For this purpose we set

$$R_m(x, T) = \sum_{n \neq m} \frac{a_n \sin(\lambda_n T - \lambda_m T)}{A(\lambda_n)(\lambda_n T - \lambda_m T)} e^{i(\lambda_n - \lambda_m)x}.$$

By virtue of (13), the last series converges uniformly in R_1. Therefore $R_m(x, T)$, as a function of x, is continuous and bounded. Also

$$A\left(\frac{1}{i} D_x\right) R_m(x, T) = r_m(x, T),$$

and, for $T \to \infty$, $R_m(x, T)$ converges uniformly to zero. This means that condition (15) is satisfied.

Remark. If $P_n(x)$, $n = 0, \pm 1, \pm 2, \ldots$, are polynomials of degrees less than a given integer and λ_n are defined as before, then the series

$$\sum_{n=-\infty}^{\infty} P_n(x) e^{i\lambda_n x}$$

converges to an ultra-distribution $\varphi(x)$. Furthermore, there exists an entire function $F(x)$ such that

$$F\left(\frac{1}{i} D\right) \varphi(x) = 0. \tag{16}$$

Conversely, in the space of ultra-distributions each solution of equation (16) has the form

$$\varphi(x) = \sum_{n=-\infty}^{\infty} P_n(x) e^{i\lambda_n x}$$

where the exponential polynomials $P_n(x) e^{i\lambda_n x}$ satisfy the equation.

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On infinite derivatives of continuous functions

by

Z. ZIELEŹNY (Wrocław)

Each distribution of L. Schwartz is locally a derivative of some order of a continuous function. The word "locally" is superfluous, if solely distributions of finite order are considered. This property is of great importance for the theory itself as well as for its applications. In particular, it is the starting point for some of the simplified approaches to the theory of distributions.

On the other hand, analytic functionals defined by L. Ehrenpreis [2], [3], and I. M. Gelfand-G. E. Šilov [4] in connection with Fourier transforms of rapidly increasing functions are not, in general, derivatives of continuous functions defined for real values of the arguments. The space D' of those functionals contains all tempered distributions and, in particular, all continuous slowly increasing functions. Furthermore, an operator of the form

$$A\left(\frac{1}{2\pi i} D\right) = \sum_k \frac{a_k}{(2\pi i)^{\sigma_k}} D^k, \tag{1}$$

where $k = (k_1, k_2, \ldots, k_q)$, D^k is the partial differential operator of order $\sigma_k = k_1 + k_2 + \ldots + k_q$, and $A(x) = \sum_k a_k x^k = \sum_{k_1, k_2, \ldots, k_q = 0}^{\infty} a_k x_1^{k_1} x_2^{k_2} \ldots x_q^{k_q}$ is an entire function, carries each element of D' into an element of D'.

In this paper we are concerned with the subspace $D'_F \subset D'$ consisting of Fourier transforms of distributions of finite order. D'_F is closed with respect to the "infinite derivation" (1) and contains all tempered distributions. We prove that each element of D'_F is an infinite derivative of a continuous slowly increasing function (of real variables). The theorem can be extended to sequences converging in D'_F, and therefore enables us to extend in a natural way the methods used by J. Mikusiński and R. Sikorski in [7], [8], so as to obtain a sequential description of the elements of D'_F. We also characterize Fourier transforms of infinitely differentiable functions as "rapidly decreasing" elements of D'_F.

For simplicity we do not define the elements of D'_F as analytic functionals. The only definition adopted in this paper for distributions (of finite order) as well as for their Fourier transforms, which we call *ultra-distributions*, is by means of sequences. We take for granted the elementary theory of distributions developed in [7] and [8], and the fundamental properties of tempered distributions given in [5] and [10]. A sequential theory of ultra-distributions based on Hadamard's "finite part" of an improper integral was given by G. Temple and the author in [12]. The present approach involves different methods. It is therefore necessary to give a detailed exposition.

Throughout the paper we use the abbreviation "iff" instead of "if and only if".

§ 1. Notation and preliminary notions. Let q be a fixed positive integer. We denote by R_q the real Euclidean q-space.

If $x = (x_1, x_2, \ldots, x_q)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_q)$ are points of R_q, then we write $x+\bar{x} = (x_1+\bar{x}_1, x_2+\bar{x}_2, \ldots, x_q+\bar{x}_q)$, $x\bar{x} = x_1\bar{x}_1+x_2\bar{x}_2+\ldots+x_q\bar{x}_q$, and also $x \geqslant \bar{x}$, if $x_j \geqslant \bar{x}_j$, $j = 1, 2, \ldots, q$. Furthermore, we write $|x| = \sqrt{x_1^2+x_2^2+\ldots+x_q^2}$ and, for any real number λ, $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_q)$.

$\mathcal{N}$ stands for the set of points $k = (k_1, k_2, \ldots, k_q)$, whose coordinates k_j are non-negative integers. The order of k is $\sigma_k = k_1+k_2+\ldots+k_q$. We use the standard notation: $k! = k_1!k_2!\ldots k_q!$, $\binom{k}{l} = \binom{k_1}{l_1}\binom{k_2}{l_2}\ldots\binom{k_q}{l_q}$, and $x^k = x_1^{k_1}x_2^{k_2}\ldots x_q^{k_q}$, where $k, l \in \mathcal{N}$ and $x \in R_q$.

For $i = 1, 2, \ldots, 2^n$, let p_i be a point of R_q, $p_i = (p_{i1}, p_{i2}, \ldots, p_{iq})$, where $p_{ij} = \pm 1$; in particular we set $p_1 = (1, 1, \ldots, 1)$. By $\mathfrak{R}_i$ we denote the subspace of R_q consisting of points $x = (x_1, x_2, \ldots, x_q)$ such that $x_j = 0$ or $\operatorname{sgn} x_j = \operatorname{sgn} p_{ij}$, $j = 1, 2, \ldots, q$.

It is also convenient to adopt the notation $\mathbf{0} = (0, 0, \ldots, 0)$ and $e_i = (d_{i1}, d_{i2}, \ldots, d_{iq})$, where $d_{ij} = 0$ for $i \neq j$ and $d_{ii} = 1$.

For any $k \in \mathcal{N}$, D^k or D_x^k is the symbol of partial derivation

$$D_x^k = \frac{\partial^{\sigma_k}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_q^{k_q}}$$

of order σ_k.

Unless the contrary is explicitly stated, all functions under consideration are complex-valued and defined on R_q.

A function $\varphi(x)$ is said to be *slowly increasing* or *of polynomial growth*, if for some integer ν, $(1+|x|^2)^\nu\varphi(x)$ is bounded in R_q. $\varphi(x)$ is *rapidly decreasing*, if for every integer ν, $(1+|x|^2)^\nu\varphi(x)$ is bounded in R_q.

The *support* of a continuous function $\varphi(x)$ is the closure of the set of points, where $\varphi(x)$ is different from zero.

We say that $\varphi(x)$ is an *entire function*, if it can be continued for all complex values of the arguments $z = (z_1, z_2, \ldots, z_q)$ and the resulting function $\varphi(z)$ is entire.

An entire function $\varphi(z)$ is *of exponential type*, if there exists a $b = (b_1, b_2, \ldots, b_q) \geqslant \mathbf{0}$ such that, for every $\varepsilon > 0$,

$$|\varphi(z)| \leqslant M_\varepsilon e^{(b_1+\varepsilon)|z_1|+(b_2+\varepsilon)|z_2|,\ldots,(b_q+\varepsilon)|z_q|},$$

where $|z_j|$ is the modulus of the complex number z_j, $j = 1, 2, \ldots, q$, and M_ε is a constant; then the type of $\varphi(z)$ is $\leqslant b$.

We use the symbol

$$\int_{-\infty}^{\infty} \varphi(x)\,dx$$

to denote the multiple Lebesgue integral of $\varphi(x)$ over R_q.

If $\varphi(x)$ is integrable over R_q, then its Fourier transform $\tilde{\varphi}(s) = \mathcal{F}\{\varphi(x)\}$, defined by the formula

$$\tilde{\varphi}(s) = \int_{-\infty}^{\infty} \varphi(x)\,e^{-2\pi isx}dx$$

is a continuous function in R_q and tends to zero as $|s| \to \infty$.

The Fourier transform $\varphi(s)$ of an entire function $\varphi(x)$ of exponential type, which is rapidly decreasing (on R_q), is infinitely differentiable and has a compact support. Conversely, if $\tilde{\varphi}(s)$ has the latter properties, then $\varphi(x)$ is an entire function of exponential type. Also Fourier's inversion theorem holds, i. e.

$$\varphi(x) = \int_{-\infty}^{\infty} \tilde{\varphi}(s)\,e^{2\pi isx}ds.$$

We write briefly $\varphi(x) = \mathcal{F}^{-1}\{\tilde{\varphi}(s)\}$.

Let now $\Phi_n(x)$ be a sequence of continuous slowly increasing functions. If, for some integer ν, the functions $(1+|x|^2)^\nu\Phi_n(x)$ are bounded in R_q and form a sequence, which converges uniformly to $(1+|x|^2)^\nu\Phi(x)$, then $\Phi(x)$ is also a continuous slowly increasing function. We say that $\Phi_n(x)$ is *ν-uniformly convergent* to $\Phi(x)$ and we write

$$\Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x).$$

Similarly

$$\Phi_n(x) \overset{\nu}{\rightrightarrows}$$

means that the sequence $\Phi_n(x)$ is ν-uniformly convergent to some function and

$$\Phi_n(x) \overset{\nu}{\rightrightarrows}\overset{\nu}{\leftleftarrows} \Psi_n(x)$$

denotes that both sequences $\Phi_n(x)$ and $\Psi_n(x)$ converge ν-uniformly to the same function.

A sequence $F_n(x)$ is said to *converge almost uniformly* in R_q, if it converges uniformly in every compact subset of R_q.

In this paper we deal only with distributions of finite order. Therefore, in contradistinction to [8], we use the concept of a fundamental sequence in the following sense:

A sequence $f_n(x)$ of infinitely differentiable functions is *fundamental*, if there exists a $k \in \mathcal{N}$ and infinitely differentiable functions $F_n(x)$, such that

$$D^k F_n(x) = f_n(x), \tag{F$_1$}$$

$$F_n(x) \text{ converges almost uniformly in } R_q. \tag{F$_2$}$$

Distributions of finite order are classes of equivalent fundamental sequences defined in the same way as in [8].

A distribution is *of compact support*, contained in $-a \leqslant x \leqslant a$, if it is determined by a fundamental sequence $f_n(x)$, such that $f_n(x) = 0$ outside $-a \leqslant x \leqslant a$.

§ 2. Two lemmas on entire functions. We need the following lemmas on entire functions.

LEMMA 1. *Let $f(x)$ be a real-valued continuous function on R_q, such that $f(x) \geqslant 1$. There exists an entire function $F(x)$, which has the following properties*:

$$|F(x)| \geqslant f(x), \tag{2}$$

and, for every $k \in \mathcal{N}$,

$$\left| D^k \left[\frac{1}{F(x)} \right] \right| \leqslant M_k, \tag{3}$$

where the M_k are constants.

Proof. Consider the function

$$g(x) = \inf_{|y| \leqslant |x|} \left[\frac{1}{f(y)} \right],$$

which is continuous in R_q, symmetric with respect to the origin and decreasing in the sense that $|x| \geqslant |y|$ implies $g(x) \leqslant g(y)$. Clearly we also have

$$0 < g(x) \leqslant 1 \tag{4}$$

and

$$\frac{1}{g(x)} \geqslant f(x). \tag{5}$$

We take the product

$$g^*(x) = \prod_{j=1}^{2^n} g(x + p_j),$$

where the p_j, $j = 1, 2, \ldots, 2^n$, are defined as in § 1. Then, for every $x \in R_q$ and every t of the q-sphere $|t| \leqslant 1$,

$$g^*(x - t) \leqslant g(x). \tag{6}$$

In fact, if $x \in \mathfrak{R}_j$ and $|t| \leqslant 1$, then

$$|x + p_j - t| \geqslant |x|.$$

Hence, by the stated properties of $g(x)$, we get

$$g(x + p_j - t) \leqslant g(x),$$

and consequently the desired inequality (6). Since $\bigcup_{j=1}^{\infty} \mathfrak{R}_j = R_q$, this is true for all $x \in R_q$.

Let now $\gamma(x)$ be a real-valued, infinitely differentiable function such that

$$\gamma(x) \geqslant 0, \quad \gamma(x) = 0 \text{ for } |x| > 1, \quad \int_{-\infty}^{\infty} \gamma(x)\,dx = 1. \tag{7}$$

Then the convolution

$$h(x) = \int_{-\infty}^{\infty} g^*(x - t)\gamma(t)\,dt$$

is an infinitely differentiable positive function. Furthermore, on account of (6) and (7),

$$h(x) \leqslant \sup_{|t| \leqslant 1} g^*(x - t) \leqslant g(x), \tag{8}$$

and also

$$|D_x^k h(x)| \leqslant \int_{-\infty}^{\infty} |D_t^k \gamma(t)|\,dt \tag{9}$$

for every $k \in \mathcal{N}$.

We now make use of a general approximation theorem of H. Whitney ([13], p. 76, lemma 6) in the case of the whole space R_q. The theorem may be formulated in a stronger form suitable for our purpose as follows:

Let $E_1, E_2, \ldots$ be bounded open sets, such that $\bar{E}_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = R_q$. Then, if $\varphi(x)$ is an infinitely differentiable function in R_q

and $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \ldots$ are given positive numbers, there is an entire function $\Phi(x)$ such that

$$|D^k \Phi(x) - D^k \varphi(x)| \leqslant \varepsilon_n \quad \text{in } R_q - E_n, \text{ for } \sigma_k \leqslant n.$$

The proof is the same as in [13].

Accordingly we can find an entire function $F(x)$ such that

$$F(x) - \frac{2}{h(x)} = \eta(x),$$

where

(10) $$|D^k \eta(x)| \leqslant 1$$

for $|x| \geqslant \sigma_k$ and every $k \in \mathcal{N}$. Hence, in view of (5) and (8), we obtain

$$|F(x)| = \left|\frac{2+\eta(x)h(x)}{h(x)}\right| \geqslant \frac{1}{h(x)} \geqslant \frac{1}{g(x)} \geqslant f(x),$$

i. e. inequality (2). It remains to show that all derivatives

$$D^k\left[\frac{1}{F(x)}\right] = D^k\left[\frac{h(x)}{2+\eta(x)h(x)}\right]$$

are bounded, but this is a consequence of the boundedness of $D^k h(x)$ and $D^k \eta(x)$ insured by (9) and (10). Lemma 1 is thus established.

LEMMA 2. *If the functions $f_1(x), f_2(x), \ldots, f_m(x)$ are continuously differentiable up to the order σ, then there exists an entire function $G(x)$ such that*

$$|G(x)| \geqslant 1 + |f_1(x)| + |f_2(x)| + \ldots + |f_m(x)|$$

and all derivatives

$$D^k\left[\frac{f_1(x)}{G(x)}\right], \quad D^k\left[\frac{f_2(x)}{G(x)}\right], \ldots, D^k\left[\frac{f_m(x)}{G(x)}\right],$$

of order $\sigma_k \leqslant \sigma$ are integrable over R_q.

Proof. Let $f(x)$ be the function defined by equation

(11) $$f(x) = g_0(x)(1+|x|^2)^{(1+\sigma)/2},$$

where

$$g_0(x) = 1 + \sum_{\sigma_k \leqslant \sigma} \sum_{j=1}^{m} |D^k f_j(x)|.$$

Then $f(x)$ is continuous and $f(x) \geqslant 1$. Therefore, by lemma 1, we can find an entire function $F(x)$ such that $|F(x)| \geqslant f(x)$ and all derivatives of $1/F(x)$ are bounded in R_q. The function

$$G(x) = F^{\sigma+1}(x)$$

satisfies the conditions of lemma 2.

For the proof we observe first that

$$|G(x)| \geqslant |F(x)| \geqslant 1 + \sum_{j=1}^{m} |f_j(x)|.$$

Now, in order to show that e. g. all derivatives $D^k[f_1(x)/G(x)]$ of order $\sigma_k \leqslant \sigma$ are integrable over R_q, we use the Leibniz formula

$$D^k\left[f_1(x)\cdot\frac{1}{G(x)}\right] = \sum_{l \leqslant k} \binom{k}{l} D^{k-l} f_1(x) D^l\left[\frac{1}{G(x)}\right].$$

Thus it is sufficient to show that, for every $k \in \mathcal{N}$ of order $\sigma_k \leqslant \sigma$, the function $D^k f_1(x)/F(x)$ is integrable and

$$D^k\left[\frac{1}{G(x)}\right] = \frac{F_k(x)}{F^{\sigma-\sigma_k+1}(x)},$$

where $F_k(x)$ is an infinitely differentiable function bounded in R_q together with all its derivatives. But these facts can be easily verified by use of equation (11) and the properties of $F(x)$.

§ 3. Generalized regular sequences. First we refer briefly to Lighthill's book [5], which contains a sequential theory of tempered distributions, based on a different idea (due to Mikusiński [6]) ([1]). "Good functions" used as the starting point of this theory are infinitely differentiable functions rapidly decreasing together with all their derivatives. We formulate the definition of a regular sequence (see [5], p. 16, definition 3) in an equivalent form as follows:

A sequence $\varphi_n(x)$ of good functions is *regular*, if there exists an integer ν, a polynomial $P(x) = \sum_{\sigma_k \leqslant \sigma} a_k x^k$ and good functions $\Phi_n(x)$ such that

(R_1) $$P\left(\frac{1}{2\pi i} D\right)\Phi_n(x) = \sum_{\sigma_k \leqslant \sigma} \frac{a_k}{(2\pi i)^{\sigma_k}} D^k \Phi_n(x) = \varphi_n(x),$$

(R_2) $$\Phi_n(x) \overset{\nu}{\rightrightarrows}.$$

One may always assume that $P(x)$ is a sufficiently high power of $(1+|x|^2)$.

We now modify slightly the above definition so as to obtain generalized regular sequences and finally ultra-distributions. We replace namely the polynomial of derivation in (R_1) by the more general operator (1), generated by an entire function. But the class of good functions is too wide for this purpose. In other words, the operator (1) cannot be

([1]) Though Lighthill's book deals only with tempered distributions of one variable, the case of several variables is analogous.

applied to every good function. Thus our first task is to find a suitable subset of the class of good functions.

In what follows an operator of the form (1), generated by an entire function $A(x)$, is called an *infinite derivative* or a *derivative of infinite order*.

We seek a class of functions, whose all infinite derivatives exist (in a certain sense) and belong to the same class. More precisely, each function $\varphi(x)$ of the class in question must be such that the series

$$\sum_k \frac{a_k}{(2\pi i)^{\sigma_k}} D^k \varphi(x),$$

where a_k are Taylor coefficients of an entire function $A(x)$ at $x=0$, converges at least almost uniformly in R_q and its sum

$$A\left(\frac{1}{2\pi i} D\right)\varphi(x)$$

is again a function of this class. A useful result in this point was given by H. Muggli [9] (see also [1], theorem 11. 7. 3.) We formulate its extension to q dimensions.

Lemma 3 (Muggli). *Suppose that $F(z) = \sum_k c_k z^k$ is an analytic function in the set $G_b = \{|z_1| \leqslant b_1,\ |z_2| \leqslant b_2, \ldots, |z_q| \leqslant b_q\}$. Then, for every entire function $\varphi(z)$ of exponential type $\leqslant b = (b_1, b_2, \ldots, b_q)$, the series*

$$\sum_k c_k D^k \varphi(z)$$

converges almost uniformly in the complex Euclidean q-space and its sum $F(D)\varphi(z)$ is again an entire function of exponential type $\leqslant b$.

The proof can be obtained in exactly the same way as in the case of one variable.

Corollary. *The infinite derivative carries each entire function of exponential type $\leqslant b$ into an entire function of exponential type $\leqslant b$.*

Moreover, if $\varphi(x)$ is an entire function of exponential type, which is rapidly decreasing (on R_q), then its infinite derivative

$$A\left(\frac{1}{2\pi i} D\right)\varphi(x)$$

is also rapidly decreasing (on R_q) (²).

In fact, by what we have said in § 1, the Fourier transform $\tilde{\varphi}(s) = \mathcal{F}\{\varphi(x)\}$ is an infinitely differentiable function with compact support.

(²) In this case the corresponding series converges in a much stronger sense.

The result follows immediately from the correspondence

(12) $$A(s)\tilde{\varphi}(s) = \mathcal{F}\left\{A\left(\frac{1}{2\pi i} D\right)\varphi(x)\right\}.$$

Entire functions of exponential type, rapidly decreasing (on R_q), form the appropriate class. Functions of this class are admitted as members of the generalized regular sequences to be defined. For brevity they are called *very good functions*.

Definition 1. A sequence $\varphi_n(x)$ of very good functions is said to be a *generalized regular sequence*, if there exists an integer ν, an entire function $A(x)$ and very good functions $\Phi_n(x)$ such that

(GR₁) $$A\left(\frac{1}{2\pi i} D\right)\Phi_n(x) = \varphi_n(x),$$

(GR₂) $$\Phi_n(x) \overset{\nu}{\rightrightarrows}.$$

It is obvious that the integer ν may always be replaced by any greater integer. We shall show later on that also $A(x)$ may be replaced by some other entire functions of greater modulus.

Each regular sequence of very good functions is a generalized regular sequence. In particular, each sequence of very good functions, which converges ν-uniformly, is a generalized regular sequence.

If $\varphi_n(x)$ is a generalized regular sequence and $A(x)$ an entire function, then

$$A\left(\frac{1}{2\pi i} D\right)\varphi_n(x)$$

is also a generalized regular sequence.

Let now $\Phi_n(x)$ be very good functions and $w(x)$ an integrable function, whose Fourier transform $\tilde{w}(s)$ is infinitely differentiable. Then the convolutions

$$\Phi_n^*(x) = \int_{-\infty}^{\infty} \Phi_n(x-t)\, w(t)\, dt$$

are also very good functions, since by the Fourier transform the convolution corresponds to the product and, as already said before, very good functions correspond to infinitely differentiable functions with compact supports. Moreover, if

(13) $$|w(x)| \leqslant M(1+|x|^2)^{-\nu-q/2-1/2},$$

where ν is an integer $\geqslant 0$ and M a constant, then

(14) $$\Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x)$$

implies

(15) $$\Phi_n^*(x) \overset{\nu}{\rightrightarrows} \Phi^*(x) = \int_{-\infty}^{\infty} \Phi(x-t)w(t)\,dt.$$

In fact,

$$1+|x-t|^2 \leqslant 2(1+|x|^2)(1+|t|^2)$$

and therefore, in view of (13),

$$\frac{|\Phi_n^*(x)-\Phi^*(x)|}{(1+|x|^2)^\nu} \leqslant 2^\nu \int_{-\infty}^{\infty} \frac{|\Phi_n(x-t)-\Phi(x-t)|}{(1+|x-t|^2)^\nu}(1+|t|^2)^\nu |w(t)|\,dt$$
$$\leqslant 2^\nu M \int_{-\infty}^{\infty} \frac{|\Phi_n(x-t)-\Phi(x-t)|}{(1+|x-t|^2)^\nu}(1+|t|^2)^{-q/2-1/2}dt.$$

By virtue of (14), the quotient under the last integral converges uniformly to zero. Hence we obtain the required convergence (15).

Next, by lemma 2 (with $m=1$), to any entire function $A(x)$ and any integer $\sigma \geqslant 0$ there exists an entire function $A^*(x)$ such that $|A^*(x)| \geqslant 1+|A(x)|$ and all derivatives $D^k[A(x)/A^*(x)]$ of order $\sigma_k \leqslant \sigma$ are integrable over R_q. We take $\sigma = 2\nu+q+1$ and $\tilde{w}(s) = A(s)/A^*(s)$. Then $w(x)$ satisfies the above conditions including inequality (13). Furthermore, since

$$A^*(s)\tilde{\Phi}_n^*(s) = A(s)\tilde{\Phi}_n(s),$$

we also get

(16) $$A^*\left(\frac{1}{2\pi i}D\right)\Phi_n^*(x) = A\left(\frac{1}{2\pi i}D\right)\Phi_n(x),$$

on account of equation (12). This shows that the entire function $A(x)$ in the definition of a generalized regular sequence may be replaced by $A^*(x)$, i. e. if $\varphi_n(x)$, ν, $\Phi_n(x)$ satisfy conditions (GR_1) and (GR_2), and if $A^*(x)$, $\Phi_n^*(x)$ are chosen as above, then

$$A^*\left(\frac{1}{2\pi i}D\right)\Phi^*(x) = \varphi_n(x)$$

and

$$\Phi_n^*(x) \overset{\nu}{\rightrightarrows}.$$

The same argument leads to the more general

PROPOSITION 1. *If $\varphi_n^{(1)}(x), \varphi_n^{(2)}(x), \ldots, \varphi_n^{(m)}(x)$ are generalized regular sequences, then there exists an entire function $A(x)$, an integer ν and very good functions $\Phi_n^{(1)}(x), \Phi_n^{(2)}(x), \ldots, \Phi_n^{(m)}(x)$, such that*

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n^{(j)}(x) = \varphi_n^{(j)}(x)$$

and

$$\Phi_n^{(j)}(x) \overset{\nu}{\rightrightarrows}, \quad j=1,2,\ldots,m.$$

§ 4. Ultra-distributions. The main advantage of our method is the fact that ultra-distributions are defined in the same way as distributions (compare with the definition in [7]). The modifications concern the operators involved, which are now more general, and the convergence, which is now stronger than the almost uniform convergence used in [7].

Definition 2. We say that two generalized regular sequences $\varphi_n(x)$ and $\psi_n(x)$ are *equivalent* and we write

$$\varphi_n(x) = \psi_n(x),$$

if $\varphi_1(x), \psi_1(x), \varphi_2(x), \psi_2(x), \ldots$ is a generalized regular sequence.

In other words, the generalized regular sequences $\varphi_n(x)$ and $\psi_n(x)$ are equivalent, iff there exists an entire function $A(x)$, an integer ν and very good functions $\Phi_n(x)$, $\Psi_n(x)$, such that

(GE_1) $$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) = \varphi_n(x), \qquad A\left(\frac{1}{2\pi i}D\right)\Psi_n(x) = \psi_n(x),$$

(GE_2) $$\Phi_n(x) \overset{\nu}{\rightrightarrows} \overset{\nu}{\leftleftarrows} \Psi_n(x).$$

Similarly as in definition 1, the integer ν in (GE_2) may be replaced by any greater integer, and also, by the results of the preceding section, $A(x)$ may be replaced by some other entire functions of greater modulus. Thus we have

PROPOSITION 2. *If $\varphi_n^{(1)}(x), \varphi_n^{(2)}(x), \ldots, \varphi_n^{(m)}(x)$; $\psi_n^{(1)}(x), \psi_n^{(2)}(x), \ldots, \psi_n^{(m)}(x)$ are generalized regular sequences and*

$$\varphi_n^{(j)}(x) \sim \psi_n^{(j)}(x), \quad j=1,2,\ldots,m,$$

then there exists an entire function $A(x)$, with $|A(x)|>1$, an integer ν and very good functions $\Phi_n^{(1)}(x), \Phi_n^{(2)}(x), \ldots, \Phi_n^{(m)}(x)$; $\Psi_n^{(1)}(x), \Psi_n^{(2)}(x), \ldots, \Psi_n^{(m)}(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n^{(j)}(x) = \varphi_n^{(j)}(x), \qquad A\left(\frac{1}{2\pi i}D\right)\Psi_n^{(j)}(x) = \psi_n^{(j)}(x),$$

and

$$\Phi_n^{(j)}(x) \overset{\nu}{\rightrightarrows} \overset{\nu}{\leftleftarrows} \Psi_n^{(j)}(x), \quad j=1,2,\ldots,m.$$

The relation $\sim$ is obviously reflexive and symmetric, i. e.

(17) $$\varphi_n(x) \sim \varphi_n(x),$$

(18) $$\varphi_n(x) \sim \psi_n(x) \text{ implies } \psi_n(x) \sim \varphi_n(x).$$

We prove now its transitivity:

(19) $\varphi_n(x) \sim \psi_n(x)$ and $\psi_n(x) \sim \vartheta_n(x)$ implies $\varphi_n(x) \sim \vartheta_n(x)$.

By proposition 2, with $m = 2$, there exists an entire function $A(x)$, $|A(x)| \geqslant 1$, an integer ν and very good functions $\Phi_n(x)$, $\Psi_n^{(1)}(x)$, $\Psi_n^{(2)}(x)$, $\Theta_n(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) = \varphi_n(x), \qquad A\left(\frac{1}{2\pi i}D\right)\Theta_n(x) = \vartheta_n(x), \tag{20}$$

$$A\left(\frac{1}{2\pi i}D\right)\Psi_n^{(1)}(x) = A\left(\frac{1}{2\pi i}D\right)\Psi_n^{(2)}(x) = \psi_n(x) \tag{21}$$

and

$$\Phi_n(x) \overset{\nu}{\Rightarrow} \overset{\nu}{\Leftarrow} \Psi_n^{(1)}(x), \qquad \Psi_n^{(2)}(x) \overset{\nu}{\Rightarrow} \overset{\nu}{\Leftarrow} \Theta_n(x). \tag{22}$$

But

$$\Psi_n^{(1)}(x) = \psi_n^{(2)}(x),$$

since from (21) we get

$$\mathcal{F}\{\Psi_n^{(1)}(x)\} = \mathcal{F}\{\Psi_n^{(2)}(x)\}.$$

Therefore, because of (22),

$$\Phi_n(x) \overset{\nu}{\Rightarrow} \overset{\nu}{\Leftarrow} \Theta_n(x),$$

and this completes the proof.

The relation $\sim$, being reflexive, symmetric and transitive, splits the set of all generalized regular sequences into disjoint classes. Two sequences belong to the same class, iff they are equivalent. These classes are called *ultra-distributions*. We denote ultra-distributions as usual functions; we also write $\varphi(x) = [\varphi_n(x)]$ to indicate that the ultra-distribution $\varphi(x)$ is determined by the generalized regular sequence $\varphi_n(x)$. It has, however, to be remarked that one cannot, in general, substitute for the variable x any point of R_q.

§ 5. Tempered distributions as ultra-distributions. Ultra-distributions determined by regular sequences of very good functions may be identified with tempered distributions. If, in particular, $\varphi(x) = [\varphi_n(x)]$, where $\varphi_n(x) \overset{\nu}{\Rightarrow}$ for some integer ν, then the ultra-distribution $\varphi(x)$ may be identified with a continuous function of polynomial growth — the limit of $\varphi_n(x)$. On the other hand, every tempered distribution may be represented by a regular sequence of very good functions. Similarly, to every continuous function $\varphi(x)$ of polynomial growth we can find a sequence $\varphi_n(x)$ of very good functions such that $\varphi_n(x) \overset{\nu}{\Rightarrow} \varphi(x)$, for a sufficiently large integer ν. We may therefore regard ultra-distributions as a generalization of tempered distributions.

In order to show that the above correspondence is one-to-one we need some further results on generalized regular sequences.

LEMMA 4. *Let $A(x)$ be an entire function, ν an integer, and $\Phi_n(x)$ a sequence of very good functions. If*

$$\Phi_n(x) \overset{\nu}{\Rightarrow} 0 \quad \textit{and} \quad A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) \overset{\nu}{\Rightarrow} \Gamma(x),$$

then $\Gamma(x) \equiv 0$.

Proof. Integrating by parts one can easily verify that, for any very good function $\Psi(x)$,

$$\int_{-\infty}^{\infty} \Psi(x) A\left(\frac{1}{2\pi i}D\right)\Phi_n(x)\,dx = \int_{-\infty}^{\infty} \Phi_n(x) A\left(\frac{-1}{2\pi i}D\right)\Psi(x)\,dx.$$

By the assumption, the integral on the right-hand side converges to zero as $n \to \infty$, and so

$$\int_{-\infty}^{\infty} \Psi(x)\Gamma(x)\,dx = 0. \tag{23}$$

Consider now the functions

$$\Psi_n(x, x_0) = n\Psi_0(nx - nx_0),$$

where x_0 is an arbitrary point of R_q and $\Psi_0(x)$ a very good functions, such that

$$\int_{-\infty}^{\infty} \Psi_0(x)\,dx = 1.$$

$\Psi_n(x, x_0)$ are very good functions, and therefore

$$\int_{-\infty}^{\infty} \Psi_n(x, x_0)\Gamma(x)\,dx = 0,$$

by equation (23). Since, on the other hand, for $n \to \infty$,

$$\int_{-\infty}^{\infty} \Psi_n(x, x_0)\Gamma(x)\,dx \to \Gamma(x_0),$$

we have $\Gamma(x_0) = 0$. Our assertion is thus proved.

PROPOSITION 3. *Let $\varphi_n(x)$ and $\psi_n(x)$ be regular sequences of very good functions. Then $\varphi_n(x) \sim \psi_n(x)$, iff the interlaced sequence $\varphi_1(x)$, $\psi_1(x)$, $\varphi_2(x)$, $\psi_2(x)$, ... is regular.*

Proof. If the interlaced sequence is regular, then it is a generalized regular sequence and therefore $\varphi_n(x) \sim \psi_n(x)$. Conversely, suppose that

$\varphi_n(x) \sim \psi_n(x)$. Then there exists an entire function $A(x)$, an integer ν, and very good functions $\Theta_n(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Theta_n(x) = \vartheta_n(x) = \varphi_n(x) - \psi_n(x), \tag{24}$$

$$\Theta_n(x) \overset{\nu}{\rightrightarrows} 0. \tag{25}$$

But $\varphi_n(x)$ and $\psi_n(x)$ are regular sequences, i. e. there exists a polynomial $P(x) = (1+|x|^2)^\mu$, an integer ν^*, and very good functions $\Phi_n^*(x)$, $\Psi_n^*(x)$, such that

$$P\left(\frac{1}{2\pi i}D\right)\Phi_n^*(x) = \varphi_n(x), \quad P\left(\frac{1}{2\pi i}D\right)\Psi_n^*(x) = \psi_n(x), \tag{26}$$

$$\Phi_n^*(x) \overset{\nu^*}{\rightrightarrows} \quad \text{and} \quad \Psi_n^*(x) \overset{\nu^*}{\rightrightarrows}. \tag{27}$$

Replacing if necessary ν or ν^* by a greater integer, we may set $\nu^* = \nu$.

For the proof it is sufficient to show that

$$\Theta_n^*(x) = \Phi_n^*(x) - \Psi_n^*(x) \overset{\nu}{\rightrightarrows} 0. \tag{28}$$

We may assume that $\mu \geqslant (1+q)/2$. Then $1/P(x)$ is integrable over R_q together with all its derivatives, so that the Fourier transform $\varrho(s) = \mathscr{F}\{1/P(x)\}$ is a continuous and rapidly decreasing function. Therefore convergence (25) implies that

$$\Theta_n^{**}(x) = \int_{-\infty}^{\infty} \Theta_n(x-t)\varrho(t)\,dt \overset{\nu}{\rightrightarrows} 0, \tag{29}$$

by the same argument as in the proof of (15). Moreover, $\Theta_n^{**}(x)$ are very good functions such that

$$P\left(\frac{1}{2\pi i}D\right)\Theta_n^{**}(x) = \Theta_n(x).$$

Hence, taking into account equations (24) and (26), we get

$$A\left(\frac{1}{2\pi i}D\right)\Theta_n^{**}(x) = \Theta_n^*(x),$$

and consequently, by application of (27), (29) and lemma 4, the convergence (28). The proof of proposition 3 is now completed.

In a similar way one can prove

PROPOSITION 4. *Let $\varphi_n(x)$ and $\psi_n(x)$ be very good functions such that*

$$\varphi_n(x) \overset{\nu}{\rightrightarrows} \varphi(x) \quad \text{and} \quad \psi_n(x) \overset{\nu}{\rightrightarrows} \psi(x)$$

for some integer ν. Then $\varphi_n(x) \sim \psi_n(x)$, iff $\varphi(x) \equiv \psi(x)$.

Two regular sequences determine the same tempered distribution, iff the interlaced sequence is regular (see [7], theorem 3.2). Thus proposition 3 shows that the stated correspondence between ultra-distributions and tempered distributions is one-to-one. Similarly, proposition 4 shows that the correspondence between ultra-distributions and continuous functions of polynomial growth is one-to-one.

A very good function $\varphi(x)$ is determined by the generalized regular sequence, all of whose terms coincide with $\varphi(x)$.

§ 6. Operations on ultra-distributions. All operations on ultra-distributions will be defined by means of the corresponding operations on functions of the generalized regular sequences, which determine the ultra-distributions. In this section we define the sum, the difference, the translation and the infinite derivative of ultra-distributions, and also the product of an ultra-distribution with a polynomial. Other operations, such as the Fourier transform, the scalar product with a very good function, the convolution and the product of an ultra-distribution with a function of a wider class, will be discussed separately.

Definition 3. For two arbitrary ultra-distributions $\varphi(x) = [\varphi_n(x)]$ and $\psi(x) = [\psi_n(x)]$ we define

$$\varphi(x) \pm \psi(x) = [\varphi_n(x) \pm \psi_n(x)],$$

$$\varphi(x+h) = [\varphi_n(x+h)],$$

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x) = \left[F\left(\frac{1}{2\pi i}D\right)\varphi_n(x)\right],$$

$$P(x)\varphi(x) = [P(x)\varphi_n(x)],$$

where $h = (h_1, h_2, \ldots, h_q)$ is a q-tuple of complex numbers, $F(x)$ an entire function and $P(x)$ a polynomial.

Proof of consistency. All terms of the new sequences are obviously very good functions. We have to verify that in each case: 1° the sequence in square brackets is a generalized regular sequence, 2° the new ultra-distribution does not depend on the choice of the generalized regular sequences representing $\varphi(x)$ and $\psi(x)$. In point of fact we need only to verify the item 1°, which already implies 2°, as it is easy to see.

From proposition 1 it follows that the sum $\varphi_n(x) + \psi_n(x)$ (and similarly the difference) is a generalized regular sequence.

The translation can be reduced to infinite differentiation, since

$$\varphi_n(x+h) = \sum_k \frac{h^k}{k!} D^k \varphi_n(x), \qquad n = 1, 2, \ldots, \tag{30}$$

the series being convergent uniformly in R_q on multiplying by any polynomial.

Next, if the functions $\varphi_n(x)$ satisfy conditions (GR$_1$) and (GR$_2$), then

$$F\left(\frac{1}{2\pi i}D\right)\varphi_n(x)$$

satisfy the same conditions with $A(x)$ replaced by the product $F(x)A(x)$. Hence

$$F\left(\frac{1}{2\pi i}D\right)\varphi_n(x)$$

is a generalized regular sequence.

Finally, let $A(x)$ be an entire function, ν an integer and $\Phi_n(x)$ very good functions such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \overset{\nu}{\Rightarrow}.$$

Then

$$x_j\varphi_n(x) = A\left(\frac{1}{2\pi i}D\right)\left(x_j\Phi_n(x)\right) + B\left(\frac{1}{2\pi i}D\right)\Phi_n(x),$$

where

$$B(x) = -\frac{1}{2\pi i}D^{e_j}A(x).$$

Since $x_j\Phi_n(x) \overset{\nu+1}{\Rightarrow}$, $x_j\varphi_n(x)$ is a generalized regular sequence. Now, given any $k = (k_1, \ldots, k_q) \in \mathcal{N}$, we repeat the above argument k_1 times with respect to x_1, k_2 times with respect to x_2, etc. It follows that $x^k\varphi_n(x)$ is a generalized regular sequence and, since the product of a generalized regular sequence with a number is again a generalized regular sequence, the same is true for $P(x)\varphi_n(x)$. Thus the consistency of all parts of definition 3 is established.

Similarly, for an $a = (a_1, \ldots, a_q) \in R_q$, $a_j \neq 0$, we set

$$\varphi(ax) = [\varphi_n(ax)];$$

the proof of consistency offers no difficulties and we leave it out.

If $\varphi(x)$, $\psi(x)$ are tempered distributions, $F(x)$ a polynomial and $h \in R_q$, then all operations in definition 3 are compatible with the same operations defined earlier for tempered distributions (see e. g. [5], definition 6).

Also, if $\varphi(x)$ is an entire function of exponential type, such that

$$\varphi(x) = O(|x|^{2\nu}) \quad \text{as} \quad |x| \to \infty,$$

and $F(x) = \sum_k c_k x^k$ an arbitrary entire function, then the infinite derivative

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x)$$

in the usual sense, i. e. regarded as the sum of the almost uniformly convergent series

$$\sum_k \frac{c_k}{(2\pi i)^{\sigma_k}} D^k\varphi(x),$$

coincides with the infinite derivative defined for ultra-distributions.

In fact, we shall show in § 10 that $\varphi(x)$ may be represented in the form

$$\varphi(x) = \int_{-\infty}^{\infty} \varphi(x-t)\chi_0(t)\,dt,$$

where $\chi_0(x)$ is some very good function. Thus, if $\chi_1(x)$ is another very good function with $\chi_1(0) = 1$, then

$$\varphi_n(x) = \int_{-\infty}^{\infty} \chi_1\left(\frac{x-t}{n}\right)\varphi(x-t)\chi_0(t)\,dt$$

is a generalized regular sequence of $\varphi(x)$, such that $\varphi_n(x) \overset{\nu}{\Rightarrow} \varphi(x)$ and also

$$F\left(\frac{1}{2\pi i}D\right)\varphi_n(x) \overset{\nu}{\Rightarrow} F\left(\frac{1}{2\pi i}D\right)\varphi(x).$$

Consequently

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x) = \left[F\left(\frac{1}{2\pi i}D\right)\varphi_n(x)\right].$$

The translation of a tempered distribution by a q-tuple of complex numbers is not, in general, a tempered distribution; it is an ultra-distribution. It can be obtained by infinite derivation as follows:

$$\varphi(x+h) = e^{hD}\varphi(x).$$

In the theory of distributions derivation (of finite order) is always feasible. Now each ultra-distribution has an arbitrary infinite derivative. Moreover, the class of all ultra-distributions is the smallest class, which 1° is closed with respect to infinite derivation, 2° contains all continuous functions of polynomial growth. We have namely

PROPOSITION 5. *Each ultra-distribution is an infinite derivative of a continuous function of polynomial growth.*

In other words, each ultra-distribution $\varphi(x)$ can be represented in the form

$$\varphi(x) = A\left(\frac{1}{2\pi i}D\right)\Phi(x), \tag{31}$$

where $A(x)$ is an entire function and $\Phi(x)$ a continuous function of polynomial growth.

Proof. If $\varphi(x) = [\varphi_n(x)]$, then there exists an entire function $A(x)$, an integer ν and very good functions $\Phi_n(x)$ such that

$$A\left(\frac{1}{2\pi i} D\right)\Phi_n(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x).$$

Here $\Phi(x) = [\Phi_n(x)]$ is a continuous function of polynomial growth and we have

$$A\left(\frac{1}{2\pi i} D\right)\Phi(x) = \left[A\left(\frac{1}{2\pi i} D\right)\Phi_n(x)\right] = [\varphi_n(x)] = \varphi(x),$$

q. e. d.

Remark. If the function $\Phi(x)$ in equation (31) is $O(|x|^{2\nu})$ as $|x| \to \infty$, and if $A^*(x)$ is an entire function making the quotient $A(x)/A^*(x)$ integrable over R_q together with all its derivatives up to the order $2\nu+q+1$, then there exists another continuous function $\Phi^*(x)$, which is $O(|x|^{2\nu})$ as $|x| \to \infty$ and satisfies the equation

$$A^*\left(\frac{1}{2\pi i} D\right)\Phi^*(x) = \varphi(x).$$

A greater rate of decrease of the quotient $A(x)/A^*(x)$ improves the smoothness of $\Phi^*(x)$. If, for example, the quotient is rapidly decreasing, then $\Phi^*(x)$ is infinitely differentiable. Thus each ultra-distribution $\varphi(x)$ admits a representation (31), where $\Phi(x)$ is an infinitely differentiable function or even an entire function.

§ 7. Another definition of ultra-distributions. Proposition 5 allows us to regard ultra-distributions as formal derivatives of slowly increasing continuous functions. This conclusion suggests an equivalent definition of ultra-distributions, which is of purely algebraic character [3]. As the starting point we use now ordered pairs $(A(x), \Phi(x))$, where $A(x)$ is an entire function and $\Phi(x)$ a continuous function of polynomial growth. The main difficulty is to define under what conditions two pairs of this kind are equivalent.

We write

$$W_{FG}(x) = \mathscr{F}\left\{\frac{F(s)}{G(s)}\right\},$$

if the quotient in curly brackets is integrable together with all its derivatives up to a certain order, so that $W_{FG}(x)$ is at least integrable over R_q. Then we have the relation

$$W_{FG}(x) * W_{GK}(x) = W_{FK}(x), \tag{32}$$

where $*$ denotes the convolution.

Let now $(A(x), \Phi(x))$ be one of the ordered pairs under consideration. Then, for some integer $\nu \geqslant 0$,

$$\Phi(x) = O(|x|^{2\nu}) \quad \text{as} \quad |x| \to \infty. \tag{33}$$

On the other hand, by lemma 2, there exists an entire function $G(x)$ such that

$$(1+|x|^2)^{\nu+q/2+1/2} W_{AG}(x) = O(1) \quad \text{as} \quad |x| \to \infty.$$

On account of (33), the latter condition insures the existence of the convolution

$$\Phi(x) * W_{AG}(x),$$

which we use always in this meaning.

We say that two pairs $(A(x), \Phi(x))$ and $(B(x), \Psi(x))$ are *equivalent* and we write

$$(A(x), \Phi(x)) \sim (B(x), \Psi(x)),$$

if there exists an entire function $F(x)$ such that

$$\Phi(x) * W_{AF}(x) = \Psi(x) * W_{BF}(x). \tag{34}$$

The relation $\sim$ is obviously reflexive and symmetric. We prove that it is also transitive, i. e. that

$$(A(x), \Phi(x)) \sim (B(x), \Psi(x)) \quad \text{and} \quad (B(x), \Psi(x)) \sim (C(x), \Theta(x))$$

implies

$$(A(x), \Phi(x)) \sim (C(x), \Theta(x)).$$

In fact, by assumption, there exist entire functions $F(x)$ and $G(x)$ such that

$$\begin{aligned} &\Phi(x) * W_{AF}(x) = \Psi(x) * W_{BF}(x), \\ &\Psi(x) * W_{BG}(x) = \Theta(x) * W_{CG}(x). \end{aligned} \tag{35}$$

Applying now lemma 2 to $F(x)$ and $G(x)$, we can find an entire function $K(x)$ such that $W_{FK}(x)$ and $W_{GK}(x)$ are integrable on multiplying by a sufficiently high power of $|x|$. Then, by virtue of (32) and (35), we obtain

$$\Phi(x) * W_{AK}(x) = \Psi(x) * W_{BK}(x),$$
$$\Psi(x) * W_{BK}(x) = \Theta(x) * W_{CK}(x),$$

[3] A similar definition of distributions of finite order was given by R. Sikorski in [11] (see also [7], § 7).

and consequently

$$\Phi(x) * W_{AK}(x) = \Theta(x) * W_{OK}(x),$$

which means that $(A(x), \Phi(x)) \sim (C(x), \Theta(x))$.

The abstraction classes (with respect to the relation $\sim$) of equivalent pairs are called *ultra-distributions*. We denote by $[A(x), \Phi(x)]$ the ultra-distribution determined by the pair $(A(x), \Phi(x))$.

Any two ultra-distributions may be represented by pairs with a common first element, since

$$(A(x), \Phi(x)) \sim (F(x), \Phi(x) * W_{AF}(x)),$$

if $F(x)$ is chosen according to lemma 2, so that the convolution on the right exists.

All operations considered in the preceding section may be defined for the new ultra-distributions in a simple way. In particular, we define the infinite derivative by equation

$$F\left(\frac{1}{2\pi i} D\right)[A(x), \Phi(x)] = [F(x)A(x), \Phi(x)],$$

where $F(x)$ is an arbitrary entire function.

If $A(x) \neq 0$ and $(A(x), \Phi(x)) \sim (A(x), \Psi(x))$, then $\Phi(x) = \Psi(x)$.

In fact, one can find an entire function $F(x)$ such that

$$\Theta(x) * W_{AF}(x) = 0, \quad \text{where} \quad \Theta(x) = \Phi(x) - \Psi(x). \tag{36}$$

Let now $\varphi(x)$ be a very good function, $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$, and $\tilde{\varphi}(s) = \mathscr{F}\{\varphi(x)\}$. We take the sequence

$$\varphi_n(x) = \mathscr{F}^{-1}\left\{\frac{F(s)}{A(s)} \tilde{\varphi}\left(\frac{s}{n}\right)\right\}.$$

Then

$$W_{AF}(x) * \varphi_n(x) = n\varphi(nx),$$

and hence, for some integer ν,

$$\Theta(x) * W_{AF}(x) * \varphi_n(x) = \Theta(x) * n\varphi(nx) \overset{\nu}{\rightrightarrows} \Theta(x).$$

Thus $\Theta(x) \equiv 0$, on account of (36).

By the above we may identify an ultra-distribution, which admits a representation of the form $[1, \Phi(x)]$, with the continuous slowly increasing function $\Phi(x)$. Moreover,

$$[A(x), \Phi(x)] = A\left(\frac{1}{2\pi i} D\right)[1, \Phi(x)],$$

and therefore each ultra-distribution is an infinite derivative of a continuous slowly increasing function. We identify an arbitrary ultra-distribution $[A(x), \Phi(x)]$ in the new sense with the infinite derivative

$$A\left(\frac{1}{2\pi i} D\right)\Phi(x)$$

in the former sense. For the proof of consistency it is sufficient to observe that, for infinite derivatives of continuous slowly increasing functions in the former sense,

$$A\left(\frac{1}{2\pi i} D\right)\Phi(x) = B\left(\frac{1}{2\pi i} D\right)\Psi(x),$$

iff there exists an entire function $F(x)$, which satisfies condition (34). This result can be easily proved by use of the Fourier transforms discussed in the next section.

§ 8. Fourier transforms of ultra-distributions. As said before, the set of Fourier transforms of very good functions consists of all infinitely differentiable functions with compact supports. Moreover, it appears that to generalized regular sequences there correspond fundamental sequences defined as in section 1.

PROPOSITION 6. *Let $\varphi_n(x)$, $n = 1, 2, \ldots$, be very good functions and $\tilde{\varphi}_n(s)$ their Fourier transforms. Then $\varphi_n(x)$ is a generalized regular sequence, iff the sequence $\tilde{\varphi}_n(s)$ is fundamental.*

Proof. Suppose first that $\varphi_n(x)$ is a generalized regular sequence. Then there exists an entire function $A(x)$, an integer $\nu \geqslant 0$ and very good functions $\Phi_n(x)$ such that

$$A\left(\frac{1}{2\pi i} D\right)\Phi_n(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \overset{\nu}{\rightrightarrows}. \tag{37}$$

The sequence

$$\Psi_n(x) = (1+|x|^2)^{-\nu-q-1}\Phi_n(x) \tag{38}$$

converges in the space L_{R_q} of integrable functions. Therefore the sequence of Fourier transforms $\tilde{\Psi}_n(s)$ converges uniformly in R_q. But from (37) and (38) we deduce that

$$\tilde{\varphi}_n(s) = A(s)\left(1 - \frac{\Delta}{4\pi}\right)^{\nu+q+1} \tilde{\Psi}_n(s),$$

where

$$\Delta = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \ldots + \frac{\partial^2}{\partial s_q^2},$$

and so $\tilde{\varphi}_n(s)$ is a fundamental sequence.

Conversely, if $\tilde{\varphi}_n(s)$ is a fundamental sequence, then there exists an $m \in \mathcal{N}$ and infinitely differentiable functions $\tilde{\Theta}_n(s)$ such that

$$D^m \tilde{\Theta}_n(s) = \tilde{\varphi}_n(s) \tag{39}$$

and $\tilde{\Theta}_n(s)$ converges almost uniformly in R_q. The functions $\tilde{\varphi}_n(s)$ have compact supports, say, contained in the q-spheres $|s| \leqslant r_n$, $n = 1, 2, \ldots$, respectively. We take an infinitely differentiable function $\alpha(s)$, which is 1 for $|s| \leqslant 1$ and vanishes for $|s| > 2$. Multiplying both sides of (39) by $\alpha_n(s) = \alpha(s/r_n)$ and applying to the left-hand side the Leibniz formula we obtain

$$\sum_{k \leqslant m} \binom{m}{k} D^{m-k}\big(\tilde{\Theta}_n(s) D^k \alpha_n(s)\big) = \tilde{\varphi}_n(s), \tag{40}$$

where, for every $k \leqslant m$, the sequence

$$\tilde{\Theta}_{k,n}(s) = \tilde{\Theta}_n(s) D^k \alpha_n(s) \tag{41}$$

converges almost uniformly in R_q. Hence all functions $\tilde{\Theta}_{k,n}(s)$ are uniformly bounded in each bounded subset of R_q. Therefore one can find a real-valued continuous function $f(s)$, which satisfies the inequalities

$$f(s) \geqslant 1 + |\tilde{\Theta}_{k,n}(s)|, \quad k \leqslant m, \ n = 1, 2, \ldots \tag{42}$$

Furthermore, by lemma 1, there exists an entire function $B(s)$ such that

$$|B(s)| \geqslant f(s)(1+|s|^2)^{(q+1)/2}. \tag{43}$$

Now, by virtue of (41), (42) and (43), we can write

$$\tilde{\Theta}_{k,n}(s) = B(s) \tilde{\Phi}_{k,n}(s), \tag{44}$$

where the functions $\tilde{\Phi}_{k,n}(s)$ have compact supports and, for every fixed $k \leqslant m$, $\tilde{\Phi}_{k,n}(s)$ converges in L_{R_q} as $n \to \infty$. Hence every sequence

$$\Phi_{k,n}(x) = \mathcal{F}^{-1}\{\tilde{\Phi}_{k,n}(s)\}$$

consists of very good functions and converges uniformly in R_q, i. e. it is a generalized regular sequence.

Finally, from (40), (41) and (44) it follows that

$$\sum_{k \leqslant m} \binom{m}{k} (-2\pi i x)^{m-k} B\left(\frac{1}{2\pi i} D\right) \Phi_{k,n}(x) = \varphi_n(x),$$

and thus $\varphi_n(x)$ is also a generalized regular sequence, q. e. d.

Two generalized regular sequences $\varphi_n(x)$ and $\psi_n(x)$ are equivalent, iff the corresponding fundamental sequences $\tilde{\varphi}_n(s)$ and $\tilde{\psi}_n(s)$ are equivalent, and so determine the same distribution of finite order. In fact, by proposition 6, $\varphi_1(x), \psi_1(x), \varphi_2(x), \psi_2(x), \ldots$ is a generalized regular sequence, iff the sequence $\tilde{\varphi}_1(s), \tilde{\psi}_1(s), \tilde{\varphi}_2(s), \tilde{\psi}_2(s), \ldots$ is fundamental.

We have thus proved the consistency of the following

Definition 4. The Fourier transform $\tilde{\varphi}(s)$ of the ultra-distribution $\varphi(x) = [\varphi_n(x)]$ is a *distribution of finite order* determined by the fundamental sequence $\tilde{\varphi}_n(s) = \mathcal{F}\{\varphi_n(x)\}$.

By proposition 6, each ultra-distribution has a distribution of finite order as its Fourier transform. Conversely, since each distribution of finite order can be represented by a fundamental sequence of infinitely differentiable functions with compact supports, it is a Fourier transform of an ultra-distribution.

The Fourier transformation just defined is obviously a linear operation, and so we can formulate

Proposition 7. *The Fourier transformation establishes an isomorphism between ultra-distributions and distributions of finite order.*

If the sequence $\varphi_n(x)$ is regular, then $\tilde{\varphi}_n(s)$ is also a regular sequence as one can easily see from the proof of proposition 6. Therefore Fourier transforms of tempered distributions are again tempered distributions. In this case our Fourier transforms coincide with the Fourier transforms defined earlier for tempered distributions (see [5] and [10]). In particular, for integrable functions our Fourier transforms coincide with the usual Fourier transforms. We shall write

$$\tilde{\varphi}(s) = \mathcal{F}\{\varphi(x)\} \quad \text{and} \quad \varphi(x) = \mathcal{F}^{-1}\{\tilde{\varphi}(s)\}$$

for an arbitrary ultra-distribution $\varphi(x)$.

Directly from definition 4 we get the following properties of the Fourier transform. Let $\varphi(x)$ be an ultra-distribution and $\tilde{\varphi}(s) = \mathcal{F}\{\varphi(x)\}$. If $F(x)$ is an entire function, $P(x)$ a polynomial and h a q-tuple of complex numbers, then

$$\mathcal{F}\left\{F\left(\frac{1}{2\pi i} D\right)\varphi(x)\right\} = F(s)\varphi(s), \tag{45}$$

$$\mathcal{F}\{P(x)\varphi(x)\} = P\left(\frac{-1}{2\pi i} D\right)\tilde{\varphi}(s), \tag{46}$$

$$\mathcal{F}\{\varphi(x-h)\} = e^{-2\pi i h s}\tilde{\varphi}(s). \tag{47}$$

If $\varphi(x) = [\varphi_n(x)]$ and $\psi(x)$ is a very good function, then the sequence

$$I_n = \int_{-\infty}^{\infty} \varphi_n(x)\psi(x)\,dx$$

converges and its limit I is independent of the generalized regular sequence representing $\varphi(x)$. We call I the scalar product of $\varphi(x)$ with $\psi(x)$ and we write

$$I=\int_{-\infty}^{\infty}\varphi(x)\psi(x)\,dx.$$

The scalar product of the distribution $\tilde{\varphi}(s)$ with $\tilde{\psi}(-s)$ can be defined in the same way and we have the Parseval theorem

$$\int_{-\infty}^{\infty}\varphi(x)\psi(x)\,dx=\int_{-\infty}^{\infty}\tilde{\varphi}(s)\tilde{\psi}(-s)\,ds.$$

Remark 1. From what we have said above it follows that there exists a one-to-one correspondence between ultra-distributions and functionals of the space D'_F introduced in [2]. In this correspondence 1° the subset of all tempered distributions remains invariant, 2° derivatives of ultra-distributions are transformed into derivatives of elements of D'_F. Thus, by proposition 5, each element of D'_F is an infinite derivative of a continuous slowly increasing function.

Remark 2. If the ultra-distribution $\varphi(x)$ is of the form

$$\varphi(x)=A\left(\frac{1}{2\pi i}D\right)\Phi(x), \tag{48}$$

where $A(x)$ is an entire function of exponential type $\leqslant b$ and $\Phi(x)$ a continuous function of polynomial growth, then there exists a continuous function $\tilde{\Psi}(s)$ of exponential growth and type $\leqslant b$ such that

$$\tilde{\varphi}(s)=D^k\tilde{\Psi}(s) \tag{49}$$

for some $k\in\mathcal{N}$ ([4]). Conversely, if $\tilde{\varphi}(s)$ has the stated property, then one can take in (48) as $A(x)$ an entire function of exponential type $\leqslant b$.

We prove now

Lemma 5. *Let $\varphi(x)$ and $\psi(x)$ be ultra-distributions and $F(x)$ an entire function. If $F(x)\neq 0$, then*

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x)=F\left(\frac{1}{2\pi i}D\right)\psi(x)$$

implies $\varphi(x)=\psi(x)$.

Proof. On account of (45) we have

$$F(s)\tilde{\varphi}(s)=F(s)\tilde{\psi}(s),$$

and since $F(s)\neq 0$, from the last equation it follows that $\tilde{\varphi}(s)=\tilde{\psi}(s)$. Consequently, $\varphi(x)=\psi(x)$, q. e. d.

([4]) A continuous function is of exponential growth and type $\leqslant b$, if it satisfies on R_q a restriction analogous to that for entire functions of exponential type $\leqslant b$.

If $F(x)\neq 0$ outside the interval $-a\leqslant x\leqslant a$, then

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x)=F\left(\frac{1}{2\pi i}D\right)\psi(x)$$

implies that $\varphi(x)$ and $\psi(x)$ differ by an entire function of exponential type $\leqslant 2\pi a$. In fact, $\tilde{\varphi}(s)$ and $\tilde{\psi}(s)$ differ by a distribution, whose support is contained in $-a\leqslant s\leqslant a$. Its Fourier transform is an entire function of exponential type $\leqslant 2\pi a$ (see [10], vol. 2, p. 128).

§ 9. Limits of ultra-distributions.

Definition 5. We say that a sequence of ultra-distributions $\varphi_n(x)$ *converges to the ultra-distribution* $\varphi(x)$ and we write

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x), \tag{50}$$

if there exists an entire function $A(x)$, an integer ν and continuous functions $\Phi_n(x)$, $\Phi(x)$ such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x)=\varphi_n(x),\qquad A\left(\frac{1}{2\pi i}D\right)\Phi(x)=\varphi(x), \tag{L$_1$}$$

$$\Phi_n(x)\overset{\nu}{\rightrightarrows}\Phi(x). \tag{L$_2$}$$

Remark. From (L$_2$) it follows that $\Phi(x)=O(|x|^{2\nu})$ as $|x|\to\infty$, and so the limit $\varphi(x)$, if it exists, is an ultra-distribution.

The integer ν and the entire function $A(x)$ may be replaced similarly as in definition 1. In particular, if $\varphi_n(x)$ and $\psi_n(x)$ are two sequences of ultra-distributions, then there exists an integer ν and an entire function $A(x)$ without real zeros, which are good for both sequences.

The limit (50) is unique. For, if

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x)\quad\text{ and }\quad\lim_{n\to\infty}\varphi_n(x)=\psi(x),$$

then there exists an entire function $A(x)$ without real zeros, an integer ν, and continuous functions $\Phi_n(x)$, $\Phi(x)$, $\Psi_n(x)$, $\Psi(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x)=A\left(\frac{1}{2\pi i}D\right)\Psi_n(x)=\varphi_n(x), \tag{51}$$

$$A\left(\frac{1}{2\pi i}D\right)\Phi(x)=\varphi(x),\qquad A\left(\frac{1}{2\pi i}D\right)\Psi(x)=\psi(x), \tag{52}$$

$$\Phi_n(x)\overset{\nu}{\rightrightarrows}\Phi(x)\quad\text{ and }\quad\Psi_n(x)\overset{\nu}{\rightrightarrows}\Psi(x). \tag{53}$$

By lemma 5, equation (51) implies that $\Phi_n(x)\equiv\Psi_n(x)$. Hence, by the convergence (53), $\Phi(x)\equiv\Psi(x)$ and finally $\varphi(x)=\psi(x)$, on account of (52).

If $\Phi_n(x)$ is a sequence of continuous functions, which satisfy condition (L_2), then evidently $\lim\limits_{n\to\infty} \Phi_n(x) = \Phi(x)$.

Also, if $\varphi_n(x)$ are tempered distributions, which satisfy the conditions (L_1) and (L_2) with a polynomial as the entire function $A(x)$, then the limit $\varphi(x)$ is a tempered distribution.

PROPOSITION 8. *If* $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ *and* $\lim\limits_{n\to\infty} \psi_n(x) = \psi(x)$, *then*

$$\lim_{n\to\infty} \{\varphi_n(x) \pm \psi_n(x)\} = \lim_{n\to\infty} \varphi_n(x) \pm \lim_{n\to\infty} \psi_n(x),$$

$$\lim_{n\to\infty} \varphi_n(x-h) = \varphi(x-h),$$

$$\lim_{n\to\infty} F\left(\frac{1}{2\pi i} D\right) \varphi_n(x) = F\left(\frac{1}{2\pi i} D\right) \varphi(x),$$

$$\lim_{n\to\infty} P(x)\varphi_n(x) = P(x)\varphi(x),$$

where h is a q-tuple of complex numbers, $F(x)$ an entire function and $P(x)$ a polynomial.

Proof. The proof of proposition 8 is much the same as the proof of consistency of definition 3. In case of the translation we have to use a representation of the form

$$\varphi_n(x) = A\left(\frac{1}{2\pi i} D\right) \Phi_n(x), \qquad \varphi(x) = A\left(\frac{1}{2\pi i} D\right) \Phi(x),$$

where $\Phi_n(x)$ and $\Phi(x)$ are entire functions. This is possible by the remark following proposition 5.

PROPOSITION 9. *A sequence of ultra-distributions $\varphi_n(x)$ converges to $\varphi(x)$, iff the sequence of Fourier transforms $\tilde{\varphi}_n(s) = \mathscr{F}\{\varphi_n(x)\}$ converges distributionally to $\tilde{\varphi}(s) = \mathscr{F}\{\varphi(x)\}$.*

Proof. In order to show that the convergence of $\varphi_n(x)$ to $\varphi(x)$ implies the distributional convergence of $\tilde{\varphi}_n(s)$ to $\tilde{\varphi}(s)$ we may proceed in the same way as in the first part of the proof of proposition 6. The proof of the converse implication is now simpler than in the case of proposition 6, because we need not to deal with very good functions. In fact, if $\tilde{\varphi}_n(s)$ converges distributionally to $\tilde{\varphi}(s)$, then

$$\tilde{\varphi}_n(s) = D^m \tilde{\Theta}_n(s), \qquad \tilde{\varphi}(s) = D^m \tilde{\Theta}(s), \tag{54}$$

where $m \in \mathscr{N}$ and $\tilde{\Theta}_n(s)$ is a sequence of continuous functions converging to $\tilde{\Theta}(s)$ almost uniformly in R_q. The functions $\tilde{\Theta}_n(s)$ being uniformly bounded in every bounded subset of R_q, there exists a continuous function $f(s)$, which satisfies the inequalities

$$f(s) > 1 + |\tilde{\Theta}_n(s)|, \qquad n = 1, 2, \ldots$$

Making use of lemma 1 one can now find an entire function $B(s)$ such that

$$|B(s)| \geqslant f(s)(1+|s|^2)^{(q+1)/2}.$$

Thus we may write

$$\tilde{\Theta}_n(s) = B(s)\tilde{\Phi}_n(s) \quad \text{and} \quad \tilde{\Theta}(s) = B(s)\tilde{\Phi}(s), \tag{55}$$

where the sequence $\tilde{\Phi}_n(s)$ consists of integrable functions and converges to $\tilde{\Phi}(s)$ in L_{R_q}.

On account of (54) and (55), the ultra-distributions $\varphi_n(x)$ and $\varphi(x)$ can be expressed as follows

$$\varphi_n(x) = (-2\pi i x)^m B\left(\frac{1}{2\pi i} D\right) \Phi_n(x), \qquad \varphi(x) = (-2\pi i x)^m B\left(\frac{1}{2\pi i} D\right) \Phi(x),$$

where $\Phi_n(x)$ are continuous functions bounded on R_q and $\Phi_n(x) \overset{0}{\rightrightarrows} \Phi(x)$. Hence, by virtue of proposition 8, $\varphi_n(x)$ converges to $\varphi(x)$ in the sense of definition 5.

For sequences of very good functions we have

PROPOSITION 10. *A sequence $\varphi_n(x)$ of very good functions converges to an ultra-distribution $\varphi(x)$, iff it is a generalized regular sequence of $\varphi(x)$, i. e. $\varphi(x) = [\varphi_n(x)]$.*

Proof. If $\varphi_n(x)$ converges to $\varphi(x)$, then there exists an entire function $A(x)$, an integer ν and continuous functions $\Phi_n(x)$ such that

$$A\left(\frac{1}{2\pi i} D\right) \Phi_n(x) = \varphi_n(x), \qquad A\left(\frac{1}{2\pi i} D\right) \Phi(x) = \varphi(x) \tag{56}$$

and

$$\Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x).$$

We may assume that $A(x) \neq 0$. Then, applying the Fourier transformation to the first equation in (56), one can easily see that $\Phi_n(x)$ are very good functions, and so $\varphi_n(x)$ is a generalized regular sequence of $\varphi(x)$. The converse implication is evident.

For a sequence of numbers (constant functions), the convergence in the sense of definition 5 coincides with the usual convergence. This follows from proposition 9, and the fact that the sequence $a_n\delta(s)$, where a_n are numbers and $\delta(s)$ is Dirac's δ-distribution, converges distributionally, iff the sequence of numbers a_n converges in the usual sense.

We also remark that, for any very good function $\psi(x)$,

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$$

implies

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x)\psi(x)\,dx = \int_{-\infty}^{\infty} \varphi(x)\psi(x)\,dx.$$

The proof can be obtained from definition 5 by integration by parts.

A series of ultra-distributions $\sum_{n=1}^{\infty}\varphi_n(x)$ is convergent, if the sequence of partial sums $\psi_n(x)=\varphi_1(x)+\varphi_2(x)+\ldots+\varphi_n(x)$ converges. We say that

$$\varphi(x)=\lim_{n\to\infty}\psi_n(x)$$

is the *sum* of the series concerned and we write

$$\sum_{n=1}^{\infty}\varphi_n(x)=\varphi(x).$$

The results of proposition 8 and proposition 9 hold also for series of ultra-distributions. For example, one may apply infinite differentiation term by term, i. e.

$$\sum_{n=1}^{\infty}\varphi_n(x)=\varphi(x)$$

implies

$$\sum_{n=1}^{\infty}F\left(\frac{1}{2\pi i}D\right)\varphi_n(x)=F\left(\frac{1}{2\pi i}D\right)\varphi(x)$$

for any entire function $F(x)$. Similarly, one may take the Fourier transform term by term, but in this case the new series consists of distributions of finite order (which need not to be ultra-distributions) and converges in the sense of distributions of finite order.

On the other hand, the infinite derivative may now be regarded as the sum of a series. This shows

PROPOSITION 11. *If $\varphi(x)$ is an ultra-distribution and $F(x)=\sum_k c_kx^k$ an entire function, then*

$$\sum_k\frac{c_k}{(2\pi i)^{\sigma_k}}D^k\varphi(x)=F\left(\frac{1}{2\pi i}D\right)\varphi(x),$$

or, in other words, the series on the left-hand side converges to the infinite derivative.

Proof. By definition 4, $\tilde{\varphi}(s)=\mathscr{F}\{\varphi(x)\}$ is a distribution of finite order. An application of equation (45) gives

$$F(s)\tilde{\varphi}(s)=\mathscr{F}\left\{F\left(\frac{1}{2\pi i}D\right)\varphi(x)\right\}$$

and for the partial sums $S_m(s)=\sum_{\sigma_k\leqslant m}c_ks^k$,

$$S_m(s)\tilde{\varphi}(s)=\mathscr{F}\left\{S_m\left(\frac{1}{2\pi i}D\right)\varphi(x)\right\}.$$

Furthermore, $S_m(s)\tilde{\varphi}(s)$ converges distributionally to $F(s)\tilde{\varphi}(s)$, whence the sequence of partial sums

$$S_m\left(\frac{1}{2\pi i}D\right)\varphi(x)$$

converges to

$$F\left(\frac{1}{2\pi i}D\right)\varphi(x),$$

on account of proposition 9, q. e. d.

Also Maclaurin's formula

$$\varphi(x+h)=\sum_k\frac{h^k}{k!}D^k\varphi(x)$$

is satisfied for every ultra-distribution $\varphi(x)$.

There is no difficulty in extending the definition of the limit to the case, where the ultra-distribution depends on a continuous complex parameter or on a point of R_q. For example, in the following definition λ is a continuous complex parameter.

Definition 5′. We say that $\varphi_\lambda(x)$ *converges to* $\varphi(x)$ *as* $\lambda\to\lambda_0$ and we write

$$\lim_{\lambda\to\lambda_0}\varphi_\lambda(x)=\varphi(x) \tag{57}$$

if there exists an entire function $A(x)$, an integer ν and continuous functions $\Phi_\lambda(x)$, $\Phi(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_\lambda(x)=\varphi_\lambda(x),\qquad A\left(\frac{1}{2\pi i}D\right)\Phi(x)=\varphi(x), \tag{L_1'}$$

$$\Phi_\lambda(x)\overset{\nu}{\rightrightarrows}\Phi(x)\quad\text{as}\quad\lambda\to\lambda_0\ (^5). \tag{L_2'}$$

All properties of the limit (50) stated above remain valid for the limit (57). The proofs are analogous.

The partial derivatives of an ultra-distribution $\varphi(x)$ may now be obtained similarly as for holomorphic functions

$$D^{e_j}\varphi(x)=\lim_{\lambda\to 0}\frac{\varphi(x+\lambda e_j)-\varphi(x)}{\lambda},\qquad j=1,2,\ldots,q.$$

§ 10. Multiplication of ultra-distributions by functions. We extend now definition 3 so as to obtain a product of ultra-distributions with functions of a wider class. Each function $\omega(x)$ of the class in question

(5) The symbol $\overset{\nu}{\rightrightarrows}$ in (L_2') means that $(1+|x|^2)^{-\nu}\Phi_\lambda(x)$ is bounded for every λ and converges, for $\lambda\to\lambda_0$, to $(1+|x|^2)^{-\nu}\Phi(x)$ uniformly in R_q.

must be such that for each very good function $\vartheta(x)$, the product $\omega(x)\vartheta(x)$ is again a very good function. This condition is satisfied, if $\omega(x)$ is an entire function of exponential type, slowly increasing on R_q. We show that, in fact, entire functions of exponential type, slowly increasing on R_q, form the desired class of multiplicators. In what follows we call them *fairly very good functions* ([6]).

A fairly very good function $\omega(x)$ is obviously an ultra-distribution; it is determined by the generalized regular sequence

$$\omega_n(x) = \vartheta_0\left(\frac{x}{n}\right)\omega(x), \tag{58}$$

where $\vartheta_0(x)$ is a very good function such that $\vartheta_0(0)=1$. If $\omega(x)$ is of exponential type $\leqslant b_1$ and $\vartheta_0(x)$ of exponential type $\leqslant b_2$ then the n-th function $\omega_n(x)$ is of exponential type $\leqslant b_1+b_2/n$. Therefore its Fourier transform $\tilde{\omega}_n(s)$ is an infinitely differentiable function, whose support is contained in the interval

$$-\frac{1}{2\pi}\left(b_1+\frac{b_2}{n}\right)\leqslant x\leqslant\frac{1}{2\pi}\left(b_1+\frac{b_2}{n}\right).$$

It follows that the distribution $\tilde{\omega}(s)=\mathscr{F}\{\omega(x)\}$, which is determined by the fundamental sequence $\tilde{\omega}_n(s)$, has a compact support contained in $-b_1/2\pi\leqslant x\leqslant b_1/2\pi$. Also the converse is true: If the distribution $\tilde{\omega}(s)$ has a compact support contained in $-b_1/2\pi\leqslant x\leqslant b_1/2\pi$, then $\omega(x)$ is a fairly very good function of exponential type $\leqslant b_1$ (see [10], vol. 2, p. 128).

Definition 6. We define the product of an ultra-distribution $\varphi(x) = [\varphi_n(x)]$ and a fairly very good function $\omega(x)$ by equation

$$\omega(x)\varphi(x) = [\omega(x)\varphi_n(x)]. \tag{59}$$

Proof of consistency. First we observe that, for every n,

$$\mathscr{F}\{\omega(x)\varphi_n(x)\} = \tilde{\omega}(s)*\tilde{\varphi}_n(s), \tag{60}$$

where, as usually, $\tilde{\omega}(s)=\mathscr{F}\{\omega(x)\}$, $\tilde{\varphi}_n(s)=\mathscr{F}\{\varphi_n(x)\}$ and the convolution on the right-hand side is well defined, since both $\tilde{\omega}(s)$ and $\tilde{\varphi}_n(s)$ have compact supports. Equation (60) involves only tempered distributions. It is a special case of a more general correspondence (see [10], vol. 2, p. 124, theorem XV). Now, on account of proposition 6, $\tilde{\varphi}_n(s)$ is a fundamental sequence. Hence, by the properties of the convolution, $\tilde{\omega}(s)*\tilde{\varphi}_n(s)$ is also fundamental. Applying again proposition 6 we conclude that $\omega(x)\varphi_n(x)$ is a generalized regular sequence.

([6]) Compare with fairly good functions defined in [5].

It remains to show that the product (60) does not depend on the choice of the sequence representing $\varphi(x)$. But this is a consequence of the fact already proved, similarly as in previous proofs of consistency.

It is easy to verify that the product (60) has the usual properties

$$\omega_1(x)\big(\omega_2(x)\varphi(x)\big) = \big(\omega_1(x)\omega_2(x)\big)\varphi(x),$$
$$\big(\omega_1(x)+\omega_2(x)\big)\varphi(x) = \omega_1(x)\varphi(x)+\omega_2(x)\varphi(x),$$
$$\omega(x)\big(\varphi(x)+\psi(x)\big) = \omega(x)\varphi(x)+\omega(x)\psi(x),$$
$$D^{e_j}\big(\omega(x)\varphi(x)\big) = D^{e_j}\omega(x)\varphi(x)+\omega(x)D^{e_j}\varphi(x).$$

If $\varphi(x)$ is a tempered distribution, then the above product coincides with the product defined in the theory of distributions. In particular, if $\varphi(x)$ is a continuous function of polynomial growth, it is the ordinary product of functions.

PROPOSITION 12. *If $\varphi(x)$ is an ultra-distribution, $\omega(x)$ a fairly very good function and $\tilde{\varphi}(s)$, $\tilde{\omega}(s)$ their Fourier transforms respectively, then*

$$\mathscr{F}\{\omega(x)\varphi(x)\} = \tilde{\omega}(s)*\tilde{\varphi}(s). \tag{61}$$

Proof. The convolution on the right-hand side is well defined because $\tilde{\omega}(s)$ is a distribution with compact support. Equation (61) follows from equation (60), proposition 6, and the fact that the convolution is a separately continuous operation.

If $\varphi(x)$ is a very good function, then from (61) we also get

$$\omega(x)*\varphi(x) = \mathscr{F}^{-1}\{\tilde{\omega}(s)\tilde{\varphi}(s)\},$$

where the product in curly brackets is well defined in the space of distribution. In particular, if $\tilde{\varphi}(s)$ is equal to 1 on the support of $\tilde{\omega}(s)$, then

$$\omega(x)*\varphi(x) = \omega(x).$$

From equation (61) and proposition 9 we obtain easily

PROPOSITION 13. *If $\varphi_n(x)$ is a sequence of ultra-distributions and*

$$\lim_{n\to\infty}\varphi_n(x) = \varphi(x),$$

then

$$\lim_{n\to\infty}\omega(x)\varphi_n(x) = \omega(x)\varphi(x)$$

for any fairly very good function $\omega(x)$.

The corresponding statement for series is evident.

Remark. Let $\varphi_n(\xi)$ and $\psi_n(\eta)$, $n=1,2,\ldots$, be very good functions of the variables $\xi=(x_1,\ldots,x_p)$, $\eta=(x_{p+1},\ldots,x_q)$. If $\varphi_n(\xi)$ and $\psi_n(\eta)$

are generalized regular sequences, then $\varphi_n(\xi)\psi_n(\eta)$ is also a generalized regular sequence. Also $\varphi_n(\xi)\sim\varphi_n^*(\xi)$ and $\psi_n(\eta)\sim\psi_n^*(\eta)$ implies $\varphi_n(\xi)\psi_n(\eta)\sim\varphi_n^*(\xi)\psi_n^*(\eta)$. Thus the product of two ultra-distributions $\varphi(\xi)=[\varphi_n(\xi)]$ and $\psi(\eta)=[\psi_n(\eta)]$ with separated variables may be defined in the usual way: $\varphi(\xi)\psi(\eta)=[\varphi_n(\xi)\psi_n(\eta)]$.

§ 11. Rapidly decreasing ultra-distributions. Each ultra-distribution is "tempered", being a derivative (of finite or infinite order) of a continuous slowly increasing function. We deduce from proposition 5 another property of ultra-distributions, which shows their "slow increase":

The translations $\varphi(x+h)$ of any ultra-distribution $\varphi(x)$ by points $h\in R_q$ are such that

$$\lim_{|h|\to\infty}\frac{\varphi(x+h)}{|h|^\mu}=0 \tag{62}$$

for some integer μ (depending on $\varphi(x)$).

In fact, suppose that

$$\varphi(x)=A\left(\frac{1}{2\pi i}D\right)\Phi(x),$$

where $\Phi(x)$ is a continuous function, bounded on dividing by $(1+|x|^2)^\nu$. Then

$$\varphi(x+h)=A\left(\frac{1}{2\pi i}D\right)\Phi(x+h)$$

and from the rate of increase of $\Phi(x)$ as $|x|\to\infty$ we infer that

$$\frac{\Phi(x+h)}{|h|^{2\nu+1}}\overset{\nu}{\rightrightarrows}0\quad\text{as}\quad|h|\to\infty.$$

This proves the convergence (62) with $\mu=2\nu+1$.

In the present section we deal with ultra-distributions, for which the μ in (62) may be zero or even an arbitrary negative integer.

Definition 7. We say that an ultra-distribution $\varphi(x)$ *vanishes at infinity*, if it is determined by a generalized regular sequence $\varphi_n(x)$, which satisfies conditions (GR_1) and (GR_2) with $\nu=0$.

Ultra-distributions vanishing at infinity can be characterized as infinite derivatives of continuous functions, which tend to zero as $|x|\to\infty$.

Proposition 14. *An ultra-distribution $\varphi(x)$ vanishes at infinity, iff it satisfies one af the following conditions:*

(V_1) *$\varphi(x)$ is an infinite derivative of a continuous function $\Phi(x)$, whose ordinary limit, as $|x|\to\infty$, is zero;*

(V_2) $$\lim_{|h|\to\infty}\varphi(x+h)=0.$$

Proof. Let $\varphi_n(x)$ be a generalized regular sequence of $\varphi(x)$, which satisfies (GR_1) and (GR_2) with $\nu=0$, i. e. there exists an entire function $A(x)$ and very good functions $\Phi_n(x)$ such that

$$\varphi_n(x)=A\left(\frac{1}{2\pi i}D\right)\Phi_n(x)\quad\text{and}\quad\Phi_n(x)\overset{0}{\rightrightarrows}\Phi(x).$$

Then

$$\varphi(x)=A\left(\frac{1}{2\pi i}D\right)\Phi(x)$$

and $\Phi(x)$ is a continuous function, whose ordinary limit, as $|x|\to\infty$, is zero. Thus condition (V_1) is satisfied. Also $\Phi(x+h)\overset{1}{\rightrightarrows}0$, and so (V_1) implies (V_2). It remains to show that condition (V_2) is sufficient for $\varphi(x)$ to be vanishing at infinity. Now, if (V_2) is satisfied, then there exists an entire function $B(x)$, continuous functions $\Phi_h(x)$, $h\in R_q$, and another continuous function $\Phi(x)$, such that

$$B\left(\frac{1}{2\pi i}D\right)\Phi_h(x)=\varphi(x+h),\qquad B\left(\frac{1}{2\pi i}D\right)\Phi(x)=0 \tag{63}$$

and for some integer ν,

$$\Phi_h(x)\overset{\nu}{\rightrightarrows}\Phi(x)\quad\text{as}\quad|h|\to\infty. \tag{64}$$

We may assume that $B(x)$ has no real zeros. Then, by lemma 5, equation (63) implies that $\Phi_h(x)=\Phi_0(x+h)$ and also $\Phi(x)\equiv 0$. Hence, substituting $x=0$ in (64), we infer that $\Phi_0(h)$ tends to zero as $|h|\to\infty$. A generalized regular sequence $\varphi_n(x)$ of $\varphi(x)$, which satisfies the condition of definition 7, is given by equation

$$\varphi_n(x)=B\left(\frac{1}{2\pi i}D\right)\left\{\vartheta_0\left(\frac{x}{n}\right)\int_{-\infty}^{\infty}n\vartheta_1(nx-nt)\Phi_0(t)\,dt\right\},$$

where $\vartheta_0(x)$ is a very good function defined as in (58) and $\vartheta_1(x)$ another very good function, such that $\int_{-\infty}^{\infty}\vartheta_1(x)\,dx=1$, q. e. d.

If $\varphi(x)$ is a distribution, which vanishes at infinity in the sense stated in [10] (see vol. 2, p. 61), i. e. $\varphi(x+h)$ converges distributionally to zero as $|h|\to\infty$, then it is an ultra-distribution vanishing at infinity in the sense of definition 7.

A fairly very good function $\varphi(x)$ vanishes at infinity, iff its ordinary limit, as $|x|\to\infty$, exists and equals zero.

In fact, if $\varphi(x)$ vanishes at infinity, then it is an infinite derivative of a fairly very good function $\Phi(x)$, which tends to zero as $|x|\to\infty$. We represent now $\Phi(x)$ in the form

$$\Phi(x)=\int_{-\infty}^{\infty}\vartheta(x-t)\Phi(t)\,dt,$$

where $\vartheta(x)$ is a very good function. Then each infinite derivative of $\Phi(x)$ is a convolution of $\Phi(x)$ with a very good function, and therefore tends to zero as $|x| \to \infty$. The converse implication is evident.

If a sequence of ultra-distributions $\varphi_n(x)$ vanishing at infinity satisfies the conditions of definition 5 with $\nu = 0$, then its limit $\varphi(x)$ also vanishes at infinity. In what follows we sometimes write

$$\lim_{\substack{n\to\infty \\ \nu=\mu}} \varphi_n(x) = \varphi(x)$$

to indicate that in definition 5, for suitably chosen $A(x)$, $\Phi_n(x)$ and $\Phi(x)$, one can take $\nu = \mu$. In particular, if $\mu = 0$, we have the proper convergence for ultra-distributions vanishing at infinity.

Definition 8. An ultra-distribution $\varphi(x)$ is said to be *rapidly decreasing*, if, for every $k \in \mathcal{N}$, $x^k \varphi(x)$ vanishes at infinity.

Similarly as before we have

Proposition 15. *An ultra-distribution $\varphi(x)$ is rapidly decreasing, iff it satisfies one of the following conditions:*

(RD_1) *For every integer $\mu \geqslant 0$ there exists a generalized regular sequence $\varphi_n(x)$ of $\varphi(x)$ satisfying conditions (GR_1) and (GR_2) with $\nu = -\mu$.*

(RD_2) *Given an arbitrary integer $\mu \geqslant 0$, $\varphi(x)$ can be represented as an infinite derivative of a continuous function $\Phi(x)$, such that $(1+|x|^2)^\mu \Phi(x)$ tends to zero as $|x| \to \infty$.*

(RD_3) *For every integer $\mu \geqslant 0$,*

$$\lim_{|h|\to\infty} (1+|h|^2)^\mu \varphi(x+h) = 0.$$

Proof. Suppose first that the ultra-distribution $\varphi(x)$ is rapidly decreasing. Then, for every $k \in \mathcal{N}$,

$$\lim_{|h|\to\infty} (x+h)^k \varphi(x+h) = 0, \tag{65}$$

by definition 8 and proposition 14. In (65) we substitute $k = 0$ and $k = e_j$, $j = 1, 2, \ldots, q$, and apply proposition 13. It follows that

$$\lim_{|h|\to\infty} h_j \varphi(x+h) = 0, \quad j = 1, 2, \ldots, q,$$

and by induction,

$$\lim_{|h|\to\infty} h_j^\mu \varphi(x+h) = 0,$$

for every integer $\mu \geqslant 0$. This proves condition (RD_3).

Next, if (RD_3) is satisfied, then there exists an entire function $A(x)$, an integer ν and a continuous function $\Phi_h(x)$ depending on h, such that

$$A\left(\frac{1}{2\pi i} D\right) \Phi_h(x) = \varphi(x+h) \tag{66}$$

$$(1+|h|^2)^\mu \Phi_h(x) \overset{\nu}{\rightrightarrows} 0. \tag{67}$$

We may assume that $A(x)$ has no real zeros. Then, by lemma 5, equation (66) implies that $\Phi_h(x) = \Phi_0(x+h)$. Hence, taking into account the convergence (67), we infer that $(1+|h|^2)^\mu \Phi_0(h)$ tends to zero as $|h| \to \infty$. Thus (RD_3) implies (RD_2). Also

$$\varphi_n(x) = A\left(\frac{1}{2\pi i} D\right)\left\{\vartheta_0\left(\frac{x}{n}\right) \int_{-\infty}^{\infty} n\vartheta_1(nx-nt)\Phi_0(t)\right\}dt,$$

where $\vartheta_0(x)$ and $\vartheta_1(x)$ are very good functions defined as in the proof of proposition 14, is a generalized regular sequence, which satisfies conditions (GR_1) and (GR_2) with $\nu = -\mu$.

It remains to show that, under assumption of (RD_1), $\varphi(x)$ is rapidly decreasing. But for this reason it is sufficient to observe that for any generalized regular sequence $\varphi_n(x)$, which satisfies (GR_1) and (GR_2) with $\nu = -\mu \leqslant 0$, $x^{2\mu p_1} \varphi_n(x)$ satisfies the same conditions with $\nu = 0$. This completes the proof of proposition 15.

If $\varphi(x)$ is a distribution, rapidly decreasing in the sense of Schwartz, then it is a rapidly decreasing ultra-distribution.

A fairly very good function is rapidly decreasing, iff it is a very good function. This is a consequence of the fact stated above that a fairly very good function vanishing at infinity has its ordinary limit zero, as $|x| \to \infty$.

Let now $\varphi_n(x)$ be a sequence of rapidly decreasing ultra-distributions. If, for every $k \in \mathcal{N}$,

$$\lim_{\substack{n\to\infty \\ \nu=0}} x^k \varphi_n(x) = x^k \varphi(x), \tag{68}$$

then $\varphi(x)$ is also rapidly decreasing. In this case we have

$$\lim_{\substack{n\to\infty \\ \nu=-\mu}} \varphi_n(x) = \varphi(x) \tag{69}$$

for every integer $\mu \geqslant 0$. The proof is similar to the proof of condition (RD_2) in proposition 15.

§ 12. The convolution of ultra-distributions.

We prove

PROPOSITION 16. *Let $\varphi(x)$ be an ultra-distribution and $\psi(x)$ a very good function. If $\varphi_n(x)$ is a generalized regular sequence of $\varphi(x)$, then for some integer ν,*

$$\varphi_n(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi_n(x-t)\psi(t)\,dt \overset{\nu}{\rightrightarrows} \chi(x) \tag{70}$$

and $\chi(x)$ is a fairly very good function. Moreover, if $\varphi(x)$ is rapidly decreasing, then $\chi(x)$ is a very good function.

Proof. By assumption there exists an entire function $A(x)$, an integer ν and very good functions $\Phi_n(x)$ such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x).$$

Hence we get

$$\varphi_n(x) * \psi(x) = \Phi_n(x) * A\left(\frac{1}{2\pi i}D\right)\psi(x) \overset{\nu}{\rightrightarrows} \Phi(x) * A\left(\frac{1}{2\pi i}D\right)\psi(x), \tag{71}$$

where the convergence follows similarly as in (15). Since

$$A\left(\frac{1}{2\pi i}D\right)\psi(x)$$

is a very good function, the limit in (71) is a fairly very good function.

If $\varphi(x)$ is rapidly decreasing, then, for any given integer μ, we can choose the entire function $A(x)$ so that $(1+|x|^2)^\mu\Phi(x)$ tends to zero as $|x| \to \infty$. Consequently $\chi(x)$ is rapidly decreasing (on R_q), and thus a very good function, q. e. d.

From (71) one can easily see that $\chi(x)$ does not depend on the sequence $\varphi_n(x)$ determining $\varphi(x)$; all sequences of $\varphi(x)$ give the same limit.

Let now $\varphi(x) = [\varphi_n(x)]$ and $\psi(x) = [\psi_n(x)]$ be ultra-distributions and assume that at least one of them, say $\varphi(x)$, is rapidly decreasing. Then, for every fixed m,

$$\varphi_n(x) * \psi_m(x) \overset{\nu}{\rightrightarrows} \chi_m(x) \tag{72}$$

as $n \to \infty$, where ν is some integer. We shall show that, in fact, one can find a ν, which is good for all m.

Definition 9. The convolution of $\varphi(x)$ and $\psi(x)$ is defined by the sequence $\chi_n(x)$ given in (72), i. e.

$$\varphi(x) * \psi(x) = [\chi_n(x)]. \tag{73}$$

Proof of consistency. By proposition 16, $\chi_n(x)$, $n = 1, 2, \ldots$, are very good functions. We assert that they form a generalized regular sequence. For, there exists an entire function $A(x)$, an integer $\nu \geqslant 0$, and very good functions $\Phi_n(x)$, $\Psi_n(x)$, such that

$$A\left(\frac{1}{2\pi i}D\right)\Phi_n(x) = \varphi_n(x), \quad A\left(\frac{1}{2\pi i}D\right)\Psi_n(x) = \psi_n(x),$$

$$\Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x) \quad \text{and} \quad \Psi_n(x) \overset{\nu}{\rightrightarrows} \Psi(x).$$

Hence, similarly as in the proof of proposition 16, we obtain the convergence (72) with

$$\chi_m(x) = A^2\left(\frac{1}{2\pi i}D\right)\{\Phi(x) * \Psi_m(x)\}. \tag{74}$$

But $\varphi(x)$ is rapidly decreasing and

$$A\left(\frac{1}{2\pi i}D\right)\Phi(x) = \varphi(x).$$

Therefore, replacing if necessary $A(x)$ by another entire function, we may assume that $(1+|x|^2)^{\nu+(q+1)/2}\Phi(x)$ tends to zero as $|x| \to \infty$. Then, for $m \to \infty$,

$$\Phi(x) * \Psi_m(x) \overset{\nu}{\rightrightarrows} \Phi(x) * \Psi(x), \tag{75}$$

and this shows, together with equation (74), that $\chi_m(x)$ is a generalized regular sequence.

As already said, the functions $\chi_m(x)$ do not depend on the sequence $\varphi_n(x)$ representing $\varphi(x)$. But the convolution (73) is also independent of the sequence $\psi_n(x)$ representing $\psi(x)$, by an argument analogous to that used in other proofs of consistency.

In particular, if $\psi(x)$ is a very good function, then

$$\varphi(x) * \psi(x) = [\chi(x)],$$

where $\chi(x)$ is defined by (70). Thus in the general case, when $\psi(x)$ is an arbitrary ultra-distribution determined by the generalized regular sequence $\psi_n(x)$, the definition (73) may be written as follows:

$$\varphi(x) * \psi(x) = [\varphi(x) * \psi_n(x)].$$

The convolution just defined has the following usual properties:

$$\varphi(x) * \psi(x) = \psi(x) * \varphi(x),$$

$$\big(\varphi_1(x) * \varphi_2(x)\big) * \psi(x) = \varphi_1(x) * \big(\varphi_2(x) * \psi(x)\big),$$

$$\big(\varphi_1(x) + \varphi_2(x)\big) * \psi(x) = \varphi_1(x) * \psi(x) + \varphi_2(x) * \psi(x),$$

$$\varphi(x) * \big(\psi_1(x) + \psi_2(x)\big) = \varphi(x) * \psi_1(x) + \varphi(x) * \psi_2(x),$$

where the ultra-distributions $\varphi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are rapidly decreasing.

Remark. If the ultra-distribution $\varphi(x)$ is rapidly decreasing, then its convolution with an arbitrary ultra-distribution is defined by formula (73). However, the convolution of $\varphi(x)$ with one particular ultra-distribution $\psi(x)$, say, can be defined under a weaker assumption on $\varphi(x)$. In fact, if $\psi(x)$ is an infinite derivative of a continuous function $\Psi(x)$, which is $O(|x|^{2\nu})$ as $|x|\to\infty$, then it is sufficient to assume that $\varphi(x)$ is an infinite derivative of a continuous function $\Phi(x)$, which is $O(|x|^{-2\nu-q-1})$ as $|x|\to\infty$. Moreover, if

$$\varphi(x) = A\left(\frac{1}{2\pi i}D\right)\Phi(x) \quad \text{and} \quad \psi(x) = B\left(\frac{1}{2\pi i}D\right)\Psi(x),$$

then the convolution may be expressed in the form

$$\varphi(x) * \psi(x) = C\left(\frac{1}{2\pi i}D\right)\{\Phi(x) * \Psi(x)\}, \tag{76}$$

where $C(x) = A(x)B(x)$.

§ 13. Fourier transforms of rapidly decreasing ultra-distributions. According to a theorem of L. Schwartz ([10], vol. 2, p. 124, theorem XV) a distribution is rapidly decreasing, iff its Fourier transform is an infinitely differentiable function, slowly increasing together with all its derivatives. Moreover, the convolution of a rapidly decreasing distribution with an arbitrary tempered distribution is transformed into the product. For rapidly decreasing ultra-distributions we have the following results:

Proposition 17. *An ultra-distribution $\varphi(x)$ is rapidly decreasing, iff its Fourier transform $\tilde{\varphi}(s)$ is an infinitely differentiable function.*

Proof. Suppose first that $\varphi(x)$ is rapidly decreasing and apply condition (RD_2) of proposition 15. Then, for any integer $\mu\geqslant 0$, there exists a representation of the form

$$\varphi(x) = A\left(\frac{1}{2\pi i}D\right)\Phi(x),$$

where $A(x)$ is an entire function and $\Phi(x)$ a continuous function, which is $O(|x|^{-2\mu-q-1})$ as $|x|\to\infty$. $\Phi(x)$ is integrable over R_q on multiplying by any power x^k with $\sigma_k\leqslant 2\mu$. Therefore its Fourier transform $\tilde{\Phi}(s)$ is continuously differentiable up to the order 2μ, and the same property has the product $\tilde{\varphi}(s) = A(s)\tilde{\Phi}(s)$. Since μ was an arbitrary integer $\geqslant 0$, we conclude that $\tilde{\varphi}(s)$ is infinitely differentiable.

Conversely, let $\tilde{\varphi}(s)$ be an infinitely differentiable function and μ a given positive integer. Then, by lemma 2, there exists an entire function $A^*(s)$ such that the quotient

$$\tilde{\Phi}^*(s) = \frac{\tilde{\varphi}(s)}{A^*(s)} \tag{77}$$

is integrable over R_q together with all its derivatives up to the order $2\mu q$. Thus the function

$$\Phi^*(x) = \mathscr{F}^{-1}\{\tilde{\Phi}^*(s)\} \tag{78}$$

is continuous and $O(|x|^{-2\mu})$ as $|x|\to\infty$. Also from (77) and (78) it follows that

$$\varphi(x) = A^*\left(\frac{1}{2\pi i}D\right)\Phi^*(x),$$

and so $\varphi(x)$ is rapidly decreasing on account of proposition 15.

Proposition 18. *Let $\varphi(x)$ and $\psi(x)$ be ultra-distributions and assume that at least one of them is rapidly decreasing. Then*

$$\mathscr{F}\{\varphi(x) * \psi(x)\} = \tilde{\varphi}(s)\tilde{\psi}(s), \tag{79}$$

where, as usually, $\tilde{\varphi}(s) = \mathscr{F}\{\varphi(x)\}$ and $\tilde{\psi}(s) = \mathscr{F}\{\psi(x)\}$.

Proof. Suppose that $\varphi(x)$ is rapidly decreasing. If $\psi(x) = [\psi_n(x)]$ then, passing in (72) to the Fourier transforms and applying proposition 6, one can easily verify that

$$\mathscr{F}\{\varphi(x) * \psi_n(x)\} = \tilde{\varphi}(s)\tilde{\psi}_n(s), \tag{80}$$

where $\tilde{\psi}_n(s) = \mathscr{F}\{\psi_n(x)\}$, $n = 1, 2, \ldots$ But $\tilde{\psi}_n(s)$ is a fundamental sequence of $\tilde{\psi}(s)$ and, by virtue of proposition 17, $\tilde{\varphi}(s)$ is an infinitely differentiable function. Therefore $\tilde{\varphi}(s)\tilde{\psi}_n(s)$ is a fundamental sequence of $\tilde{\varphi}(s)\tilde{\psi}(s)$. On the other hand, by definition 9, $\varphi(x)*\psi_n(x)$ is a generalized regular sequence of $\varphi(x)*\psi(x)$. The result (79) follows now from definition 4 and equation (80).

If both $\varphi(x)$ and $\psi(x)$ are rapidly decreasing ultra-distributions, then the convolution $\varphi(x)*\psi(x)$ is also rapidly decreasing, on account of proposition 17 and proposition 18.

If $\varphi(x)$ is rapidly decreasing, then

$$\lim_{n\to\infty}\psi_n(x) = \psi(x)$$

implies

$$\lim_{n\to\infty}\varphi(x) * \psi_n(x) = \varphi(x) * \psi(x).$$

Let now $\varphi_n(x)$ be a sequence of rapidly decreasing ultra-distributions and $\tilde{\varphi}_n(s) = \mathscr{F}\{\varphi_n(x)\}$. We prove

Proposition 19. *The following conditions are equivalent:*

$$\lim_{\substack{n\to\infty \\ \nu=-\mu}}\varphi_n(x) = \varphi(x), \qquad \mu = 1, 2, \ldots; \tag{C_1}$$

(C_2) *for every $k\in\mathscr{N}$, $D_s^k\tilde{\varphi}_n(s)$ converges to $D_s^k\tilde{\varphi}(s)$ almost uniformly in R_q.*

Proof. Condition (C_1) means that to any integer ν there exists an entire function $A(x)$ and continuous functions $\Phi_n(x)$, $\Phi(x)$, such that

$$\varphi_n(x) = A\left(\frac{1}{2\pi i}D_x\right)\Phi_n(x), \qquad \varphi(x) = A\left(\frac{1}{2\pi i}D_x\right)\Phi(x), \tag{81}$$

$$\Phi_n(x) \overset{\nu}{\rightrightarrows} \Phi(x). \tag{82}$$

Let us take $\nu = -\mu-(q+1)/2$, where μ is a given positive integer, and consider the Fourier transforms $\tilde{\Phi}_n(s) = \mathscr{F}\{\Phi_n(x)\}$, $\tilde{\Phi}(s) = \mathscr{F}\{\Phi(x)\}$. Then from (82) it follows that, for any $k\in\mathscr{N}$ of order $\sigma_k \leqslant 2\mu$, the sequence $D_s^k\tilde{\Phi}_n(s)$ converges to $D_s^k\Phi(s)$ uniformly in R_q. But equations (81) are transformed into

$$\tilde{\varphi}_n(s) = A(s)\tilde{\Phi}_n(s), \qquad \tilde{\varphi}(s) = A(s)\tilde{\Phi}(s).$$

Hence, for every $k\in\mathscr{N}$ of order $\sigma_k \leqslant 2\mu$, $D_s^k\tilde{\varphi}_n(s)$ converges to $D_s^k\tilde{\varphi}(s)$ almost uniformly in R_q. μ being an arbitrary integer, we have proved that (C_1) implies (C_2).

Conversely, suppose that condition (C_2) is satisfied. Then, for every $k\in\mathscr{N}$, the functions $D_s^k\tilde{\varphi}_n(s)$, $n = 1, 2, \ldots$, are uniformly bounded in every bounded subset of R_q. Thus one can find a continuous function $f_k(s)$ majorating the functions concerned, i. e.

$$|D_s^k\tilde{\varphi}_n(s)| \leqslant f_k(s), \qquad n = 1, 2, \ldots,$$

for every $k\in\mathscr{N}$. We proceed now similarly as in the proof of lemma 2. Accordingly, given a positive integer μ, we set

$$f(s) = \Big(1+\sum_{\sigma_k\leqslant 2\mu} f_k(s)\Big)\,(1+|x|^2)^{(q+1)/2}.$$

By lemma 1 there exists an entire function $F(s)$ such that $|F(s)| > > f(s)$ and all derivatives $D_s^k[1/F(s)]$ are bounded in R_q.

Let us take the entire function

$$G(s) = F^{2\mu q+1}(s).$$

Using the same argument as in the proof of lemma 2 one shows that the functions

$$\tilde{\Phi}_n^*(s) = \frac{\tilde{\varphi}_n(s)}{G(s)}, \qquad \tilde{\Phi}^*(s) = \frac{\tilde{\varphi}(s)}{G(s)} \tag{83}$$

are integrable over R_q together with all their derivatives up to the order $2\mu q$. Moreover, for every $k\in\mathscr{N}$ of order $\sigma_k \leqslant 2\mu q$, the sequence $D_s^k\tilde{\Phi}_n^*(s)$ converges to $D_s^k\tilde{\Phi}^*(s)$ in L_{R_q}. This implies that

$$\Phi_n^*(x) = \mathscr{F}^{-1}\{\tilde{\Phi}_n^*(s)\}, \qquad \Phi^*(x) = \mathscr{F}^{-1}\{\tilde{\Phi}^*(s)\}$$

are continuous functions and

$$(1+|x|^2)^\mu\Phi_n^*(x) \overset{0}{\rightrightarrows} (1+|x|^2)^\mu\Phi^*(x). \tag{84}$$

Also from equations (83) it follows that

$$\varphi_n(x) = G\left(\frac{1}{2\pi i}D_x\right)\Phi_n^*(x), \qquad \varphi(x) = G\left(\frac{1}{2\pi i}D_x\right)\Phi^*(x),$$

and so

$$\lim_{\substack{n\to\infty\\ \nu=-\mu}} \varphi_n(x) = \varphi(x),$$

in view of the convergence (84). Since μ is an arbitrary positive integer, condition (C_1) is proved.

Corollary. *If $\varphi_n(x)$, $n = 1, 2, \ldots$, are rapidly decreasing ultra-distributions, then*

$$\lim_{\substack{n\to\infty\\ \nu=-\mu}} \varphi_n(x) = \varphi(x), \qquad \mu = 1, 2, \ldots,$$

implies

$$\lim_{n\to\infty} \varphi_n(x) * \psi(x) = \varphi(x) * \psi(x)$$

for an arbitrary ultra-distribution $\psi(x)$.

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