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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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On a generalization of regularly increasing functions

by

W. MATUSZEWSKA (Poznań)

1. In this section we shall denote by $f, g, h, \ldots$ real functions defined and non-decreasing in $(-\infty, \infty)$. The following notation will be used:

$$\bar{\varrho}_f(\mu) = \limsup_{u\to\infty}\big(f(u+\mu)-f(u)\big),$$

$$\underline{\varrho}_f(\mu) = \liminf_{u\to\infty}\big(f(u+\mu)-f(u)\big).$$

We denote by C_0 the space whose elements are functions $\mu(\cdot)$ continuous in $(-\infty, \infty)$ and converging to 0 with $u\to\infty$ and to a finite limit with $u\to-\infty$. Equipped with the usual metric defined by $d\big(\mu_1(\cdot), \mu_2(\cdot)\big) = \|\mu_1(\cdot)-\mu_2(\cdot)\|$, where $\|\mu(\cdot)\| = \sup_{-\infty\leqslant t<\infty}|\mu(t)|$ for $\mu(\cdot)\in C_0$, C_0 is a complete metric space. We write $\mu(\cdot)\in C_0^+$ if $\mu(\cdot)\in C_0$ and $\mu(u)>0$ everywhere.

The aim of section 1 is to present some lemmas the use of which simplifies the proofs of the theorems given further in section 3 and 4.

1.1. *The following equalities are satisfied for any function f:*

$$(*)\quad \lim_{\mu\to 0+}\frac{\bar{\varrho}_f(\mu)}{\mu} = \sup_{\mu>0}\frac{\bar{\varrho}_f(\mu)}{\mu},\qquad (**)\quad \lim_{\mu\to\infty}\frac{\bar{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0}\frac{\bar{\varrho}_f(\mu)}{\mu};$$

$$(+)\quad \lim_{\mu\to 0+}\frac{\underline{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0}\frac{\underline{\varrho}_f(\mu)}{\mu},\qquad (++)\quad \lim_{\mu\to\infty}\frac{\underline{\varrho}_f(\mu)}{\mu} = \sup_{\mu>0}\frac{\underline{\varrho}_f(\mu)}{\mu}.$$

The proofs of (**), (++) can be found in [2], the proofs of (*), (+) run on the same lines.

1.2. *If for any $\mu(\cdot)$ in C_0^+ there exists a limit*

$$(+)\qquad \lim_{u\to\infty}\big[f\big(u+\mu(u)\big)-f(u)\big] = g\big(\mu(\cdot)\big),$$

then

(a) $g\big(\mu(\cdot)\big)=0$ *for any* $\mu(\cdot)\in C_0$;

(b) *for any* $\varepsilon>0$ *there exist a* $\delta>0$ *and* u_0 *such that the inequality*

$$|f(u+\mu)-f(u)|<\varepsilon$$

holds for $|\mu|\leqslant\delta$ *and* $u\geqslant u_0$.

Evidently (b) $\Rightarrow$ (a) and so it is enough to prove (b). If (b) is not satisfied, there exist an $\varepsilon_0 > 0$, a sequence of u_n converging to ∞ and a sequence of positive numbers μ_n which tends to 0 such that

$$|f(u_n+\mu_n)-f(u_n)| \geqslant \varepsilon_0.$$

Take a function $\mu_0(\cdot)$ such that $\mu_0(u_n) = \mu_n$, $\mu_0(v_n) = c_n$ assuming $v_n = (u_n+u_{n+1})/2$, $c_n > 0$, $|f(v_n+c_n)-f(v_n)| < \varepsilon_0/2$, and assuming $\mu_0(\cdot)$ to be linear in the intervals $\langle 0, u_1\rangle$, $\langle u_n, v_n\rangle$, $\langle v_n, u_{n+1}\rangle$ and constant in $(-\infty, 0\rangle$. Evidently $\mu_0(\cdot)\in C_0^+$ and the limit (+) does not exist, which gives a contradiction.

1.3. *If f is continuous, $\bar{\varrho}_f(\mu)$ and $\underline{\varrho}_f(\mu)$ are both continuous for $\mu = 0$, then for any $\varepsilon > 0$ there exist $\delta > 0$ and u_0 such that relation* 1.2, (b) *holds.*

Under the assumption of continuity of $\bar{\varrho}_f(\mu)$, $\underline{\varrho}_f(\mu)$ for $\mu = 0$ we have

$$|f(u+\mu)-f(u)| \leqslant \varepsilon/2 \quad \text{for} \quad u \geqslant u(\mu),$$

and for any μ satisfying the inequality $|\mu| \leqslant \mu_0$.

Let us define $A_n = \{\mu\colon |f(u+\mu)-f(u)| \leqslant \varepsilon/2 \text{ for } u \geqslant n,\ |\mu| \leqslant \mu_0\}$ where $n = 1, 2, \ldots$ Since the sets A_n are closed and $A_1 \cup A_2 \cup \ldots = \langle -\mu_0, \mu_0\rangle$, there exist an integer m and an interval $\langle \mu_1, \mu_2\rangle$ contained in $\langle -\mu_0, \mu_0\rangle$ such that $\langle \mu_1, \mu_2\rangle \in A_m$. Consequently for any $\mu', \mu'' \in \langle \mu_1, \mu_2\rangle$ we have $|f(u+\mu''-\mu')-f(u-\mu')| \leqslant \varepsilon/2$, $|f(u-\mu')-f(u)| \leqslant \varepsilon/2$ for $u \geqslant m+\mu_2$, which implies

$$|f(u+\mu''-\mu')-f(u)| \leqslant \varepsilon \quad \text{for} \quad u \geqslant m+\mu_2.$$

Using the last inequality for $\mu = \mu''-\mu'$, where $\mu', \mu'' \in \langle \mu_1, \mu_2\rangle$, we obtain 1.2,(b) with $\delta = \mu_2-\mu_1$, $u_0 = m+\mu_2$.

Remark. Lemma 1.3 remains valid if in place of the continuity of f we assume only it measurability.

1.4. *If f is continuous and C_{0f} denotes the collection of all $\mu(\cdot)$ in C_0 for which the limit* 1.2 (+) *with $g(\mu(\cdot)) = 0$ exists, then C_{0f} is either of the first category in C_0 or identical with the whole space C_0.*

Assuming C_{0f} be of the second category, by arguments analogous to those used in 1.3 we can prove what follows:

There exist $\mu_0(\cdot)\in C_0$, $\delta > 0$ and u_0 such that the inequality

$$|f(u+\mu(u)+\mu_0(u))-f(u)| \leqslant \varepsilon$$

holds for $u \geqslant u_0$ and $\|\mu(\cdot)\| \leqslant \delta$. Let $\bar{\mu}(\cdot)$ be a given function in C_0 and suppose the inequality $|\bar{\mu}(u)-\mu_0(u)| \leqslant \delta/2$ is satisfied for $u \geqslant \bar{u}$. Let us define a function $\bar{\bar{\mu}}(\cdot)\in C_0$ as $\bar{\bar{\mu}}(u) = \bar{\mu}(u)-\mu_0(u)$ for $u \geqslant 2\bar{u}$, $\bar{\bar{\mu}}(u) = 0$ for $u \leqslant \bar{u}$ and $\bar{\bar{\mu}}(\cdot)$ as a linear function in the interval $\langle u, 2\bar{u}\rangle$. Evidently $\|\bar{\bar{\mu}}(\cdot)\| \leqslant \delta$, $\bar{\bar{\mu}}(u)+\mu_0(u) = \bar{\mu}(u)$ for $u \geqslant 2\bar{u}$, hence $|f(u+\bar{\mu}(u))-f(u)| \leqslant \varepsilon$ for $u \geqslant \sup(u_0, 2\bar{u})$, and consequently $f(u+\bar{\mu}(u))-f(u) \to 0$ as $u \to \infty$.

2. According to the terminology of [2] a function φ continuous and non-decreasing for $u \geqslant 0$, vanishing for $u = 0$ only and tending to infinity as $u \to \infty$ will be called a *φ-function*. The following will show the usefulness of the substitution (φ):

$$(*\varphi) \qquad f(u) = \lg\varphi(e^u), \qquad\qquad (**\varphi) \qquad e^{\mu(u)} = \lambda(u),$$

which reduces the investigation of φ-functions to the functions we have considered previously. Given a φ-function φ, we define the following extended-valued functions:

$$\underline{h}_\varphi(\lambda) = \liminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad \bar{h}_\varphi(\lambda) = \limsup_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}.$$

Using the substitution (φ) we obtain from 1.1 the following statements:

2.1. *There exist limits*

$$(1) \qquad s_\varphi = \lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda} = \sup_{0<\lambda<1} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda},$$

$$(1') \qquad s_\varphi^1 = \lim_{\lambda\to 1-} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda} = \inf_{0<\lambda<1} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda},$$

$$(2) \qquad \sigma_\varphi = \lim_{\lambda\to 0+} \frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda} = \inf_{0<\lambda<1} \frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda},$$

$$(2') \qquad \sigma_\varphi^1 = \lim_{\lambda\to 1-} \frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda} = \sup_{0<\lambda<1} \frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda}$$

(cf. [1]). As regards the meaning of the above formulae we shall keep the conventions $\lg 0 = -\infty$, $\lg\infty = \infty$, and the same conventions are adopted in analogous situations.

2.1.1. Let us call attention to some differences between the properties of the indices s_φ, σ_φ and those of s_φ^1, σ_φ^1. The values of s_φ, σ_φ do not change if we replace φ by a φ-function ψ such that $\varphi \overset{l}{\sim} \psi$ (as regards the notation of l-equivalency; cf. [3]); on the contrary, s_φ^1, σ_φ^1 are not invariant with respect to l-equivalency. However, it is readily seen that $\varphi \sim \psi$ (i. e. $\varphi(u)/\psi(u) \to g$ as $u \to \infty$, where $g > 0$) implies $s_\varphi^1 = s_\psi^1$, $\sigma_\varphi^1 = \sigma_\psi^1$.

2.2. *If*

$$\psi(u) = \int_0^u \varphi(t)\,dt, \qquad h(u) = u\varphi(u)/\psi(u) \qquad \textit{for} \qquad u > 0,$$

then

$$1 \leqslant \liminf_{u\to\infty} h(u) \leqslant s_\psi^1 \leqslant s_\psi \leqslant \sigma_\psi \leqslant \sigma_\psi^1 \leqslant \limsup_{u\to\infty} h(u).$$

This follows from the inequality (+) in [3], p. 336, and the trivial remark $\psi(u) \leqslant u\varphi(u)$.

2.3. Assuming $s_\varphi^1 = \sigma_\varphi^1 = r_\varphi$, $r_\varphi < \infty$, we obtain by 2.1 $\bar{h}_\varphi(\lambda) = \underline{h}_\varphi(\lambda) = \lambda^{-r_\varphi}$, which means that in this case φ is a *regularly increasing* function with the index r_φ in the sense of Karamata, and especially a *slowly varying* function if $r_\varphi = 0$.

Conversely, if φ is a regularly increasing φ-function, then $s_\varphi^1 = \sigma_\varphi^1$.

2.4. We shall denote by K_c the class of all those φ-functions φ for which the relation $\varphi(u\alpha(u))/\varphi(u) \to 1$ as $u \to \infty$ holds if $\alpha(u)$ is a function continuous and positive in $\langle 0, \infty)$ and such that $\alpha(u) \to 1$ as $u \to \infty$.

2.5. The following properties are equivalent:

(a) $\varphi \in K_c$;

(b) $\bar{h}_\varphi(\lambda)$, $\underline{h}_\varphi(\lambda)$ *are continuous for* $\lambda = 1$;

(c) *for any φ-functions φ_1, φ_2 the relation $\varphi_1 \simeq \varphi_2$ implies $\varphi(\varphi_1) \simeq \varphi(\varphi_2)$* ($\varphi' \simeq \varphi''$ means the asymptotical equality of the functions φ', φ'' for large u).

In order to prove the implication (a) $\Rightarrow$ (b) let us use the substitution (φ). Evidently, the assumptions $\varphi \in K_c$ and $f(u+\mu(u)) - f(u) \to 0$ as $u \to \infty$ for an arbitrary $\mu(\cdot)$ in C_0, are equivalent. Since $\lg \bar{h}_\varphi(\lambda) = \bar{\varrho}_f(-\mu) = -\underline{\varrho}_f(\mu)$, where $e^\mu = \lambda$, the continuity of $\bar{h}_\varphi(\lambda)$ for $\lambda = 1$ follows by 1.2. In a similar way one can prove the continuity of $\underline{h}_\varphi(\lambda)$ for $\lambda = 1$. Let us now assume that condition (b) is satisfied. Applying the substitution ($*\varphi$) and 1.3 we obtain:

For any $\varepsilon > 0$ there exist $\delta(\varepsilon)$, $v(\varepsilon)$ such that

$$(+) \qquad 1-\varepsilon < \frac{\varphi(v)}{\varphi(\lambda v)} < 1+\varepsilon,$$

for $|\lambda - 1| \leqslant \delta(\varepsilon)$, $v \geqslant v(\varepsilon)$. Suppose $\varphi_1 \simeq \varphi_2$ or equivalently $\varphi_2(u) = \alpha(u)\varphi_1(u)$ for $u \geqslant u_0$, where $\alpha(u)$ is continuous and positive for $u \geqslant 0$ and $\alpha(u) \to 1$ as $u \to \infty$. Taking u sufficiently large so that $\varphi_1(u) \geqslant v(\varepsilon)$, $|\alpha(u) - 1| \leqslant \delta(\varepsilon)$, we obtain from (+)

$$1-\varepsilon < \frac{\varphi(\varphi_1(u))}{\varphi(\varphi_2(u))} < 1+\varepsilon,$$

which means $\varphi(\varphi_1) \simeq \varphi(\varphi_2)$ and consequently (b) $\Rightarrow$ (c).

For the proof of (c) $\Rightarrow$ (a) it is sufficient to put $\varphi_2(u) = u$, $\varphi_1(u) = \alpha(u)u$, where $\alpha(u)$ has the same meaning as above.

2.6. *A necessary and sufficient condition for a φ-function φ to belong to K_c is that the inequalities*

$$(*) \qquad c(a)\varphi(u) \leqslant \varphi(au) \leqslant d(a)\varphi(u)$$

hold for $u \geqslant u(a)$ and for every $a > 1$, where $1 < d(a) < \infty$, $d(a) \to 1$ as $a \to 1$, $1 \leqslant c(a)$, $c(a) \to 1$ as $a \to 1$.

Necessity. By 2.5 we have $h_\varphi(1/a) \to 1$ as $a \to 1+0$ if $\varphi \in K_c$. Since $\bar{h}_\varphi(1/a) = \limsup_{u\to\infty} \varphi(au)/\varphi(u)$ and since it is easily seen that $\bar{h}_\varphi(1/a) < \infty$ for $a > 1$, the right-hand inequality of (*) is satisfied, if we assume, say, $d(a) = \bar{h}_\varphi(1/a)a$. Analogously one can assume in the left-hand inequality of (*), $c(a) = \sup(\underline{h}_\varphi(1/a)a^{-1}, 1)$.

Sufficiency. Assuming (*) to be satisfied we obtain for $0 < a < 1$,

$$(**) \qquad \frac{1}{d(1/a)}\varphi(u) \leqslant \varphi(au) \leqslant \frac{1}{c(1/a)}\varphi(u) \quad \text{for} \quad u \geqslant u_1(a) = u(1/a)/a,$$

whence putting $\lambda = 1/a$ we have $c(1/\lambda)$ or $(d(\lambda))^{-1} \leqslant \bar{h}_\varphi(\lambda) \leqslant d(1/\lambda)$ or $(c(\lambda))^{-1}$ and consequently $\bar{h}_\varphi(\lambda) \to \bar{h}_\varphi(1) = 1$ as $\lambda \to 1$. The proof of the continuity of $\underline{h}_\varphi(\lambda)$ for $\lambda = 1$ follows by analogous arguments. Now, it suffices to apply 2.5.

Any regularly increasing or slowly varying φ-function belongs to the class K_c. This follows immediately from a well-known theorem which says that $\varphi(\lambda u)/\varphi(u)$ tends uniformly to λ^{r_φ} on any interval $\langle \lambda', \lambda'' \rangle$, $\lambda' > 0$, for any measurable regularly increasing function. Other examples of φ-functions of the class K_c can be obtained if we define a φ-function by the formula

$$\varphi(u) = \varphi_0(u)\exp\int_1^u \varepsilon(t)t^{-1}dt,$$

where $\varepsilon(u)$ denotes an arbitrary function, measurable and bounded in $(0, \infty)$ such that

$$\int_1^u \varepsilon(t)t^{-1}dt \to \infty \quad \text{as} \quad u \to \infty,$$

and $\varphi_0(u)$ is continuous and non-decreasing on $\langle 0, \infty)$, vanishes only for $u = 0$ and tends to a finite limit with u tending to ∞.

2.6.1. A φ-function φ is said to satisfy the *condition* (Δ_a) *for large* u if $a > 1$ and if the inequality $\varphi(au) \leqslant d_a\varphi(u)$ holds for $u \geqslant u_0(a)$ and for a constant d_a. It is said to satisfy the *condition* (Λ_a) *for large* u if $a > 1$ and $\varphi(u)c_a \leqslant \varphi(au)$ for $u \geqslant u_1(a)$ and for a constant $c_a > 1$.

It follows from 2.6 that any $\varphi \in K_c$ satisfies the condition (Δ_a) for every $a > 1$ with a constant d_a which can be chosen so as to satisfy $d_a \to 1$ if $a \to 1+0$. The condition (Λ_a) is not necessarily fulfilled in general. In fact, for slowly varying functions the condition (Λ_a) is not satisfied for any $a > 1$ and nevertheless they belong to K_c.

2.6.2. Let us denote by K_c^* the subclass of K_c consisting of those φ-functions for which the condition (Λ_a) is satisfied for any $a > 1$.

A φ-function φ belongs to K_c^* if and only if the inequality $(*)$ holds for any $a > 1$ with a constant $c(a)$ which has the properties mentioned in 2.6 and satisfies in addition the inequality $c_a > 1$.

2.6.3. If $\sigma_\varphi^1 < \infty$, *then the condition* (Δ_a) *is satisfied for any* $a > 1$ *with a constant* $d(a)$ *such that* $d(a) \to 1$ *as* $a \to 1+0$; *if* $s_\varphi^1 > 0$, *then the condition* (Λ_a) *is satisfied for any* $a > 1$; *consequently if* $0 < s_\varphi^1 \leqslant \sigma_\varphi^1 < \infty$, *then* $\varphi \in K_c^*$.

Suppose $\sigma_\varphi^1 < \infty$, $\sigma_\varphi^1 < \sigma$. Since 2.1, (2′) imply the inequality $\limsup_{u\to\infty} \varphi(au)/\varphi(u) < a^\sigma$ for $a > 1$, the condition (Λ_a) is satisfied with the constant $d_a = a^\sigma$. One can prove analogously the condition (Λ_a) for any $a > 1$ under the hypothesis $s_\varphi^1 > 0$.

2.7. *For a strictly increasing φ-function $\varphi \in K_c$ both inclusions $\varphi^{-1} \in K_c$ and $\varphi \in K_c^*$ are equivalent.*

Let $\mu(v)$ be a continuous and positive function for $v \geqslant 0$ which tends to 1 as $v \to \infty$ and let $v = \varphi(u)$,

$$a(v) = \frac{\varphi^{-1}\big(\mu(v)v\big)}{\varphi^{-1}(v)}.$$

Suppose $\varphi \in K_c^*$ and $a > 1$. If $a(v) \geqslant a > 1$ for infinitely many v tending to ∞, then in view of 2.6 we have

$$c(a)\varphi(u) \leqslant \varphi\big(a(v)u\big) = \mu(v)\varphi(u)$$

for some sufficiently large u which implies $\mu(v) \geqslant c(a) > 1$. This contradicts $\mu(v) \to 1$. Consequently we have $a(v) < a$ for large v. Therefore $\limsup_{v\to\infty} a(v) \leqslant 1$. By analogous arguments and by 2.6, $(**)$ we shall prove that $\liminf_{v\to\infty} a(v) \geqslant 1$ so that finally $\lim_{v\to\infty} a(v) = 1$, $\varphi^{-1} \in K_c$. Suppose now $\varphi^{-1} \in K_c$; then for any $\beta > 1$ the inequality

$$\varphi^{-1}(\beta v) \leqslant d(\beta)\varphi^{-1}(v)$$

holds for $v \geqslant v_0(\beta)$. Therefore $\beta\varphi(u) = \beta v \leqslant \varphi\big(d(\beta)u\big)$ for $u \geqslant u_0 = \varphi^{-1}(v_0)$. If, given $a > 1$, we choose β in such a way that $d(\beta) \leqslant a$ and define c_a to be equal to β, the condition (Λ_a) will be satisfied for φ, which means $\varphi \in K_c^*$.

2.8. *If $\psi(u) = \int_0^u \varphi(t)\,dt$ the following inequalities are satisfied:*

(a) $$s_\psi^1 \geqslant 1 + s_\varphi^1,$$

(b) $$\sigma_\psi^1 \leqslant 1 + \sigma_\varphi^1.$$

Applying the generalized L'Hospital rule to the ratio $\psi(u)/\psi(\lambda u)$ we obtain $\underline{h}_\psi(\lambda) \geqslant \underline{h}_\varphi(\lambda)/\lambda$, whence (a) immediately follows. The proof of (b) is analogous.

2.8.1. (a) *If $\varphi \in K_c$, and ψ means the same φ-function as in 2.8, then $\psi \in K_c^*$;* (b) *if $\varphi \in K_c$ and is convex φ-function, then $\varphi \in K_c^*$.*

Owing to $\varphi \in K_c$, the inequality $\varphi(au) \leqslant d(a)\varphi(u)$ holds for $u \geqslant u(a) = u_0$ where $d(a) \to 1$ as $a \to 1$. Because of the equality

$$\frac{\psi(au) - \psi(au_0)}{\psi(u) - \psi(u_0)} = a\frac{\varphi\big(av(u)\big)}{\varphi\big(v(u)\big)} \leqslant ad(a),$$

which holds for suitably chosen $v(u)$, $u_0 \leqslant v(u) \leqslant u$, we obtain $\psi(au) \leqslant a^2 d(a)\psi(u)$ for sufficiently large u, whence by 2.6, $\psi \in K_c$. ψ being convex, we have $\psi(au) > a\psi(u)$ for any $a > 1$; so the condition (Λ_a) is satisfied for any $a > 1$ and consequently $\psi \in K_c^*$.

2.9. (a) *If $\varphi, \psi \in K_c$, then $\varphi\psi \in K_c$;* (b) *if $\varphi \in K_c$, $a > 0$, $k > 0$, then $a\varphi^k \in K_c$;* (c) *if $\varphi, \psi \in K_c$, then $\varphi(\psi) \in K_c$.*

The above theorems remain true if we replace K_c by K_c^*.

Theorems (a), (b) follow directly from the definitions of the class K_c and the class K_c^* respectively. In order to prove (c) note that for φ inequality 2.6, $(*)$ holds, and an analogous one holds also for the function ψ

$$(+)\qquad \psi(u)\bar{c}(a) \leqslant \psi(au) \leqslant \bar{d}(a)\psi(u) \quad \text{for} \quad u \geqslant \bar{u}(a).$$

Defining $\gamma(a)$ for $a > 1$ by $\psi(au) = \gamma(a)\psi(u)$ we obtain from (+) $\bar{c}(a) \leqslant \gamma(u) \leqslant \bar{d}(a)$ and by 2.6, $(*)$

$$c\big(\bar{c}(a)\big)\varphi\big(\psi(u)\big) \leqslant \varphi\big(\psi(au)\big) \leqslant d\big(\bar{d}(a)\big)\varphi\big(\psi(u)\big),$$

where the constants $c\big(\bar{c}(a)\big)$, $d\big(\bar{d}(a)\big)$ assuming that $c(1) = 1$, satisfy the assumptions of Theorem 2.6.

If φ, $\psi \in K_c^*$ then $c(a) > 1$, $\bar{c}(a) > 1$ for $a > 1$ and therefore also $c\big(\bar{c}(a)\big) > 1$.

3. In this section we always assume φ to satisfy the following conditions: $\varphi(u)/u \to 0$ as $u \to 0$, $\varphi(u)/u \to \infty$ as $u \to \infty$. Under these assumptions the function

$$\varphi^*(v) = \sup_{u\geqslant 0}\big(uv - \varphi(v)\big),$$

complementary to the function φ, may be defined. As is well known, φ^* is a convex φ-function, and for a convex φ-function φ we have $(\varphi^*)^* = \varphi$. It is also known that for $\varphi_1(u) = a\varphi(bu)$, where $a, b > 0$, $\varphi_1^*(u) = a\varphi^*\big(u(ab)^{-1}\big)$, and from the inequality $\varphi_1(u) \geqslant \varphi(u)$ for $u \geqslant u_0$ the inequality $\varphi^*(u) \geqslant \varphi_1^*(u)$ for $u \geqslant u_0^*$ follows ([1], [4]).

3.1. *If* $\sigma^1_\varphi < \infty$, *then* $\sigma^1_\varphi \geqslant 1$ *and*

$$\frac{1}{s^1_{\varphi*}} + \frac{1}{\sigma^1_\varphi} \leqslant 1; \tag{+}$$

if $\infty > s^1_\varphi > 1$, *then*

$$\frac{1}{s^1_\varphi} + \frac{1}{\sigma^1_{\varphi*}} \geqslant 1. \tag{++}$$

If $\sigma^1_\varphi = 1$, *then* $s^1_{\varphi*} = \infty$; *if* $s^1_\varphi = \infty$, *then* $\sigma^1_{\varphi*} = 1$, *so that the inequalities* (+), (++) *remain true also in this limiting case.*

Let us assume $\sigma^1_\varphi < \infty$, $\sigma^1_\varphi < \sigma$. In view of 2.1, (2′) the last inequality is equivalent to $\bar{h}_\varphi(\lambda) < a^\sigma$ for any λ, where $a = 1/\lambda$, $0 < \lambda < 1$. It follows that

$$\varphi(au) \leqslant a^\sigma \varphi(u) \quad \text{for} \quad u \geqslant u_0(a),$$

$$\varphi^*(u/a) \geqslant a^\sigma \varphi^*(u/a^\sigma) \quad \text{for} \quad u \geqslant u_0^*(a),$$

whence

$$\varphi^*(a^{\sigma-1}u) \geqslant a^\sigma \varphi^*(u) \quad \text{for} \quad u \geqslant u_0^*(a). \tag{o}$$

The last inequality implies $\sigma \geqslant 1$ such that $\sigma^1_\varphi \geqslant 1$. In fact, if $\sigma < 1$, then from the convexity of φ^* follows $a^{\sigma-1}\varphi^*(u) \geqslant \varphi^*(a^{\sigma-1}u)$, which is contradictory to (o) and $a > 1$.

From (o) we obtain therefore for $\sigma > 1$

$$\frac{\lg \underline{h}_{\varphi*}(a^{\sigma-1}u)}{\lg a^{\sigma-1}} \geqslant \frac{\sigma}{\sigma - 1},$$

$$s^1_{\varphi*} \geqslant \sigma^1_\varphi/\sigma^1_\varphi - 1, \; s^1_{\varphi*} = \infty \text{ if } \sigma^1_\varphi = 1.$$

The proof of (++) is analogous.

3.2. Let us consider two properties of φ-functions:

A. *φ satisfies the condition* (Λ_a) *for any* $a > 1$ *with a constant* $c_a > a$;

B. *φ satisfies the condition* (Δ_a) *for any* $a > 1$ *with a constant* $d_a > 1$, $d_a \to 1$ *as* $a \to 1+0$.

If φ has the property A *or* B, *then φ^* has the property* B *or* A *respectively.*

Suppose φ has the property A, in other words for any $a > 1$ the inequality $\varphi(u)c_a \leqslant \varphi(au)$ holds for $u \geqslant u_0(a)$ and $c_a > a$. For the complementary function we have $\varphi^*(a^{-1}c_a u) \leqslant c_a\varphi^*(u)$ for $u \geqslant u_0^*(a)$. We can always assume that $c_a \to 1$ as $a \to 1$. For any $\beta > 1$ within a sufficiently small neighbourhood of 1 we can choose $a(\beta)$ in such a manner that $\beta a(\beta) \leqslant c_{a(\beta)}$, $a(\beta) \to 1$ as $\beta \to 1$. Defining $d_\beta = c_{a(\beta)}$ we obtain $\varphi^*(\beta u) \leqslant d_\beta \varphi^*(u)$ for large u and so, (Δ_β) being satisfied for φ^* for small β, it is automatically satisfied for all $\beta > 1$.

Assuming φ to have the property B one can prove analogously to the above $\varphi^*(a^{-1}d_a u) \geqslant d_a\varphi(u)$ for $u \geqslant u(a)$.

It does not mean any restriction if we assume for d_a an arbitrary number $> a$ and not less than the originally given d_a. This being so, we choose, for any $\beta > 1$ an $a(\beta) > 1$, $d_{a(\beta)}$ in such a manner that $d_{a(\beta)} = \beta a(\beta)$, $a(\beta) \to 1$ as $\beta \to 1$. Hence φ^* satisfies the condition (Λ_β) for $\beta > 1$, because of $\varphi^*(\beta u) \geqslant c_\beta \varphi^*(u)$ for large u and with $c_\beta = d_{a(\beta)}$, $c_\beta > \beta$.

3.3. It has been remarked in 2.3 that for a φ-function φ a necessary and sufficient condition to be regularly increasing with the index r_φ is $s^1_\varphi = \sigma^1_\varphi = r_\varphi$. This remark and 3.1 imply the following theorem:

If φ is regularly increasing and $r_\varphi > 1$, then φ^* is regularly increasing and the indices $r_\varphi, r_{\varphi*}$ are related to each other by $1/r_\varphi + 1/r_{\varphi*} = 1$ (see [2]).

3.4. *If φ is a convex φ-function,* $s^1_\varphi > 1$ *and* $\sigma^1_\varphi < \infty$, *then the formulae*

$$\frac{1}{s^1_\varphi} + \frac{1}{\sigma^1_{\varphi*}} = 1, \tag{o}$$

$$\frac{1}{s^1_{\varphi*}} + \frac{1}{\sigma^1_\varphi} = 1, \tag{oo}$$

are satisfied.

If $s^1_\varphi > 1$, then according to 3.1, (++), $\sigma^1_{\varphi*} < \infty$ and since $(\varphi^*)^* = \varphi$, we obtain by 3.1, (+), $1/s^1_\varphi + 1/\sigma^1_{\varphi*} \leqslant 1$, from which, together with 3.1, (++), the relation (o) follows. Under the assumption $\sigma^1_\varphi < \infty$ the proof of (oo) remains the same.

3.5. *If* $s^1_\varphi > 1$, *then* $\varphi^* \epsilon K_c^*$.

From $s^1_\varphi > 1$ follows the property A defined in 3.2 and hence φ^* has the property B. It suffices now to apply 2.6 and 2.8.1.

References

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[2] — *Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces*, Studia Math. 21 (1962), p. 317-344.

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[4] — *A note on the theory of s-normed spaces of φ-integrable functions*, Studia Math. 21 (1961), p. 107-115.

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