$, and at each point $x \epsilon E$*

([5]) This is a special case of a more general result of Dr. Mary Weiss, [14]. For $a = 0$ it reduces to Lemma 13.

(32) $$\int_0^h [F(x+t)+F(x-t)]^2 dt = O(h^\alpha) \quad (h \to 0).$$

Then at almost all points $x \in E$ we have

(33) $$\int_{-h}^h F^2(x+t)\,dt = O(h^\alpha).$$

If $\alpha \geqslant 1$ we can replace the O in the last equation by o. The conclusions hold if the integrand in (32) is $[F(x+t)-F(x-t)]^2$.

We write $E = \sum E_n$, where E_n is the set of points $x \in E$ such that

(34) $$\int_0^h [F(x+t)+F(x-t)]^2 dt \leqslant nh^\alpha \quad \text{for} \quad 0 < h \leqslant 1/n,$$

and we will show that (33) holds at the points of density of each E_n. We fix n, write $E_n = \mathcal{E}$, and suppose, for example, that $x = 0$ is a point of density of $\mathcal{E}$. Let $\mathcal{E}(h)$ be the part of $\mathcal{E}$ situated in the interval $(-h, h)$. Integrating (34) over $\mathcal{E}(h)$ we have (for $h \leqslant 1/n$)

$$\int_{\mathcal{E}(h)} dx \int_{-h}^h [F(x+t)+F(x-t)]^2 dt = O(h^{\alpha+1}).$$

Hence, with $x+t = u$, $x-t = v$,

$$\iint_{\frac12|u-v|\leqslant h,\ \frac12(u+v)\in\mathcal{E}(h)} [F(u)+F(v)]^2\,du\,dv = O(h^{\alpha+1}).$$

We adjoin on the left the condition $v \in \mathcal{E}(h)$ and restrict u (which is, anyway, contained in $(-2h, 2h)$) to the interval $(-h, h)$. Since then $F(v) = 0$, we obtain, a fortiori,

(35) $$\int_{-h}^h F^2(u)\Big\{\int_{\frac12|u-v|\leqslant h,\ v\in\mathcal{E}(h),\ \frac12(u+v)\in\mathcal{E}(h)} dv\Big\}\,du = O(h^{\alpha+1}).$$

But if $u \in (-h, h)$ and $v \in \mathcal{E}(h)$, then, necessarily, $\frac12|u-v| \leqslant h$, so that the latter condition can be dropped in the last formula and the the cofactor of $F^2(u)$ in the integrand can be written

$$\int_{-h}^h \varphi(v)\varphi\big(\tfrac12(u+v)\big)\,dv,$$

where φ designates the characteristic function of the set $\mathcal{E}$. But if $u \in (-h, h)$ the last integral is asymptotically equal to $2h$ so that (35) implies $\int_{-h}^h F^2(u)\,du = O(h^\alpha)$. Hence (33) holds almost everywhere in E.

Suppose now that $\alpha \geqslant 1$, and let P be any closed subset of E where (33) holds uniformly; in other words,

(36) $$\int_{-h}^h F^2(x+t)\,dt \leqslant Mh^\alpha \quad \text{for} \quad 0 < h \leqslant \eta,\ x \in P,$$

with M independent of x. At every point x_0 which is a point of density of P we necessarily have $\int_{-h}^h F^2(u)\,du = o(h^\alpha)$. For if $0 < h \leqslant \eta$, the left side of (36), with $x = x_0$, is majorized by $M\sum(b_n-a_n)^\alpha$, where (a_n, b_n) are the intervals contiguous to P which overlap with (x_0-h, x_0+h). Since $\alpha \geqslant 1$, we have $\sum(b_n-a_n)^\alpha \leqslant \{\sum(b_n-a_n)\}^\alpha = o(h^\alpha)$, x_0 being a point of density of P. It is now enough to observe that $|E-P|$ can be arbitrarily small.

Remark. Lemma 14 holds, of course, if the exponent 2 on the left is replaced by any $p \geqslant 1$.

LEMMA 15. *Let P be a closed set situated in a finite interval (a, b). Let $\chi(x)$ be the function equal to 0 in P and equal to $\beta-\alpha$ for x in (α, β), if (α, β) is any interval contiguous to P. Let $\delta(t)$ denote the distance of the point t from P. Then for any $\lambda > 0$ the integrals*

$$\int_a^b \frac{\chi^\lambda(t)}{|x-t|^{\lambda+1}}\,dt, \qquad \int_a^b \frac{\delta^\lambda(t)}{|x-t|^{\lambda+1}}\,dt$$

converge at almost all points $x \in P$.

This is well known; see e. g., [11_1], p. 130. If x is a point of density of P, the convergence of either integral is equivalent to that of $\sum[(b_n-a_n)/d_n(x)]^{\lambda+1}$ where $d_n(x)$ is the distance of (a_n, b_n) from x.

The lemma which follows is an analogue of Lemma 12 for derivatives in L^2. Its proof may be found in [1], p. 186-189, Theorem 9 and Corollary. The formulation there is very general, valid for derivatives in L^p and functions of n variables. The special case we need is as follows:

LEMMA 16. *Let $F(x)$ be a function defined in a finite interval and suppose that at each point x of a set E, $|E| > 0$, F has a k-th derivative in L^2. Then for every $\varepsilon > 0$ we can find a closed subset P of E with $|E-P| < \varepsilon$, a positive number η, and a decomposition $F(x) = G(x)+H(x)$ such that $G \in C^k$, $G = F$ on P, and*

$$\int_{-h}^h H^2(x+t)\,dt \leqslant Mh^{2k+1} \quad \textit{for} \quad x \in P,\ 0 \leqslant h \leqslant \eta,$$

with M independent of x.

9. We can now pass to the proofs of the theorems enunciated in Chapter I, beginning with Theorem 1. For the sake of definiteness, we assume that k is odd; for k even the proof is similar.

Suppose, therefore, that for each $x_0 \epsilon E$ we have (cf. (2))

$$\varphi_{x_0}(t; F) = a_0(x_0) + \frac{a_2(x_0)}{2!}t^2 + \ldots + \frac{a_{k-1}(x_0)}{(k-1)!}t^{k-1} + \frac{\varepsilon_{x_0}(t)}{k!}t^k,$$

with $\varepsilon_{x_0}(t)$ bounded as $t \to 0$ and $\varepsilon_{x_0}^2(t)/t$ integrable near $t = 0$. We have to show that F has a k-th differential almost everywhere in E. The boundedness of $\varepsilon_{x_0}(t)$ as $t \to 0$ implies that $F(x_0+t)+F(x_0-t)-2F(x_0)$ tends to 0 as $t \to 0$, so that F is continuous almost everywhere in E (Lemma 9), and since the problem of the differentiability of F is local, we may assume that F is periodic (of period 2π), bounded, and continuous in E. Let $\tilde{F}(x)$ be the function conjugate to F, $F(\varrho, x)$ the Poisson integral of F, and $\tilde{F}(\varrho, x)$ the conjugate Poisson integral of F.

By Lemma 1, for each $x_0 \epsilon E$ the function

$$\left|\frac{\partial^{k+1}}{\partial x^{k+1}}F(\varrho, x_0+\xi) + \frac{\partial^{k+1}}{\partial x^{k+1}}F(\varrho, x_0-\xi)\right|^2$$

is integrable over any domain $|\xi| \leqslant C(1-\varrho)$, $\frac{1}{2} \leqslant \varrho < 1$, and by Lemma 8 the same can be said, for almost all $x_0 \epsilon E$, about the function

$$\left|\frac{\partial^{k+1}}{\partial x^{k+1}}F(\varrho, x_0+\xi)\right|^2$$

and so also, by Lemma 2, about the function

$$\left|\frac{\partial^{k+1}}{\partial x^{k+1}}\tilde{F}(\varrho, x_0+\xi)\right|^2.$$

This implies, by Lemma 3, that the function

$$\frac{\partial^k}{\partial x^k}\{F(\varrho, x) + i\tilde{F}(\varrho, x)\},$$

which is regular inside the unit circle, has a non-tangential limit at almost all points $x_0 \epsilon E$. In particular, the radial limit

$$\lim_{\varrho \to 1}\frac{\partial^k}{\partial x^k}\tilde{F}(\varrho, x)$$

exists an is finite almost everywhere in E.

Denote by S the Fourier series of F, by $\tilde{S}$ the series conjugate to S, and by $S^{(k)}$ and $\tilde{S}^{(k)}$, respectively, the series S and $\tilde{S}$ differentiated termwise k times. We have just proved that $\tilde{S}^{(k)}$ is Abel summable almost everywhere in E. By Lemmas 4 and 5, $\tilde{S}^{(k)}$ is summable $(C, k+1)$ almost everywhere in E, and, by Lemma 7, the same holds for $S^{(k)}$. The latter series has coefficients $o(n^k) = o(n^{k+1})$.

Let T denote the series obtained by integrating $S^{(k)}$ termwise $k+3$ times, and let Φ be the sum of T (observe that T converges absolutely and uniformly). By Lemmas 6 and 10, Φ has at almost all points of E, a differential of order $k+3$. Clearly, Φ is a third integral of F.

By Lemma 12, for any $\varepsilon > 0$ there is a closed subset P of E, with $|E-P| < \varepsilon$, and a decomposition

$$\Phi = \Psi + X \tag{37}$$

where $\Psi = \Phi$ on P, $\Psi \epsilon C^{k+3}$ and

$$|X(x)| \leqslant C\{\delta(x)\}^{k+3} \qquad (\delta(x) = \delta(x, P)) \tag{37a}$$

except, perhaps, for a finite number of intervals contiguous to P.

Without loss of generality we may assume that $\Phi_{(k+3)}(x)$ exists everywhere in P; hence $X_{(k+3)}(x)$ exists everywhere in P. The inequality for X shows that $X_{(j)}(x) = 0$ for $j \leqslant k+3$ at each point of density of P. We may also assume that $F_{(k-1)}(x)$ exists everywhere in P.

Let us differentiate the equation $\Phi = \Psi + X$ three times. Since Φ is a third integral of F, and F is continuous in $E \supset P$, we have $\Phi'''(x) = F(x)$ in P (6). Hence, with $\Psi''' = G \epsilon C^k$ and $H = F - G$, we have the identity

$$F(x) = G(x) + H(x) \tag{38}$$

where $H(x) = X'''(x)$ in P.

The function H satisfies, like F, condition Λ_k at each point of P. It also has, like F, a differential of order $k-1$ everywhere in P. Since, clearly, X is a third integral of H, the differential of order $k+2$ of X at any point $x_0 \epsilon P$ is obtained by integrating the differential of order $k-1$ of H at x_0 three times. But we observed that $X_{(j)}(x_0) = 0$ for $0 \leqslant j \leqslant k+2$ and almost all $x_0 \epsilon P$. Hence

$$H_{(j)}(x_0) = 0 \quad \text{for} \quad 0 \leqslant j \leqslant k-1 \text{ and almost all } x_0 \epsilon P.$$

This, together with the fact that H satisfies condition Λ_k at each point of P, shows that for almost all points $x \epsilon P$ we have

$$H(x+t) + H(x-t) = O(t^k). \tag{39}$$

(For k even we would get, instead, $H(x+t) - H(x-t) = O(t^k)$.) Hence, by Lemma 11, $H_{(k)}(x)$ exists almost everywhere in P. It follows that $F_{(k)}(x)$ exists almost everywhere in P, and so also almost everywhere in E.

(6) The continuity of F in E is not really indispensable in the argument. The conclusion $\Phi''' = F$ in P can be reached if F is continuous in the mean at each $x \epsilon E$, i. e., if $\int_0^h |F(x+t) - F(x)|\,dt = o(h)$, and this holds, anyway, for almost all x.

10. This completes the proof of Theorem 1. Assuming the validity of Theorem 2, we may also consider Theorems 3 and 3′ as established.

We now pass to the proof of Theorems 4 and 4′. The theorems say essentially the same thing and we may confine our attention to Theorem 4 and assume for the sake of definiteness that k is odd; we may also assume that $0 < |E| < \infty$. We begin with the second part of the theorem which asserts that if for each $x_0 \epsilon E$ the function F has a k-th differential in L^2, if this differential is

$$P_{x_0}(t) = \sum_{j=0}^{k} \frac{a_j(x_0)}{j!} t^j$$

and if $\varepsilon_{x_0}(t)$ is defined by the equation

$$F(x_0+t) = P_{x_0}(t) + \varepsilon_{x_0}(t)\frac{t^k}{k!},$$

then $\varepsilon_{x_0}(t) - \varepsilon_{x_0}(-t)$ satisfies condition N at almost all points of E.

By Lemma 16, we can find a closed subset P of E, with $|E-P| < \varepsilon$ and a decomposition $F = G+H$ such that $G \epsilon C^k$, $G = F$ in P, and if $\{(a_n, b_n)\}$ is the sequence of intervals contiguous to P we have

$$\int_{a_n}^{b_n} H^2(t)\,dt \leqslant M(b_n - a_n)^{2k+1}, \tag{40}$$

except possibly for a finite number of these intervals; the constant M is independent of n. In view of Theorem 2, it is enough to prove the required result of $F = H$ and $E = P$. Since $H = 0$ on P, the last inequality implies that for every x_0 which is a point of density of P we have

$$\int_{-h}^{h} H^2(x_0+t)\,dt = o(h^{2k+1})$$

so that the k-th differential $P_{x_0}(t)$ of H in L^2 is identically 0 and $\varepsilon_{x_0}(t) = k!H(x_0+t)t^{-k}$. It is therefore enough to show that for almost all $x_0 \epsilon P$ the function $H^2(x_0+t)|t|^{-(2k+1)}$ is integrable near $t = 0$.

Take any point x_0 of density of P and $\eta > 0$ so small that for all the intervals (a_n, b_n) situated in $I = (x_0 - \eta, x_0 + \eta)$ we have (40). By reducing η still more we may assume that if $d_n = d_n(x_0)$ is the distance of (a_n, b_n) from x_0, then $d_n \geqslant b_n - a_n$ for all (a_n, b_n) in I. We may also assume that $x_0 \pm \eta \epsilon P$. Let $\chi(x)$ be the function equal to 0 in P and to $b_n - a_n$ in the intervals (a_n, b_n).

Then

$$\int_{-\eta}^{\eta} \frac{H^2(x_0+t)}{|t|^{2k+1}}\,dt = \sum \int_{a_n}^{b_n} \frac{H^2(x)}{|x-x_0|^{2k+1}}\,dx \leqslant \sum d_n^{-2k-1} \int_{a_n}^{b_n} H^2(x)\,dx$$

$$\leqslant M \sum d_n^{-2k-1}(b_n - a_n)^{2k+1} \leqslant M\cdot 2^{2k+1} \sum \int_{a_n}^{b_n} \frac{\chi^{2k}(x)}{|x-x_0|^{2k+1}}\,dx$$

$$= M \int_{x_0-\eta}^{x_0+\eta} \frac{\chi^{2k}(x)}{|x-x_0|^{2k+1}}\,dx$$

and, by Lemma 15, the last integral is finite for almost all x_0 in P.

11. It remains to prove the first part of Theorem 4, namely that if at each point of a set E (assuming. e. g., k odd) the function $\varepsilon_{x_0}(t)$ which appears in

$$\varphi_{x_0}(t; F) = a_0(x) + \frac{a_2(x)}{2!}t^2 + \ldots + \frac{a_{k-1}(x_0)}{(k-1)!}t^{k-1} + \varepsilon_{x_0}(t)\frac{t^k}{k!}$$

satisfies condition N, then F has a k-th differential in L^2 at almost all points of E.

The proof is to a considerable degree parallel to that of Theorem 1. The integrability of $\varepsilon_{x_0}^2(t)/|t|$ near $t = 0$ implies, firstly, that $\int_0^h |\varepsilon_{x_0}(t)|\,dt = o(h)$ and, secondly, that $\{F(x_0+t)+F(x_0-t)\}^2$ is integrable near $t = 0$. By Lemma 13, F^2 is integrable near almost all points of E. We may therefore assume from the start that $F(x)$ is periodic and of the class L^2, and arguing as in the case of Theorem 1 we have the decomposition (37) with Ψ and X having the same properties as before (cf. also the footnote in the proof of Theorem 1). By differentiating we again obtain (38) with G in C^k and $H = X'''$ in P. As before, the $(k-1)$-th differential of H is identically 0 at almost all points of P, and, by Theorem 2 applied to G, we have

$$\int_0^{\pi} \frac{[H(x+t)+H(x-t)]^2}{t^{2k+1}}\,dt < \infty \tag{35a}$$

(instead of (39)) for almost all points $x \epsilon P$. Hence, a fortiori,

$$\int_0^h \{H(x+t)+H(x-t)\}^2\,dt = o(h^{2k+1})$$

almost everywhere in P, and so also, by Lemma 14 ([6a]),

$$\int_{-h}^{h} H^2(x+t)\,dt = o(h^{2k+1}) \tag{35b}$$

([6a]) Added in proof. This is the only place where Lemma 14 is used, but we could do without it and use the key Lemma 8 instead, with $f(x, y) = H(x)y^{-k-1}$. For then (35a) implies the integrability of functions $H^2(x \pm t)t^{-2k-1}$ near $t = 0$, and so also (35b), almost everywhere in P.

almost everywhere in P. It follows that H, and so also $F = G+H$, has a k-th differential in L^2 at almost all points of P, and therefore also at almost all points of E. This completes the proof of Theorem 4.

Chapter III

1. Let us return to the definition of the derivative in L^p and suppose that $1 \leqslant p < \infty$. It can be shown ([7]) that if, say, k is even and for each $x_0 \in E$ the even part $\varphi_{x_0}(t)$ of F has at $t=0$ a k-th differential in L^p, then F itself has a k-th differential in L^p at almost all points of E; the same conclusion holds if k is odd and we replace $\varphi_{x_0}(t)$ by $\psi_{x_0}(t)$. If we interchange the roles of φ_{x_0} and ψ_{x_0}, without changing the parity of k, we are led to the notions of conditions λ_k and Λ_k in the metric L^p.

Suppose that F belongs to L^p in the neighborhood of x_0 and that the even part $\varphi_{x_0}(t)$ has at $t=0$ a differential in L^p of odd order k, or that the odd part $\psi_{x_0}(t)$ has a differential of even order k. It is not difficult to see that in either case the k-th derivative must be 0. For if, for example, k is odd (the definitions and arguments which follow are analogous for k even) and $U(t)$ is the k-th differential of $\varphi_{x_0}(t)$ at $t=0$, the hypothesis

$$\Big\{h^{-1}\int_{-h}^{h} |\varphi_{x_0}(t)-U(t)|^p\,dt\Big\}^{1/p} = o(h^k)$$

and the even character of φ_{x_0} imply that $U(t)$ is also even, and so is of degree $k-1$; hence the k-th derivative of $\varphi_{x_0}(t)$ at $t=0$ is actually 0. We therefore have

$$\varphi_{x_0}(t, F) = U(t)+\varepsilon_{x_0}(t)\frac{t^k}{k!}, \tag{1}$$

where

$$\Big\{h^{-1}\int_0^h |\varepsilon_{x_0}(t)|^p t^{kp}\,dt\Big\}^{1/p} = o(h^k),$$

a condition which is easily seen to be equivalent to

$$\Big\{h^{-1}\int_0^h |\varepsilon_{x_0}(t)|^p\,dt\Big\}^{1/p} = o(1). \tag{2}$$

If (2) holds we shall say that F *satisfies condition* λ_k^p *at* x_0, and by replacing here o by O we define condition Λ_k^p.

If F satisfies condition Λ_k^p at each point of E, it does not necessarily follow that that F has a k-th derivative in L^p almost everywhere in E, though the existence of the $(k-1)$-th derivative is assured by the remark made at the beginning of the section. We have, however, the following

THEOREM 5. *If F satisfies condition Λ_k^p, $1 \leqslant p < \infty$, in a set E, then the necessary and sufficient condition for F to have a k-th differential in L^p almost everywhere in E is that the function*

$$\varepsilon_{x_0}^*(t) = \frac{1}{t}\int_0^t \varepsilon_{x_0}(s)\,ds \tag{2a}$$

(see (1)) *satisfies condition* N *almost everywhere in* E.

The definition of condition Λ_k^p presupposes (if k is odd) that $\varepsilon_{x_0}(t)$ is in L^p near $t=0$. By an analogue of Lemma 9, with exponent p, F itself is in L^p near almost all points of E. Hence, without any loss of generality, we may assume that F is in L^p over the whole interval of definition.

Let F_1 be the indefinite integral of F. Integrating (1) with respect to t we get (omitting the subscript x_0 on the right)

$$\psi_{x_0}(t; F_1) = U_1(t)+\eta(t)\frac{t^{k+1}}{(k+1)!}, \tag{3}$$

where

$$U_1(t) = \int_0^t U(s)\,ds, \qquad \eta(t) = \frac{k+1}{t^{k+1}}\int_0^t \varepsilon(s)s^k\,ds.$$

Hence

$$|\eta(t)| \leqslant \frac{k+1}{t^{k+1}}\int_0^t |\varepsilon(s)|s^k\,ds \leqslant (k+1)\frac{1}{t}\int_0^t |\varepsilon(s)|\,ds \tag{3a}$$

$$\leqslant (k+1)\Big\{\frac{1}{t}\int_0^t |\varepsilon(s)|^p\,ds\Big\}^{1/p},$$

so that if F satisfies condition Λ_k^p at x_0, F_1 satisfies condition Λ_{k+1}.

Next, $\varepsilon^*(t)$ satisfies condition N if and only if $\eta(t)$ does. This follows from the two formulas

$$(k+1)\varepsilon^*(t) = \eta(t)+\frac{k}{t}\int_0^t \eta(s)\,ds,$$

$$\frac{\eta(t)}{k+1} = \varepsilon^*(t)-\frac{k}{t^{k+1}}\int_0^t \varepsilon^*(s)s^k\,ds,$$

which are easily obtainable by integration by parts. Both integrals on the right are absolutely convergent, as may be easily seen from the first inequality (3a) and (2) (with O instead of o).

[7] See [14].

We shall now prove the sufficiency of the condition in Theorem 5. Suppose that F satisfies condition Λ_k^p in E and that $\varepsilon_{x_0}^*(t)$ satisfies condition N there. Hence the indefinite integral F_1 of F satisfies condition Λ_{k+1} in E and we have (3) with η satisfying condition N. By Theorem 1, F_1 has a Peano $(k+1)$-th derivative almost everywhere in E. By Lemma 6, we have a decomposition $F_1 = G_1+H_1$, where $G_1 \in C^{k+1}$, $G_1 = F_1$ in a closed set $P \subset E$ with $|E-P|$ arbitrarily small, and the derivatives of order $\leqslant k+1$ of H_1 vanish almost everywhere on P. By differentiating the equation $F_1 = G_1+H_1$ we obtain $F = G+H$, where $G = G_1' \in C^k$ and $H = H_1'$ vanishes almost everywhere on P.

Consider now any point $x_0 \in P$ where F_1 has a differential of order $k+1$ and H_1 has a differential of order $k+1$ vanishing identically; at x_0 the differentials of order $k+1$ of F_1 and G_1 are equal. At such an x_0 we must have $\eta_{x_0}(t) = o(1)$ since the left-hand side of (3) is an odd function of t, $U_1(t)$ is an odd polynomial of degree $\leqslant k$, and $k+1$ is even. If, for the same x_0, we write $\varphi_{x_0}(t, G_1) = V_1(t)+o(t^{k+1})$, then $U_1(t) = V_1(t)$. It follows that the differential of order k of $\varphi_{x_0}(t, G)$ at $t = 0$ is $V_1'(t) = U_1'(t) = U(t)$. Hence, by (1),

$$\varphi_{x_0}(t, H) = \varphi_{x_0}(t, F)-\varphi_{x_0}(t, G) = \varepsilon_{x_0}(t)\frac{t^k}{k!}+o(t^k),$$

$$\left\{\frac{1}{h}\int_0^h |\varphi_{x_0}(t, H)|^p dt\right\}^{1/p} \leqslant h^k\left\{\frac{1}{h}\int_0^h |\varepsilon_{x_0}(t)|^p dt\right\}^{1/p} + o(h^k)$$
$$= O(h^k)+o(h^k) = O(h^k).$$

Since $H = 0$ almost everywhere in P, an application of the analogue of Lemma 14 with exponent p instead of 2, shows that at almost all points $x \in P$ we have

$$\left\{\frac{1}{2h}\int_{-h}^h |H(x+t)|^p dt\right\}^{1/p} = o(h^k).$$

At such a point x the function $F = G+H$ has a k-th differential in L^p. Hence F has such a differential almost everywhere in P, and so also almost everywhere in E. This completes the proof of the sufficiency part of Theorem 5.

The proof of the necessity of the condition in Theorem 5 is simple. Suppose that F has a k-th differential in L^p in E and satisfies condition Λ_k^p there. It is immediate that the indefinite integral F_1 of F has a $(k+1)$-th Peano differential in E. Moreover, we have (3), where $U_1(t) = \int_0^t U(s)\,ds$ is a polynomial of degree k and $\eta(t) = o(1)$. By Theorem 2, $\eta(t)$ satisfies condition N almost everywhere in E and this implies, as we indicated above, that $\varepsilon_{x_0}^*(t)$ satisfies condition N almost everywhere in E.

The following result is a corollary of Theorem 5:

THEOREM 6. *Suppose that $F \in L^p$, $1 \leqslant p < \infty$. The necessary and sufficient condition for F to have a k-th derivative in L^p almost everywhere in a set E is that, almost everywhere in E,*

a) *F satisfies condition Λ_k^p;*

b) *the function $\varepsilon_{x_0}^*(t)$ defined by* (2a) *satisfies condition* N.

That condition Λ_k^p is satisfied at each point where the k-th derivative in L^p exists, is clear, and then the necessity of condition b) follows from Theorem 5. The latter theorem also implies the sufficiency of conditions a) and b).

2. In the case $p \geqslant 2$ Theorem 6 can be stated in a different form

THEOREM 7. *Suppose that $F \in L^p$, $2 \leqslant p < \infty$. The necessary and sufficient condition for F to have a k-th derivative almost everywhere in a set E is that, almost everywhere in E,*

a) *F satisfies condition Λ_k^p;*

b) *the function $\varepsilon_{x_0}(h)$ in* (1) *(for k odd, with a corresponding modification for k even) satisfies condition* N.

The necessity of condition b) (condition a) here is the same as in Theorem 5) follows from Theorem 4 and the fact that, since $p \geqslant 2$, differentiability in L^p implies differentiability in L^2. The sufficiency of the conditions follows from Theorem 6 observing that if $\varepsilon_{x_0}(t)$ satisfies condition N so does $\varepsilon_{x_0}^*(t)$ (a simple consequence of Schwarz's inequality).

3. For our next theorem we need the following lemma:

LEMMA 17. *Suppose that $F \in L^p$, $p \geqslant 2$, and that (for k odd, with the corresponding modification for k even) the function $\varepsilon_{x_0}(t)$ in* (1) *satisfies condition* N *in a set E. Then the following two conditions are equivalent almost everywhere in E:*

a) *F satisfies condition Λ_k^p;*

b) *the integral $\int_0 t^{-1}|\varepsilon_{x_0}(t)|^p dt$ is finite.*

It is clear that condition b) implies a) at each point. It is the converse that requires proof. It will be convenient to assume that F is periodic of period 2π.

Since $\varepsilon_{x_0}(t)$ satisfies condition N in E, F has a k-th derivative in L^2 almost everywhere in E (Theorem 4). Let us apply to F Lemma 16 and keep the notation of that lemma. Without loss of generality we may assume that the function G is not only in C^k but also of period 2π. Then the function $\varepsilon_x(t)$ of (1) but corresponding to G ($\varepsilon_x(t) = \varepsilon_x(t, G)$) satisfies the inequality

$$\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varepsilon_x^2(t,G)\frac{dx\,dt}{t}\right\}^{1/2}\leqslant A\left\{\int_{-\pi}^{\pi}g^2(x)\,dx\right\}^{1/2}\qquad (g=G^{(k)}),$$

with A independent of G. This is proved in [13], and in the case $k=1$ pretty familiar (see [4]). We claim that if $2<p<\infty$, then

$$\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\varepsilon_x(t,G)|^p\frac{dx\,dt}{t}\right\}^{1/p}\leqslant A\left\{\int_{-\pi}^{\pi}|g|^p\,dx\right\}^{1/p}.$$

For $p=2$ this is the preceding inequality, and it also holds for $p=\infty$ if we interpret is as $\operatorname{Ess\,sup}_{x,t}|\varepsilon_x(t,G)|\leqslant A\operatorname{Ess\,sup}_{x}|g(x)|$. Hence, by M. Riesz' convexity theorem, it is valid for $2<p<\infty$ (for $k=1$, the result is already in [4]). It follows that

$$\int_0^{\pi}|\varepsilon_x(t,G)|^p t^{-1}dt<\infty$$

for almost all x.

In particular, G satisfies condition A_k^p almost everywhere. Hence, if F satisfies condition A_k^p almost everywhere in P, so does the function H in the decomposition $F=G+H$. Hence, by the Remark to Lemma 14 (with $a=kp+1$),

$$\left\{\frac{1}{h}\int_0^h|H(x\pm t)|^p\,dt\right\}^{1/p}=o(h^k)$$

for almost all $x\in P$. This shows (since $H=0$ in P) that the function $|H(x\pm t)|^p t^{-(kp+1)}$, and so also the function $|\varepsilon_x(t,H)|^p t^{-1}$, is integrable near $t=0$ for almost all $x\in P$. Hence the same holds for $\varepsilon_x(t,F)$ and the lemma is established.

THEOREM 8. *Suppose that $F\in L^p$, $p\geqslant 2$, and k is odd (the corresponding result holds for k even), then F has a k-th derivative in L^p almost everywhere in E if and only if the two integrals*

$$\int_0\{\varepsilon_x(t)\}^2\frac{dt}{t},\qquad \int_0|\varepsilon_x(t)|^p\frac{dt}{t}$$

are finite almost everywhere in E.

If the two integrals are finite almost everywhere in E, then, by Theorem 7, F has a k-th derivative in L^p at almost all points of E. The converse follows from Theorem 7 and Lemma 17.

If $k=1$, Theorem 8 asserts that, for $p\geqslant 2$, the simultaneous existence of the two integrals

$$\int_0\frac{[F(x+t)+F(x-t)-2F(x)]^2}{t^3}\,dt,\qquad \int_0\frac{|F(x+t)+F(x-t)-2F(x)|^p}{t^{p+1}}\,dt$$

almost everywhere in E is both necessary and sufficient for the differentiability of F in L^p almost everywhere in E.

4. The following result is an immediate corollary of Theorem 4′ and of known results:

THEOREM 9. *Suppose that $F(x)$ is in $L^2(-\infty,+\infty)$ and has a $(k-1)$-th Peano derivative at each point of a set E. Suppose also that the function $\omega_x(t)$ defined by the equation*

$$F(x+t)=\sum_{j=0}^{k-1}\frac{1}{j!}F_{(j)}(x)t^j+\omega_x(t)\frac{t^k}{k!}\qquad (x\in E)$$

satisfies at each point $x\in E$ the condition

$$\int_0^{\infty}\frac{[\omega_x(t)-\omega_x(-t)]^2}{t}\,dt<\infty. \tag{4}$$

Then the integral

$$\int_0^{\infty}\frac{\omega_x(t)-\omega_x(-t)}{t}\,dt=\lim_{\varepsilon\to+0}\int_{\varepsilon}^{\infty} \tag{5}$$

exists almost everywhere in E.

Proof. The finiteness of the integral (4) implies that F has a k-th differential in L^2 at almost all points of E. At each point at which this occurs, and so almost everywhere in E, the function $\Phi(x)=\int_0^x F(t)\,dt$ has a $(k+1)$-th Peano derivative. By the main result of [10], the integral (5) exists almost everywhere in E.

In the case $k=1$, Theorem 9 asserts that if $F\in L^2(-\infty,+\infty)$, the existence of the integral of Marcinkiewicz

$$\int_0^{\infty}\frac{[F(x+t)+F(x-t)-2F(x)]^2}{t^3}\,dt$$

for $x\in E$, implies the existence of

$$\int_0^{\infty}\frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,dt$$

almost everywhere in E. The converse is not true; to see this, it is enough to construct an $F\in L^2(-\infty,+\infty)$, such that $\Phi(x)=\int_0^x F(t)\,dt$ has a second Peano derivative almost everywhere while F is almost everywhere without a first derivative in L^2. The construction is not difficult and is omitted here. A similar argument shows that the converse of Theorem 9 is false for each k.

5. Some of the results proved in this paper have extensions to functions of several variables. We give here one of such extensions which is an analogue of Theorem 3′.

THEOREM 10. *Suppose that* $F(x) = F(x_1, x_2, \ldots, x_n) \epsilon L^2(E^n)$, *and that it satisfies condition* Λ *for each* $x_0 \epsilon E \subset E^n$, *i. e., we have*

$$F(x_0+h)+F(x_0-h)-2F(x_0) = O(|h|), \tag{6}$$

where $h = (h_1, h_2, \ldots, h_n)$, $|h| = (\sum h_j^2)^{1/2}$. *Then the necessary and sufficient condition for* F *to have a first total differential almost everywhere in* E *is that*

$$\int_{E^n} \frac{[F(x_0+h)+F(x_0-h)-2F(x_0)]^2}{|h|^{n+2}}\, dh < \infty \tag{7}$$

almost everywhere in E.

The integral in (7) is the n-dimensional analogue of the integral of Marcinkiewicz. The part of it extended over $|h| \geqslant \varepsilon > 0$ is always finite.

The necessity of the condition is proved exactly as in the one-dimensional case, by decomposing F into a "good" and "bad" part (see [1], p. 189). We have here even a somewhat stronger result: we have (7) almost everywhere in E if we merely assume that F has at each point of E a total differential in the sense of L^2.

The sufficiency of the condition is a relatively simple consequence of the corresponding result for the one-dimensional case. We sketch the proof, and we omit routine arguments involving the measurability of the sets which occur in the proof. Without loss of generality we may replace (6) by

$$|F(x_0+h)+F(x_0-h)-2F(x_0)| \leqslant M|h| \qquad (x_0 \epsilon E;\ |h| \leqslant \delta).$$

Suppose we have (7) for a fixed x_0 and let $e_1, e_2, \ldots, e_n$ be a system of n mutually orthogonal unit vectors. Then, by Fubini's theorem, we have

$$\sum_{k=1}^{n} \int_0^\infty t^{-3}[F(x_0+e_k t)+F(x_0-e_k t)-2F(x_0)]^2 dt < \infty \tag{8}$$

for almost all choices of the frame $e_1, e_2, \ldots, e_n$. Using Fubini's theorem again, we obtain the existence of a *fixed* frame $e_1, e_2, \ldots, e_n$ such that (8) holds for almost all $x_0 \epsilon E$. Applying a rotation, we may assume that this frame lies along the $x_1, x_2, \ldots, x_n$ axes.

By our one-dimensional theorem, the partial derivative $(\partial/\partial x_k)F$ $(k = 1, \ldots, n)$ exist almost everywhere in E. This implies that F has an approximate total differential almost everywhere in E, i. e., for almost all $x_0 \epsilon E$ there is a set H_{x_0} having 0 as a point of density and such that

$F(x_0+h) = \sum h_k(\partial/\partial x_k)F(x_0)+o(|h|)$, provided h tends to 0 through the set H_{x_0}. Take a fixed point x_0 at which this occurs. We may assume without loss of generality that $x_0 = 0$ and that $(\partial/\partial x_k)F(x_0) = 0$, $k = 1, 2, \ldots, n$. Write H_0 for H_{x_0}. Since 0 is a point of density for H_0, for any h with $|h|$ sufficiently small, and for any $\varepsilon > 0$, we can find two points h_1 and h_2 in H_0 such that, in the first place, h_1 is the mid-point of the segment (h, h_2) and, second, $|h-h_1| \leqslant \varepsilon|h|$. Since $F(h_1) = o(|h_1|) = o(|h|)$ and, similarly, $F(h_2) = o(|h|)$, and since

$$|F(h)-2F(h_1)+F(h_2)| \leqslant M|h-h_1| \leqslant M\varepsilon|h|,$$

it follows that $|F(h)| \leqslant 2\varepsilon M|h|$ for h arbitrary and sufficiently small. Hence $F(h) = o(|h|)$ and the theorem is established.

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