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On two-functional spaces

by

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1. Introduction. This paper is concerned with some properties of two abstract functional spaces defined by a certain class of asymptotic approximation functions. One of the two spaces considered includes the Orlicz space as a particular case, while the other one is a generalization of a functional space defined by Lorentz [9].

Let $\varphi(u)$ be non-negative, convex, vanishing at the origin and such that $\varphi(u)/u \to \infty$ with u. Let $\psi(u)$ denote the complementary function of $\varphi(u)$ in the sense of Young. Consider the measurable functions $x(t)$, $a \leqslant t \leqslant b$, such that the product $x(t)y(t)$ is integrable over (a, b) for every measurable function $y(t) \in L_\psi(a, b)$ and set

$$\|x(t)\| = \|x(t)\|_\varphi = \sup_y \left| \int_a^b x(t)y(t)\,dt \right|, \tag{1.1}$$

the sup being with respect to all y with

$$\varrho_y = \int_a^b \psi(|y(t)|)\,dt \leqslant 1. \tag{1.2}$$

The class of such x will be denoted by L_φ^* which is usually called the *Orlicz space* (1). If $\varphi(u)$ satisfies the Δ_2-condition which was defined by Krasnosel'skii and Rutickii [7]: $\varphi(2u) \leqslant K\varphi(u)$, then L_φ and L_φ^* are identical (2). In this paper, we shall define a new functional space L_φ^{**} which is wider than the Orlicz space L_φ^*.

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(1) See [18], p. 378, Notes to Chap. IV, section 10. But this is not the original space which means $\varphi(u)$ should satisfy the Δ_2-condition: $\varphi(2u) \leqslant K\varphi(u)$. For detailed properties of Orlicz spaces, cf. [3], [10], [13], [14], [18], p. 170-174, and the book by Krasnosel'skii and Rutickii [7].

(2) Cf. [18], p. 172. Throughout in this paper, $\varphi(x)$ will denote a non-negative even function so that we do not need to restrict ourselves the Δ_2-condition only for $u \geqslant 0$. By the variable x we always mean it is a non-negative variable.

Lorentz [9] has considered linear transformations between classes of Fourier series of functions satisfying certain Lipschitz conditions and some other functional spaces. This paper also contains generalizations of Lorentz's theorems which are related to functions satisfying "generalized Lipschitz conditions". Lorentz has found some relations between the Fourier constants of a function belonging to the Lipschitz class $\mathrm{Lip}\,\alpha$ or $\mathrm{Lip}(\alpha, p)$. The functions $f(x)$ satisfying the Lipschitz condition $\mathrm{Lip}(\alpha, p)$ form a linear normed space and the norm for each element is defined by

$$\|f\|_{\mathrm{Lip}(\alpha,p)} = \sup_h \frac{\{\int_{-\pi}^{\pi} |f(t+h)-f(t)|^p dt\}^{1/p}}{|h|^\alpha}. \tag{1.3}$$

For functions satisfying $\mathrm{Lip}\,\alpha$, the norm is defined by

$$\|f\|_{\mathrm{Lip}\,\alpha} = \sup_{h,t} \frac{|f(t+h)-f(t)|}{|h|^\alpha}, \tag{1.4}$$

which corresponds to the particular case in $\mathrm{Lip}(\alpha, p)$ when $p=\infty$. Lorentz has also considered the spaces $M_{\alpha,p}$, $R_{\alpha,p}$ for all real number sequences $y = \{a_1, b_1, \ldots, a_n, b_n, \ldots\}$, such that

$$\mathfrak{M}_{\alpha,p}[y] = \sup_n \frac{\{\sum_{k=1}^{n} (|a_k|^p + |b_k|^p)\}^{1/p}}{n^\alpha} < \infty \tag{1.5}$$

and

$$\mathfrak{R}_{\alpha,p}[y] = \sup_n \frac{\{\sum_{k=n}^{\infty} (|a_k|^p + |b_k|^p)\}^{1/p}}{n^{-\alpha}} < \infty \tag{1.6}$$

are satisfied, respectively, where $0<\alpha\leqslant 1$ and $1\leqslant p\leqslant\infty$. The quantities on the left of (1.5) and (1.6) are defined as the norms of the corresponding spaces.

Notation. By $\varphi(x) \sim [p_1, p_2]$, $0\leqslant p_1 \leqslant p_2 < \infty$ (or similarly, $-\infty < p_1 \leqslant p_2 \leqslant 0$) we denote the non-negative even function $\varphi(x)$, such that $\varphi(x)x^{-p_1}$ is non-decreasing and $\varphi(x)x^{-p_2}$ is non-increasing, as x is increasing in $(0,\infty)$. By $\varphi(x)\sim[p_1,\infty]$ we denote $\varphi(x)$ such that $\varphi(x)x^{-p_1}$ is non-decreasing, and by $\varphi(x)\sim[p_1,\infty\rangle$ we denote $\varphi(x)$ such that $\varphi(x)\sim[p_1, N]$ for some positive constant $N\geqslant p_1$. By $\varphi(x)\sim\langle p_1, p_2\rangle$, $0\leqslant p_1 < p_2 < \infty$ (or similarly for $-\infty < p_1 < p_2 \leqslant 0$), we mean $\varphi(x) \sim [p_1+\varepsilon, p_2-\varepsilon]$ for some $\varepsilon > 0$. We define $\varphi(x)\sim[p_1,p_2\rangle$, $\varphi(x)\sim\langle p_1, p_2]$, where $p_1<p_2$, in a similar way. By K we denote a certain positive constant, so that two different positive constants may be denoted by the same K. By $\varphi(x)\,\epsilon\, M(a,b)$, $1\leqslant a < b < \infty$, we denote the non-negative continuous non-decreasing function $\varphi(x)$, $0\leqslant x<\infty$, satisfying $\varphi(0)=0$ and the following conditions:

$$\varphi(2u) = O\{\varphi(u)\}, \tag{1.7}$$

$$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}\,dt = O\left\{\frac{\varphi(u)}{u^b}\right\}, \tag{1.8}$$

$$\int_1^u \frac{\varphi(t)}{t^{a+1}}\,dt = O\left\{\frac{\varphi(u)}{u^a}\right\} \tag{1.9}$$

as $u\to\infty$. By $\varphi(x)\,\epsilon\, Z(a,b)\subset M(a,b)$ we denote $\varphi(x)$ satisfying the above conditions and also the following conditions:

$$\varphi(2u) = O\{\varphi(u)\}, \tag{1.10}$$

$$\int_u^1 \frac{\varphi(t)}{t^{b+1}}\,dt = O\left\{\frac{\varphi(u)}{u^b}\right\}, \tag{1.11}$$

$$\int_0^u \frac{\varphi(t)}{t^{a+1}}\,dt = O\left\{\frac{\varphi(u)}{u^a}\right\}, \tag{1.12}$$

as $u\to+0$.

In a previous paper [4] the author has defined the *generalized Lipschitz condition*. By $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(\delta)$, $\alpha(\delta)\sim\langle 0,1\rangle$, we mean $\omega(\delta)\leqslant K\alpha(\delta)$, where $\omega(\delta)=\omega(\delta;f)=\max_{x_1,x_2}|f(x_1)-f(x_2)|$, $|x_1-x_2|\leqslant\delta$. Similarly, by $f(x)\,\epsilon\,\mathrm{Lip}\big(\alpha(\delta),p\big)$, $p>1$, we mean

$$\|f\| = \sup_h \frac{\{\int_{-\pi}^{\pi} |f(t+h)-f(t)|^p dt\}^{1/p}}{\alpha(h)} < \infty, \tag{1.13}$$

where $\|f\|$ is the norm of the space $\mathrm{Lip}\big(\alpha(\delta),p\big)$. We shall define two spaces $M_{\psi(n),p}$ and $R_{\psi(n),p}$ which reduce to $M_{\alpha,p}$ and $R_{\alpha,p}$ respectively, when $\psi(\delta)=\delta^\alpha$. These spaces have already been defined in [9], p. 137, but have not yet been intensively investigated. For all real number sequences $y=\{a_1,b_1,\ldots,a_n,b_n,\ldots\}$ let us now define the spaces $M_{\psi,p}$, $R_{\psi,p}$ for which the conditions

$$\mathfrak{M}_{\psi,p}[y] = \sup_n \frac{\{\sum_{k=1}^{n} |a_k|^p + |b_k|^p\}^{1/p}}{\psi(n)} < \infty, \tag{1.14}$$

$$\mathfrak{R}_{\psi,p}[y] = \sup_n \frac{\{\sum_{k=n}^{\infty} |a_k|^p + |b_k|^p\}^{1/p}}{\psi(1/n)} < \infty \tag{1.15}$$

are satisfied, respectively, where $\psi(\delta)\sim\langle 0,1]$, $1\leqslant p<\infty$.

2. The functional space L_φ^{**}. We now prove

LEMMA 1. *If $\varphi(x) \sim [a, b\rangle$, $0 \leqslant a < b < \infty$, then $\varphi(x)$ is absolutely continuous in $(-N, N)$ where N is any positive constant. If $\varphi(x) \sim \langle -\infty, 0]$, then $\varphi(x)$ is absolutely continuous in (ε, ∞), where ε is any positive constant.*

Proof. We prove the first part, while the proof of the second part for $\varphi(x) \sim \langle -\infty, 0]$ follows in a similar way. Without loss of generality we take $\varphi(0) = 0$. Since $\varphi(x)$ increases in $(0, \infty)$, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\delta = \delta(\varphi, \varepsilon)$, $\varphi(\delta) < \varepsilon/2$. Let us consider the interval (δ, N). If $\delta \leqslant x_1 < x_2 \leqslant N$, then

$$\frac{\varphi(x_2)}{x_2^b} \leqslant \frac{\varphi(x_1)}{x_1^b}, \tag{2.1}$$

$$\varphi(x_2)-\varphi(x_1) \leqslant \varphi(x_1)\{(x_2/x_1)^b-1\} \leqslant \frac{\varphi(x_1)}{x_1^b}|x_2^b-x_1^b| \tag{2.2}$$

$$\leqslant \{\varphi(\delta)/\delta^b\}|x_2^b-x_1^b| = K_{\delta,\varphi}|x_2^b-x_1^b|.$$

Since $y = x^b$ is absolutely continuous in $(0, N)$, we have

$$\sum_{\nu=1}^{n} |\varphi(x_{\nu+1})-\varphi(x_\nu)| \leqslant \varphi(\delta)+K_{\delta,\varphi}\sum_{\nu=1}^{n}|x_{\nu+1}^b-x_\nu^b| \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \tag{2.3}$$

for any integer n provided that

$$\sum_{1}^{n}|x_{\nu+1}-x_\nu| < \delta' = \delta'(\varphi, \varepsilon), \qquad x_\nu \in (0, N).$$

Since $\varphi(x)$ is even, it is absolutely continuous in $(-N, N)$.

As a direct consequence of Lemma 1, the first part of Theorem 1 in [4] may be improved as follows:

THEOREM 1. *If $\varphi(x) \sim \langle a, b\rangle$, where $1 \leqslant a < b < \infty$, then $\varphi(x) \in Z(a, b)$. The class of functions $\sim \langle a, b\rangle \subset Z(a, b) \subset$ the class of irregular increasing functions bounded by Kx^a and Kx^b.*

We are now in a position to define a new space L_φ^{**} which is wider than the Orlicz space L_φ^{*} and has almost all the properties of L_φ^{*}. Let $\varphi(x) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, with $\varphi_1(t) = \varphi(t)/t \to \infty$ as $t \to \infty$ (the condition is certainly satisfied if $1 < p_1 \leqslant p_2 \leqslant \infty$), and let $\psi_1(t)$ be the inverse function of the positive non-decreasing continuous function $\varphi_1(t) = \varphi(t)/t$. Write

$$\Phi_1(x) = \int_0^x \varphi_1(t)\,dt \quad \text{and} \quad \Psi_1(x) = \int_0^x \psi_1(t)\,dt.$$

Then $\Phi_1(x)$ is a convex function, and hence $\Phi_1(x)$ and $\Psi_1(x)$ are complementary functions in the sense of Young. Consider the functions $x(t)$, $a \leqslant t \leqslant b$, such that the product $x(t)y(t)$ is integrable over the interval (a, b) for every $y(t) \in L_{\Psi_1}(a, b)$ and set

$$\|x(t)\| = \|x(t)\|_{\langle\varphi\rangle} = \sup_y \Big|\int_a^b x(t)y(t)\,dt\Big|, \tag{2.4}$$

the sup being with respect to all y with

$$\varrho_y = \int_a^b \Psi_1(|y(t)|)\,dt \leqslant 1. \tag{2.5}$$

The class of such $x(t)$ is denoted by $L_\varphi^{**} = L_\varphi^{**}[a, b]$. It is easy to see that if $\varphi(x) = x^p$, $p > 1$, then the norm thus defined differs only from the ordinary p-norm in L_p-space by a constant factor depending on p. We require the following lemmas.

LEMMA 2. *If $\varphi(x)$ is non-negative and is convex such that $\varphi(0) = 0$, $\varphi(2x) \leqslant k\varphi(x)$, then there exists a constant $m < \infty$ such that $\varphi(x) \sim [1, m\rangle$.*

Proof. Since $\varphi(0) = 0$, $\varphi(x) \geqslant 0$ and is convex, $\varphi(x)/x$ is non-decreasing as x is increasing. The condition that $\varphi(x)/x$ is non-decreasing is obvious. To show that $\varphi(x)/x^m$ is decreasing, it is sufficient to consider the case $0 < x_1 < x_2 < \infty$, $1 < x_2/x_1 < 2$. Write $x_2 = x_1+\Delta x$, where $\Delta x < x_1$, and $\varphi(x) = \int_0^x p(t)\,dt$, where $p(t)$ is non-negative and non-decreasing [3]. The function $F(a, x) \equiv (1+x)^a-(1+ax)$ is positive for $a > 1$ and $x > 0$, since $F(a, 0) = 0$ and $\partial F/\partial x = a(1+x)^{a-1}-a > 0$. Then we have

$$\frac{\varphi(x_2)}{\varphi(x_1)} = \frac{\varphi(x_1+\Delta x)}{\varphi(x_1)} = \frac{\varphi(x_1)+\int_{x_1}^{x_1+\Delta x} p(t)\,dt}{\int_0^{x_1} p(t)\,dt} \tag{2.6}$$

$$< 1+\frac{\int_{x_1}^{x_1+\Delta x} p(x_2)\,dt}{\int_{x_1/2}^{x_1} p(x_1/2)\,dt} = 1+\left\{\frac{2p(x_2)}{p(x_1/2)}\right\}\left(\frac{\Delta x}{x_1}\right),$$

$$2x_1p(x_2) \leqslant 2x_1p(2x_1) < \left(\int_{2x_1}^{4x_1}+\int_0^{2x_1}\right)p(t)\,dt = \varphi(4x_1) \tag{2.7}$$

$$\leqslant k^3\varphi(x_1/2) = k^3\int_0^{x_1/2} p(t)\,dt \leqslant \frac{x_1}{2}k^3p(x_1/2).$$

(3) Cf. [10], p. 187-189, and [18], p. 24 and p. 25.

It follows that

$$p(x_2) \leqslant \tfrac{1}{4}k^3 p(x_1/2) \tag{2.8}$$

and

$$\frac{\varphi(x_2)}{\varphi(x_1)} < 1+\tfrac{1}{2}k^3\left(\frac{\Delta x}{x_1}\right) < 1+(1+\tfrac{1}{2}k^3)\left(\frac{\Delta x}{x_1}\right) < \left\{1+\frac{\Delta x}{x_1}\right\}^{(1+\frac{1}{2}k^3)} = (x_2/x_1)^m, \tag{2.9}$$

where $m = 1+\frac{1}{2}k^3$. The case when $x_2/x_1 > 2$ can be reduced to $(x_2/x_1) = (x_2/x_3)(x_3/x_4)\dots(x_\nu/x_1)$. This proves Lemma 2 with $m = 1+\frac{1}{2}k^3$.

LEMMA 3. *Let $\varphi(x) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 < \infty$. Then*

$$p_1 \leqslant \frac{\varphi(x)}{\int_0^x \{\varphi(t)/t\}\,dt} = \frac{\varphi(x)}{\int_0^x \varphi_1(t)\,dt} = \frac{\varphi(x)}{\Phi_1(x)} \leqslant p_2. \tag{2.10}$$

Proof. Since $\varphi(x)/x^{p_1}$ is non-decreasing and $\varphi(x)/x^{p_2}$ is non-increasing, by differentiating these expressions we obtain, for almost all x,

$$p_1 \leqslant \frac{\varphi'(t)}{\{\varphi(t)/t\}} = \frac{\varphi'(t)}{\varphi_1(t)} \leqslant p_2. \tag{2.11}$$

Since $\varphi(x)$ is absolutely continuous, we have $\varphi(x) = \int_0^x \varphi'(t)\,dt$. Hence the result.

THEOREM 2. (a) *If $\varphi(x)$ satisfies the following conditions:*

(i) *$\varphi(x)$ is a convex function;*

(ii) *$0 = \varphi(0) \leqslant \varphi(x) \leqslant \varphi(y)$ when $0 < x < y$;*

(iii) *$\varphi(x)/x \uparrow \infty$, as x increases from 0 to ∞;*

(iv) *Δ_2-condition: $\varphi(2u) \leqslant K\varphi(u)$ for all $u > 0$;*

then $L_\varphi = L_\varphi^* = L_\varphi^{**}$.

In other words, for convex functions $\{\varphi(x)\}$, the classes of spaces $\{L_\varphi\}$, $\{L_\varphi^*\}$, $\{L_\varphi^{**}\}$ for different $\varphi(x)$'s thus defined are identical.

(b) *In the general case, if each $\varphi(x)$ out of the class $\{\varphi(x)\}$ satisfies the following conditions:*

(i) *$\varphi(x) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$;*

(ii) *$\varphi_1(t) = \varphi(t)/t \uparrow \infty$, as t increases from 0 to ∞;*

then $L_\varphi^* \subset L_\varphi^{**}$. More precisely, there exists $\varphi(x)$ satisfying (i) *and* (ii), *such that $L_\varphi^* \subset L_\varphi^{**}$, $L_\varphi^* \neq L_\varphi^{**}$.*

In other words, the class of spaces $\{L_\varphi^{**}\}$ corresponding to different $\varphi(x)$'s thus defined includes the class of spaces $\{L_\varphi^*\}$ as a proper subclass and there exists a function $\varphi(x)$ satisfying conditions (i) *and* (ii) *such that L_φ^* is not defined (i. e. L_φ^* is a void set).*

Proof. (a) If $\varphi(x)$ satisfies the Δ_2-condition, then the classes L_φ^* and L_φ are identical ([18], p. 172). Since $\Phi_1(x) = \int_0^x \varphi_1(t)\,dt$ is the integral of a non-decreasing function, $\Phi_1(x)$ is a convex function, and hence $\Phi_1(x)/x$ is non-decreasing. By Lemma 2 and Lemma 3, $\{\Phi_1(x)/x\} \to \infty$, since $\{\varphi(x)/x\} \to \infty$. It is easy to see that $\Phi_1(x)$ also satisfies the Δ_2-condition, for

$$\Phi_1(2u) = \Phi_1(u) + \int_u^{2u} \varphi_1(t)\,dt \leqslant \Phi_1(u) + K\int_{u/2}^{u} \varphi_1(t)\,dt \leqslant K\Phi_1(u).$$

Hence, by our definition for L_φ^{**}, we have $L_\varphi^{**} = L_{\Phi_1}^* = L_{\Phi_1}$, and from Lemma 3, we obtain $L_\varphi^{**} = L_{\Phi_1} = L_\varphi$.

(b) In the space L_φ^*, for convex φ, $\{\varphi(x)/x\} \to \infty$, as $x \to \infty$. Let $\psi_1(t)$ and $\psi(t)$ denote the inverse functions of $\varphi_1(t)$ and $\varphi'(t)$, respectively. From (2.11) and assumptions of Lemma 3 we have $\varphi_1(t) \leqslant \varphi'(t)$. It follows that $\psi(t) \leqslant \psi_1(t)$, and therefore

$$\Psi(x) = \int_0^x \psi(t)\,dt \leqslant \int_0^x \psi(t)\,dt = \Psi_1(x) \qquad (x > 0).$$

It follows that the class $\{y\}$ satisfying (2.5) is a subclass of $\{y\}$ satisfying (1.2). Hence $L_\varphi^* \subset L_\varphi^{**}$ and the norm of x with respect to L_φ^{**} is bounded by the norm of x with respect to L_φ^*. The relation $L_\varphi \subset L_\varphi^*$, $L_\varphi \neq L_\varphi^*$, has been established in [7], p. 75. To show that L_φ^{**} is practically wider than L_φ^*, it is sufficient to consider a function $\varphi(x) \sim [1, m\rangle$, such that $\varphi(x)$ is not convex, hence L_φ^* does not exist. Let us take $\varphi(x) \sim \langle 2, 3\rangle$. Then

$$\frac{d}{dx}\left\{\frac{\varphi(x)}{x^2}\right\} = \left\{\frac{\varphi'(x)}{x^2} - \frac{2\varphi(x)}{x^3}\right\} > 0, \tag{2.12}$$

$$\frac{d}{dx}\left\{\frac{\varphi(x)}{x^3}\right\} = \left\{\frac{\varphi'(x)}{x^3} - \frac{3\varphi(x)}{x^4}\right\} < 0, \tag{2.13}$$

where the function $\varphi'(x)$ exists almost everywhere and

$$2\{\varphi(x)/x\} = 2\varphi_1(x) < \varphi'(x) < 3\varphi_1(x) = 3\{\varphi(x)/x\}. \tag{2.14}$$

Conversely, if (2.14) is satisfied, then $\varphi(x)/x^2 \nearrow$, $\varphi(x)/x^3 \searrow$, and $\varphi(x) \sim \langle 2, 3\rangle$. Now if we construct the function $\varphi(x)$ so that $\varphi'(x)$ is bounded by the function $2\{\varphi(x)/x\}$ and $3\{\varphi(x)/x\}$ and such that $\varphi'(x)$ is not always non-decreasing, then $\varphi(x)$ is not convex [4]. To this end, we divide all members of (2.14) by $\varphi(x)$ and then integrate the three members from 1

[4] If $\varphi(x)$ is convex, then $\varphi(x) = \int_0^x p(t)\,dt$, where $p(t)$ is non-decreasing.

to x, when $x > 1$. Then we have

$$x^2 < \varphi(x)/\varphi(1) < x^3 \qquad (x > 1). \tag{2.15}$$

Assuming $\varphi(1) = 1$, and set

$$\varphi(x) = \begin{cases} x^{5/2}, & 0 \leqslant x \leqslant 1, \\ x^{9/4}, & x > 1. \end{cases} \tag{2.16}$$

The function $\varphi(x)$ thus defined satisfies (2.14). On the other hand, $\varphi'(x)$ exists everywhere except at $x = 1$, where $\varphi'(1^-) = 5/2$ and $\varphi'(1^+) = 9/4$, so that $\varphi(x)$ is not equal to the integral of a non-decreasing function. Hence $\varphi(x)$ is not a convex function.

From Theorem 2, we conclude that L^{**}_φ is wider than the Orlicz space and that L_φ is equivalent to L^{**}_φ if the above defined p_2 is finite. We now see that the space L^{**}_φ has almost all the properties of L^*_φ. In fact, by similar arguments as in [18], p. 170-175, and in virtue of $\varphi(t)/t$ is non-decreasing and tending to ∞, as $t \to \infty$, we have the following similar results for L^{**}_φ (the detailed proofs are omitted here):

THEOREM 3. *The space L^{**}_φ is a complete space.*

If there is a number $\theta > 0$ such that $\theta x(t) \in L_\varphi$ for $\varphi(t) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 < \infty$, then $x(t) \in L^{**}_\varphi$. Conversely, if $x(t) \in L^{**}_\varphi$, with finite p_2, then there is a constant $\theta > 0$ such that $\theta x(t) \in L_\varphi$. More precisely, we have

THEOREM 4. *If $x(t) \in L^{**}_\varphi$, with finite p_2, $\|x(t)\|_{\langle\varphi\rangle} \neq 0$, then*

$$\int_a^b \varphi\{|x(t)|/\|x(t)\|_{\langle\varphi\rangle}\}\,dt \leqslant K.$$

This follows from an argument in [18], p. 171, and from (2.11).

THEOREM 5. (a) *If*

$$u(x) = \int_a^b x(t)y(t)\,dt$$

is bounded for every $x(t) \in L^{**}_\varphi(a, b)$, then $y(t) \in L^{**}_{\Psi_1} = L^{**}_{\Psi_1}(a, b)$.

(b) *If the sequence*

$$u_n(x) = \int_a^b x(t)y_n(t)\,dt$$

is bounded for every $x(t) \in L^{**}_\varphi$, then $\|y_n(t)\|_{\Psi_1} = O(1)$, as $n \to \infty$, with respect to the space $L^{**}_{\Psi_1}$.

(c) *If the sequence in* (b) *is bounded for every $x(t) \in L_\varphi$, then there is a constant $\theta > 0$ such that*

$$\int_a^b \Psi_1\{\theta\,|y_n(t)|\}\,dt = O(1), \qquad as \quad n \to \infty.$$

We now present a comparison between J. Lamperti's results [8] and some results in [4] concerning conjugate functions. Let $\varphi(t)$ be non-negative, convex, and be defined for $t \geqslant 0$ with $\varphi(0) = 0$, $\varphi(2t) \leqslant K\varphi(t)$. Let the classes A to E be classes of $\{\varphi(t)\}$ defined as follows:

Class A: $\varphi'(t)$ is concave, and $\varphi(t^\theta)$ is convex for some $\theta < 1$.

Class B: $\varphi'(t)$ is convex such that $\varphi'(0) = 0$ (5), and $\varphi(t^{1-\theta})$ is concave for some $\theta < 1$.

Class C: $\varphi(t) = \varphi_1(t)+\varphi_2(t)$, where $\varphi_1 \in A$, $\varphi_2 \in B$.

Class D: $0 < a \leqslant \varphi(t)/\varphi_1(t) \leqslant b < \infty$ for all large t, where $\varphi_1 \in C$.

Class E: $\varphi(t) = t^p L(t)$ where $1 < p \neq 2$ and $L(t)$ is slowly varying in the sense of Karamata [5, 6].

Lamperti proved that $A \cup B \subset C \subset D$, $E \subset D$, and that if $\varphi(t)$ belongs to any one of the classes A to E, $f(x) \in L_\varphi$, then $\|\bar{f}(x)\|_\varphi \leqslant K\|f(x)\|_\varphi$, where $\bar{f}(x)$ is the conjugate function of $f(x)$.

To compare the results it is sufficient to consider functions belonging to A or B, while the extensions to classes C, D, E are trivial. Since $\varphi(t)$ is convex, $\varphi(0) = 0$, $\varphi(2t) \leqslant K\varphi(t)$, therefore by our results, $\varphi(t)$ is absolutely continuous and $\varphi(t) \sim [1, m\rangle$, where $1 < m < \infty$. If $\varphi(t) \in A$, then $\varphi'(t)$ is concave and $\varphi(t^\theta)$ is convex for some $\theta < 1$. It follows that $\varphi(t^\theta) \sim [1, m\rangle$ and $\varphi(t) \sim [1/\theta, m/\theta\rangle = [p_1, p_2\rangle$, where $1 < p_1 < p_2 < \infty$. Hence $\varphi(t) \sim \langle a, p_2\rangle$, where $1 < a < p_2 < \infty$. By Theorem 1 and [4], Theorem 1, $\varphi \in Z(a, p_2)$, where $1 < a < p_2 < \infty$. If $\varphi(t) \in B$, then $\varphi'(t)$ is convex and $\varphi(t^{1-\theta})$ is concave for some $\theta < 1$. So as in footnote (5), we may, without loss of generality, take $\varphi''(t) > 0$, $t > 0$, and $\varphi'(0) = 0$. It follows that $\varphi'(t) = \int_0^t \varphi''(s)\,ds$, where $\varphi''(x)$ is non-decreasing. Let us now only consider the case when $\varphi'(t)$ is convex, $\varphi''(t) > 0$, $t > 0$, which is in a sense wider than the class B. Then

$$\varphi'(x) = \int_0^x \varphi''(s)\,ds \leqslant \int_0^x \varphi''(x)\,ds = x\varphi''(x), \qquad \varphi'(x)/x \leqslant \varphi''(x)$$

(since $\varphi'(0) = 0$, $\varphi'(x)$ is convex, non-negative and $\varphi''(x)$ increases).

(5) $\varphi'(0) = 0$ has not been assumed in [8]. But, however, it is easy to see that this condition is indispensable. For if $\varphi(t) = t$, then $\varphi'(t) = 1$, and $\varphi(t)$ is convex, $\varphi(t^{1-\theta}) = t^{1-\theta}$ is concave for $\theta = 1/2$ and the result is certainly invalid. It should be further remarked that in Lemma 3 of [8], we should make use of Theorem 7.6.5 of Zygmund's *Trigonometrical Series* (1st ed.). Hence $\varphi'(t)$ must satisfy Young's conditions (i. e. $\varphi'(t)$ be continuous, vanishing at the origin, non-decreasing, tending to ∞) and therefore we must assume $\varphi'(0) = 0$. Since $\varphi''(t)$ is non-decreasing, consequently, it is trivial when $\varphi'(0) = 0$, $\varphi''(t) = 0$ for $0 < t < \delta$, and so we may, without loss of generality, take $\varphi''(t) > 0$ ($\varphi''(t)$ is non-decreasing).

We then obtain

$$(2.17)\qquad \varphi(t) = \int_0^t \varphi'(u)\,du \leqslant \int_0^t u\varphi''(u)\,du = t\varphi'(t) - \int_0^t \varphi'(u)\,du,$$

$$(2.18)\qquad \varphi'(t) \geqslant 2\varphi(t)/t.$$

It follows that $\varphi(t)/t^2$ is non-decreasing. On the other hand, since $\varphi(x)$ is convex, $\varphi(0)=0$, $\varphi(2x)\leqslant K\varphi(x)$, by Lemma 2, $\varphi(x)\sim[1, m\rangle$, $m<\infty$. On account of $\varphi(x)/x^2$ is non-decreasing, it follows that $\varphi(x)\sim[2, m\rangle$, $m<\infty$. Hence by Theorem 1 in [4] and Lemma 1, Theorem 1, $\varphi(x)\in Z(\frac{3}{2}, m+1)$. Collecting the results just obtained, if $\varphi(x)\in A$ or $\varphi(x)\in B$, then $\varphi(x)\in Z(a,b)$, where $1<a<b<\infty$. Hence Lamperti's results are effective special cases of Marcinkiewicz-Zygmund's results, a fortiori, particular cases of results in [4].

3. Some properties of the space $L_\varphi^{**}(G)$. Let G be a bounded closed set in a finite dimensional Euclidean space. Let $\varphi(x)\sim[p_1,p_2]$, $1\leqslant p_1\leqslant p_2\leqslant\infty$, and let $\varphi_1(t)=\varphi(t)/t$ be non-decreasing as t is increasing in $(0,\infty)$. Let $\psi_1(t)$ be the inverse function of $\varphi_1(t)$. Write

$$\Phi_1(u) = \int_0^u \varphi_1(t)\,dt \quad\text{and}\quad \Psi_1(u) = \int_0^u \psi_1(t)\,dt.$$

We now replace (1.2) by

$$\varrho_y = \int_G \varphi(|y(t)|)\,dt \leqslant 1,$$

also (1.1) by

$$\|x(t)\|_\varphi = \sup_y \left|\int_G x(t)y(t)\,dt\right|,$$

and $L^*(G)$ is defined in a similar way as before but with respect to G in the n-dimensional Euclidean space. Then $L_\varphi^{**}(G)=L_{\Phi_1}^*(G)$, where G is a bounded closed set in the n-dimensional Euclidean space. We shall denote by $\|f(x)\|_{\langle\varphi\rangle}=\|f(x)\|_{L^*_{\Phi_1}}$ the norm of the space $L_\varphi^{**}(G)$.

Definition 1. The sequence of functions $u_n(x)\in L_\varphi^{**}(G)$ is said to be *mean convergent to zero* if

$$\lim_{n\to\infty}\varrho(u_n;\Phi_1)=0,\qquad\text{where}\qquad \varrho(u_n;\Phi_1)=\int_G \Phi_1[|u_n(x)|]\,dx<\infty.$$

Theorem 6. *If $\varphi(x)\sim[p_1,p_2]$, $1\leqslant p_1\leqslant p_2<\infty$, and if $\{u_n(x)\}$ converges in mean to zero in $L_\varphi^{**}=L_\varphi^{**}(G)$, then $\{u_n\}$ converges in mean to zero in L_φ.*

Proof. By Lemma 3, it follows that if $p_2<\infty$, then

$$(3.1)\qquad p_1\Phi_1(x)\leqslant\varphi(x)\leqslant p_2\Phi_1(x).$$

Hence $\varrho(u_n,\Phi_1)\to 0$ implies $\varrho(u_n,\varphi)\to 0$, as $n\to\infty$.

Definition 2. By $M_1(u)\prec M_2(u)$ we denote, if there exist, positive constants u and k such that $M_1(u)\leqslant M_2(ku)$, $u>u_0$, as defined in [7], p. 15.

Definition 3. By $M_1(u)\sim M_2(u)$ we mean $M_1(u)$ and $M_2(u)$ are *equivalent*, i. e. $M_1(u)\prec M_2(u)$ and $M_2(u)\prec M_1(u)$.

By the definition of $L_\varphi^{**}(G)$ (for simplicity, we use the same notation L_φ^{**}), it is easy to see that

(i) $\|u\|_{\langle\varphi\rangle}=0$ if and only if $u(x)=0$ almost everywhere;

(ii) $\|\alpha u\|_{\langle\varphi\rangle}=|\alpha|\,\|u\|_{\langle\varphi\rangle}$, where α is any constant;

(iii) $\|u_1+u_2\|_{\langle\varphi\rangle}\leqslant\|u_1\|_{\langle\varphi\rangle}+\|u_2\|_{\langle\varphi\rangle}$.

The following theorem is a consequence of Theorem 3 (cf. [11], p. 72, and [12], p. 45, Theorem 1):

Theorem 7. *If $\varphi(x)\sim[p_1,p_2]$, $1\leqslant p_1\leqslant p_2\leqslant\infty$, then the set $L_\varphi^{**}(G)$ is a normed linear space. Moreover, it is a Banach space.*

From Lemma 9.2 in [7], we obtain

Theorem 8. *If $\|u\|_{\langle\varphi\rangle}\leqslant 1$, where $\varphi(x)\sim[p_1,p_2]$, $1\leqslant p_1\leqslant p_2\leqslant\infty$, then $u(x)\in L_{\Phi_1}(G)$, where $\Phi_1(x)=\int_0^x\{\varphi(t)/t\}\,dt$, and $\varrho(u;\Phi_1)\leqslant\|u\|_{\langle\varphi\rangle}$. Moreover, if $u\in L_\varphi^{**}$, then*

$$(3.2)\qquad \int_G \Phi_1\left[\frac{u(x)}{\|u\|_{\langle\varphi\rangle}}\right]dx\leqslant 1.$$

In addition, it follows from (3.1) *that if $p_2<\infty$, then*

$$(3.3)\qquad \int_G \varphi\left[\frac{u(x)}{\|u\|_{\langle\varphi\rangle}}\right]dx\leqslant p_2.$$

From Theorem 8 we conclude that if $p_2<\infty$ and if $\{u_n(x)\}$ converges in norm to $u_0(x)$ in L_φ^{**}, then $\{u_n(x)\}$ is also mean convergent to $u_0(x)$ in L_φ^{**}. Furthermore, from Theorem 4, (3.2) and (3.3), we see that the space L_φ^{**} is the linear hull ([6]) of the class L_{Φ_1} and it is also the linear hull of the class L_φ provided that $p_2<\infty$, since $\Phi_1(2u)\leqslant\Phi_1(u)+2^{p_2}\Phi_1(u)=K\Phi_1(u)$, so that the Δ_2-condition is satisfied. In other words, if p_2 is finite, then L_φ^{**} consists of all the functions which are product of some constants of L_{Φ_1}. More precisely, if we set $p_2<\infty$, then we have:

(α) If $f(x)\in L_\varphi^{**}$, where $\varphi\sim[p_1,p_2]$, $1\leqslant p_1\leqslant p_2<\infty$, then there exists $k>0$ such that $kf\in L_{\Phi_1}$;

(β) If $f(x)\in L_{\Phi_1}$, with $p_2<\infty$, then $kf\in L_\varphi^{**}$ for any positive constant k.

Theorem 9. *The space L_φ^{**} is a linear set if and only if $\varphi(x)$ satisfies the Δ_2-condition: $\varphi(2u)\leqslant K\varphi(u)$.*

([6]) Cf. [7], p. 75; here we mean "linear covering".

The result follows immediately from Theorem 8.2 in [7] and the following

LEMMA 4. *If $\varphi(t) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, and if*

$$\Phi_1(x) = \int_0^x \{\varphi(t)/t\}\,dt = \int_0^x \varphi_1(t)\,dt,$$

then $\Phi_1(x)$ satisfies the Δ_2-condition if and only if $\varphi(x)$ satisfies the Δ_2-condition.

Proof. If $\varphi(x)$ satisfies the Δ_2-condition, then

$$\Phi_1(2x) = \int_0^{2x} \varphi_1(t)\,dt = \Phi_1(x) + \int_x^{2x} \{\varphi(t)/t\}\,dt \tag{3.4}$$
$$\leqslant \Phi_1(x) + K\int_{x/2}^{x} \{\varphi(t)/t\}\,dt \leqslant K\Phi_1(x).$$

It follows that $\Phi_1(x)$ satisfies the Δ_2-condition. Conversely, if $\Phi_1(x)$ satisfies the Δ_2-condition, then, by Lemma 2, there exists a positive constant m, such that $\Phi_1(x) \sim [1, m\rangle$. It follows that by differentiating $\Phi_1(x)/x$ and $\Phi_1(x)/x^m$,

$$1 \leqslant \{\Phi_1'(t)/[\Phi_1(t)/t]\} = \{\varphi(t)/\Phi_1(t)\} \leqslant m. \tag{3.5}$$

This means $\varphi(t) \sim \Phi_1(t)$, and therefore $\varphi(t)$ satisfies the Δ_2-condition. The following theorem is a consequence of Theorem 9.4 in [7]:

THEOREM 10. *If the function $\varphi(x)$ satisfies the Δ_2-condition, then convergence in norm in L_φ^{**} is equivalent to mean convergence in L_φ^{**}.*

Let $K(x; \mathscr{E})$ denote the characteristic function of the set $\mathscr{E} \subset G$. Since $L_\varphi^{**}(G) = L_{\Phi_1}^{*}(G)$, by formula (9.11) in [7], we obtain:

THEOREM 11. *The norm of the characteristic function $K(x; \mathscr{E})$ is given by the formula*

$$\|K(x; \mathscr{E})\|_{\langle\varphi\rangle} = \operatorname{mes}\mathscr{E}\cdot\Psi_1^{-1}\left\{\frac{1}{\operatorname{mes}\mathscr{E}}\right\}, \tag{3.6}$$

where $\Psi_1(x)$ is the function $\int_0^x \psi_1(t)\,dt$, and $\psi_1(t)$ is the inverse function of $\varphi_1(t) = \varphi(t)/t$.

THEOREM 12. *The space L_φ^{**} defined above is separable if and only if the function $\varphi(t)$ satisfies the Δ_2-condition.*

This follows from Lemma 1 and Theorem 10.2 in [7].

THEOREM 13. *If the space L_φ^{**} is generated by $\varphi(x) \sim [p_1, p_2]$, where $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, then it is an Orlicz space $L_{\Phi_1}^{*}$, where $\Phi_1(x) \sim [p_1, p_2]$, i. e. with the same p_1 and p_2, and $\Phi_1(x) = \int_0^x \{\varphi(t)/t\}\,dt$.*

In fact, the result is an immediate consequence of the following

LEMMA 5. *If the function $\varphi(t)$ satisfies $\varphi(t) \sim [p_1, p_2]$, where $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, then $\Phi_1(t)$ also satisfies $\Phi_1(t) \sim [p_1, p_2]$, where $\Phi_1(t) = \int_0^t \{\varphi(u)/u\}\,du$.*

Proof. If $\varphi(t) \sim [p_1, p_2]$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, and if $0 < t < x$, then

$$\frac{\varphi(x)}{x}\left(\frac{t}{x}\right)^{p_2-1} \leqslant \frac{\varphi(t)}{t} \leqslant \frac{\varphi(x)}{x}\left(\frac{t}{x}\right)^{p_1-1}. \tag{3.7}$$

It follows that

$$\frac{1}{p_2}\,\frac{\varphi(x)}{x} \leqslant \frac{1}{x}\int_0^x \{\varphi(t)/t\}\,dt \leqslant \frac{1}{p_1}\,\frac{\varphi(x)}{x} \tag{3.8}$$

and

$$p_1\Phi_1(x)/x \leqslant \{\varphi(x)/x\} = \Phi_1'(x) \leqslant p_2\Phi_1(x)/x. \tag{3.9}$$

This means $\Phi_1(x)x^{-p_1}$ is non-decreasing and $\Phi_1(x)x^{-p_2}$ is non-increasing. Hence $\Phi_1(x) \sim [p_1, p_2]$. Now if $p_2 = \infty$, then one of the inequalities in each case of (3.7), (3.8), (3.9) holds. This implies $\Phi_1(x) \sim [p, \infty]$.

Definition 4. Let the Luxemburg norm (7) of $f(x) \in L_M^*$ be denoted by $\|f(x)\|_{(M)}$, which is equal to $\inf k$ where the infimum is taken over all $k > 0$ such that

$$\varrho\left(\frac{u(x)}{k}; M\right) = \int_G M\left[\frac{u(x)}{k}\right]dx \leqslant 1. \tag{3.10}$$

We define

$$\|u(x)\|_{[\varphi]} = \|u(x)\|_{(\Phi_1)}, \tag{3.11}$$

where $\|u\|_{(\Phi_1)}$ is the Luxemburg norm of $L_{\Phi_1}^{*} = L_\varphi^{**}$. It follows then from formula (9.24) in [7] that

$$\|u(x)\|_{[\varphi]} \leqslant \|u(x)\|_{\langle\varphi\rangle} \leqslant 2\,\|u(x)\|_{[\varphi]}, \tag{3.12}$$

and the *strengthened Hölder's inequality* (8) is

$$\left|\int_G u(x)v(x)\,dx\right| \leqslant \|u\|_{\langle\varphi\rangle}\|v\|_{(\Psi_1)}, \tag{3.13}$$

where $u \in L_{\Phi_1}^{*} = L_\varphi^{**}$, $v \in L_{\Psi_1}^{*}$,

$$\left|\int_G u(x)v(x)\,dx\right| \leqslant \|u\|_{[\varphi]}\|v\|_{\Psi_1}, \tag{3.14}$$

(7) For the definition, see [7], formulas (9.18) and (9.19).
(8) The results are obtained from (9.26) and (9.27) in [7].

where $u \in L_\varphi^{**}$, $v \in L_{\Psi_1}$. Corresponding to the theory of Orlicz spaces, we have corresponding formulas for the norm of the space L_φ^{**}:

THEOREM 14. *The norm of the space L_φ^{**} satisfies*

$$(3.15)\qquad \|u\|_{\langle\varphi\rangle} \leqslant 1 + \int_G \Phi_1[u(x)]\,dx = 1 + \int_G dx \int_0^{u(x)} \{\varphi(t)/t\}\,dt,$$

$$(3.16)\qquad \int_G \Phi_1\left[\frac{|u(x)|}{\|u\|_{\langle\varphi\rangle}}\right]dx = \int_G dx \int_0^{|u(x)|/\|u\|_{\langle\varphi\rangle}} \{\varphi(t)/t\}\,dt \leqslant 1.$$

Moreover, if $\|u\|_{\langle\varphi\rangle} \leqslant 1$, then

$$(3.17)\qquad \int_G \Phi_1[|u(x)|]\,dx = \int_G dx \int_0^{|u(x)|} \{\varphi(t)/t\}\,dt \leqslant \|u\|_{\langle\varphi\rangle}.$$

The results are immediate consequences of formulas (9.12), (9.14) and (9.21) in [7].

THEOREM 15. *In the space L_φ^{**}, a necessary and sufficient condition that convergence in norm is equivalent to mean convergence in L_φ^{**} is that the function $\varphi(t)$ should satisfy the Δ_2-condition.*

Since $L_\varphi^{**} = L_{\Phi_1}^{*}$, the result is immediate consequence of Lemma 4 and results in Chap. II, section 6 of [7].

Definition 5. By E_φ we denote, as in [7], section 10, the closure in L_φ^{*} of the set of bounded functions in the sense of convergence in norm. In other words, if

(i) $u_n(x) \to u_0(x)$, $u_n(x)$ are bounded;

(ii) $\|u_n(x) - u_0(x)\|_\varphi \to 0$, as $n \to \infty$,

then $u_0(x) \in E_\varphi$.

It is very curious that if $u_0(x) \in L_\varphi$ and if we set

$$(3.18)\qquad u_n(x) = \begin{cases} u_0(x) & \text{if} \quad |u_0(x)| \leqslant n, \\ 0 & \text{if} \quad |u_0(x)| > n, \end{cases}$$

then $\|u_n(x) - u_0(x)\|_\varphi$ may not tend to zero, in spite of the fact that $\varrho[u_n - u_0] = \int_G \varphi[u_n - u_0]\,dx$ tends to zero, as $n \to \infty$ (9).

Definition 6. By E'_φ we denote the closure in $L_\varphi^{**} = L_{\Phi_1}^{*}$ of the set of bounded functions in the sense of convergence in norm.

The following theorem is a consequence from Theorem 10.1 in [7] and Lemma 4:

THEOREM 16. *If $\varphi(x)$ does not satisfy the Δ_2-condition, then the set of bounded functions is nowhere dense in L_φ^{**} in the sense of convergence in norm. But if $\varphi(u)$ satisfies the Δ_2-condition, then $E_\varphi = L_\varphi^{*} = L_\varphi^{**} = E'_\varphi$.*

(9) See Lemma 10.1 in [7].

The set E'_φ may be considered as the maximal linear subspace of E_φ^{**} which is contained in L_{Φ_1}, since E_φ is the maximal linear subspace of L_φ^{*} contained in L_φ. This follows from the fact that if $\lambda u(x) \in L_\varphi$ for all values of λ, then $u(x) \in E_\varphi$ (10). To justify this assertion, we consider any given $\varepsilon > 0$, and set $\lambda = \varepsilon/2$. Since $u(x)/\lambda \in L_\varphi$, there exists a bounded function $v(x) = u_1(x)/\lambda$ such that

$$(3.19)\qquad \varrho\left[v - \frac{u}{\lambda}; \varphi\right] = \int_G \varphi\left[\frac{u_1(x) - u(x)}{\lambda}\right]dx \leqslant 1\ (11).$$

But $v(x) - u(x)/\lambda = (\lambda v - u)/\lambda = (u_1 - u)/\lambda$, where $u_1(x)$ is a bounded function. It follows that the Luxemburg norm satisfies

$$(3.20)\qquad \|u_1(x) - u(x)\|_{(\varphi)} \leqslant \lambda = \varepsilon/2.$$

Hence by formula (9.24) in [7], we obtain

$$(3.21)\qquad \|u_1(x) - u(x)\|_\varphi < \varepsilon.$$

This implies that there exists a sequence of bounded functions $\{u_n(x)\}$ such that

$$(3.22)\qquad \|u_n(x) - u(x)\|_\varphi \to 0, \quad \text{as} \quad n \to \infty.$$

Hence $u(x) \in E_\varphi$.

From Lemma 4 and Theorem 10.2 in [7], we obtain

THEOREM 17. *The space L_φ^{**} is separable if and only if $\varphi(x)$ satisfies he Δ_2-condition.*

4. Applications. It is of interest to consider some alternative forms of the norm in L_φ^{**}, since the usual formula does not allow us to actually carry out the computation of the norm.

THEOREM 18. *Let $u(x) \in L_\varphi^{**}$ and suppose there exists a positive number k^* such that*

$$(4.1)\qquad \int_G \Psi_1[\varphi(k^*|u(x)|)/k^*|u(x)|]\,dx = 1,$$

then

$$(4.2)\qquad \|u\|_{\langle\varphi\rangle} = \|u\|_{\Phi_1} = \frac{1}{k^*}\int_G \varphi[k^*|u(x)|]\,dx.$$

The results are direct consequences of Theorem 10,4 in [7]. We have another alternative expression of the norm in L_φ^{**}:

(10) This assertion is given, but without detailed proof in [7], p. 84 (English ed.).
(11) Cf. the proof of Theorem 10.1 in [7].

THEOREM 19. *Let* $u(x) \in L_{\varphi}^{**}$. *Then*

$$(4.3)\qquad \|u(x)\|_{\langle\varphi\rangle} = \|u(x)\|_{\Phi_1} = \inf_{k>0} \frac{1}{k}\Big(1+\int_G \Phi_1[k|u(x)|]\,dx\Big).$$

This coincides with Theorem 10.5 in [7] if we replace $M(u)$ by $\Phi_1(u)$.

THEOREM 20. *There exists a basis in the space* $E'_{\varphi}(G)$. (For the space $E'_{\varphi}(G)$, cf. Definition 6 above.)

This follows from Theorem 12.1 in [7] and the fact that $L_{\varphi}^{**}(G) = L_{\Phi_1}^{*}(G)$.

Concerning the space E'_{φ}, it is of interest to find a necessary and sufficient condition for the function $u(x)$ belonging to E'_{φ}.

THEOREM 21. *A necessary and sufficient condition that the function* $u(x) \in L_{\varphi}^{**}$ *should belong to* E'_{φ} *is that the space* L_{φ}^{**} *has an absolutely continuous norm, i. e. for every* $\varepsilon > 0$, *there is a* $\delta > 0$, *such that*

$$(4.4)\qquad \|u(x)K(x;\mathscr{E})\|_{\langle\varphi\rangle} < \varepsilon$$

is satisfied whenever mes $\mathscr{E} < \delta$, *where* $K(x;\mathscr{E})$ *is the characteristic function of* $\mathscr{E}$.

This follows from [7], Theorem 10.3, For a comparison of the spaces E'_{φ} and L_{Φ_1}, if we denote the totality of functions for which

$$(4.5)\qquad d(u, E'_{\varphi}) = \inf_{W \in E'_{\varphi}} \|u-w\|_{\langle\varphi\rangle} < r$$

by $\Pi(E'_{\varphi}; r)$, then we have

THEOREM 22. *Suppose that the function* $\varphi(x)$ *does not satisfy the* Δ_2-*condition; then*

$$(4.6)\qquad \Pi(E'_{\varphi};1) \subset L_{\Phi_1} \subset \overline{\Pi}(E'_{\varphi};1), \qquad \Pi(E'_{\varphi};1) \neq L_{\Phi_1} \neq \overline{\Pi}(E'_{\varphi};1).$$

This follows from Lemma 4 and Theorem 10.1 in [7].

Corresponding to A. N. Kolmogorov's compactness criterion for the space E_{φ} we have immediately the following criterion for the space E'_{φ} (cf. Theorem 11.1 in [7]):

THEOREM 23. *A family* $\mathfrak{N}$ *of functions of the space* E'_{φ} *is compact* (*with respect to* L_{φ}^{**}) *if and only if*

(a) $\|u\|_{\langle\varphi\rangle} \leqslant A$, $u(x) \in \mathfrak{N}$;

(b) *for arbitrary* $\varepsilon > 0$, *a* $\delta > 0$ *can be found such that the condition* $r < \delta$ *implies that* $\|u-u_r\|_{\langle\varphi\rangle} < \varepsilon$ *for all functions of the family* $\mathfrak{N}$. *Here* $u_r = u_r(x)$ *is the Steklov function defined by*

$$u_r(x) = \frac{1}{m_r}\int_{T_r(x)} u(t)\,dt,$$

where $T_r(x)$ *is the n-dimensional sphere with radius r and centre at x and* m_r *is the volume of this sphere* (cf. [7], formula (11.3)).

Corresponding to F. Riesz's compactness criterion for the space E_{φ}, we also have the following criterion for E'_{φ} (cf. Theorem 11.4 in [7]):

THEOREM 24. *A family* $\mathfrak{N}$ *of functions in the space* E'_{φ} *is compact* (*with respect to* L_{φ}^{**}) *if and only if the following conditions are satisfied*:

(a) $\|u\|_{\langle\varphi\rangle} \leqslant A$, $u(x) \in \mathfrak{N}$;

(b) *for arbitrary* $\varepsilon > 0$, *a* $\delta > 0$ *can be found such that* $d(h,0) < \delta$ *implies* $\|u(x+h)-u(x)\|_{\langle\varphi\rangle} < \varepsilon$ *for all* $u(x) \in \mathfrak{N}$, *where* $d(h,0)$ *is the distance in the n-dimensional Euclidean space* (*in case if x is a single variable*, $n = 1$) *between h and zero.*

Now suppose that as above defined $\Phi_1(x)$ and $\Psi_1(x)$ are mutually complementary functions where $\Phi_1(x) = \int_0^x \{\varphi(t)/t\}\,dt$. Let

$$(4.7)\qquad \mathscr{L}(u) = [u, v] = \int_G u(x)v(x)\,dx, \qquad u(x) \in L_{\varphi}^{**},$$

where $v(x)$ is a fixed function in $L_{\Psi_1}^{*}$. Then it follows from (1.13) and Banach-Steinhaus theorem [12] that $\mathscr{L}(u)$ is a linear functional defined on the entire space $L_{\varphi}^{**}(G)$. Let

$$(4.8)\qquad \|\mathscr{L}\| = \sup_{\|u\|_{\langle\varphi\rangle}\leqslant 1} |\mathscr{L}(u)|.$$

Then from [7], formula (14.2), we have

$$(4.9)\qquad \|\mathscr{L}\| \leqslant \|v\|_{\Psi_1} \leqslant 2\|\mathscr{L}\|.$$

Let $K(v) = \|v\|_{\Psi_1}/\|\mathscr{L}\|$. Then $1 \leqslant K(v) \leqslant 2$. To compute the value of $K(v)$ for L_{φ}^{**} defined by the function $\varphi(u) = |u|^{\alpha}$, $\alpha > 1$: since $\Phi_1(u) = u^{\alpha}/\alpha$, it follows that

$$(4.10)\qquad \|u\|_{\langle\varphi\rangle} = \|u\|_{\Phi_1} = \alpha^{1/\alpha}\beta^{1/\beta}\Big\{\int_G \Phi_1[|u|]\,dx\Big\}^{1/\alpha}.$$

The following estimate is obtained from the result in [7], section 14, p. 125:

$$(4.11)\qquad \|\mathscr{L}\| = \sup_{\|u\|_{\Phi_1}\leqslant 1}\Big|\int_G u(x)v(x)\,dx\Big| = \frac{\|v\|_{\Psi_1}}{\alpha^{1/\alpha}\beta^{1/\beta}}.$$

Hence we have

$$(4.12)\qquad K(v) = \alpha^{1/\alpha}\beta^{1/\beta}, \qquad v(x) \in \mathscr{L}_{\Psi_1}.$$

THEOREM 25. *Suppose that* $\varphi(u)$ *does not satisfy the* Δ_2-*condition. Then* (4.7) *is not the general form of a linear functional on* L_{φ}^{**}.

This follows from Theorem 14.1 in [7] and Lemma 4.

[12] See for example, [8], p. 135, and [2], p. 54.

THEOREM 26. *The formula* (4.7), *where* $v(x)\in L^*_{\Psi_1}$, *yields the general form of a linear functional on* E'_φ.

This follows from our definition for E'_φ and Theorem 14.2 in [7].

We come now the study of linear operators A, operating from one space $L^{**}_{\varphi_1}$ into another space $L^{**}_{\varphi_2}$. Basically we shall consider integral operators of the form

$$Au(x) = \int_G k(x, y)u(y)dy. \tag{4.13}$$

Definition 7. By $\hat{G}$ we denote the topological product $G\times G$ equipted with the natural measure. By $\hat{L}_\Phi$, $\hat{L}^*_\Phi$, $\hat{L}^{**}_\varphi$, $\hat{E}'_\varphi$ we shall denote the corresponding classes and spaces $L_\Phi(\hat{G})$, $L^*_\Phi(\hat{G})$, $L^{**}_\varphi(\hat{G})$ and $E'_\varphi(\hat{G})$.

The following theorem is an immediate consequence of Theorem 15.1 in [7]:

THEOREM 27. *Let* $\varphi(u)\sim[p_1, p_2]$, $1\leqslant p_1\leqslant p_2<\infty$, *be such that for* $u(x)\in L^{**}_{\varphi^{(1)}}$, $v(x)\in L_{\Psi_1^{(2)}}$

$$w(x, y) = u(y)v(x)\in \hat{L}^{**}_\varphi \tag{4.14}$$

and

$$\|w(x, y)\|_{\langle\hat{\varphi}\rangle}\leqslant l\|u\|_{\langle\varphi_1^{(1)}\rangle}\|v\|_{\Psi_1^{(2)}}, \tag{4.15}$$

where l is a constant. Suppose further that the kernel $k(x, y)$ of the linear integral (4.14) *belongs to the space* $\hat{L}^*_{\Psi_1}$, *where* $\Psi_1(v)$ *is the complementary N-function to the N-function*

$$\Phi_1(u) = \int_0^u \{\varphi(t)/t\}dt \qquad (u\geqslant 0),$$

and $\varphi = \varphi^{(1)}\times\varphi^{(2)}$. *Then the operator* $Au(x)$ *belongs to* $\{L^{**}_{\varphi(1)}\to L^{**}_{\varphi(2)}; c\}$ (*i. e. the operation mapping from* $L^{**}_{\varphi(1)}$ *to* $L^{**}_{\varphi(2)}$ *is continuous*). *There exists a function* φ *such that* (4.14) *and* (4.15) *are satisfied.*

We now consider an application to singular integrals. Let $L^{**}_\varphi(D)$ be the space defined by the function $\hat{\varphi}(D)$ on the bounded measurable set D of m-dimensional space, where $\hat{\varphi}(\bar{x}) = \varphi(x_1)\times\varphi(x_2)\times\ldots\times\varphi(x_m)$, with $\varphi(t)\sim[p_1, p_2]$, $1\leqslant p_1\leqslant p_2\leqslant\infty$. By our definition,

$$L^{**}_{\hat{\varphi}}(D) = L^*_{\hat{\Phi}_1}(D), \qquad \text{where} \qquad \Phi_1(x) = \int_0^x \{\varphi(t)/t\}dt.$$

Simonenko [15] has obtained a result concerning boundedness of singular integrals in Orlicz spaces. He also remarked that the result is applicable to estimating the higher order derivatives of elliptic equations, in the theory of one-dimensional singular equations and in boundary value problems for analytic functions. It is used in discussing some problems in mechanics. Let the singular integral Kf be defined by

$$Kf = \int_D \frac{\Omega(P;\theta)}{|P-Q|^m} f(Q)dQ, \tag{4.16}$$

where D is a bounded measurable set of m-dimensional space E_m, and $\Omega(P;\theta)$ satisfies the following conditions:

(i) $\int_{s_1}\Omega(P;\theta)ds_\theta = 0$, where s_1 is the unit sphere;

(ii) $\Omega(P;\theta)$ is continuous in θ for a fixed P;

(iii) $|\Omega(P;\theta_1)-\Omega(P;\theta_2)|\leqslant\omega(|\theta_1-\theta_2|)$, where ω does not depend on P and satisfies the condition

(iv) $\int_0^1 (\omega(t)/t)dt<\infty$;

(v) $|\Omega(P_1;\theta)-\Omega(P_2;\theta)|\leqslant B|P_1-P_2|^\alpha$, where $\alpha>0$ and B is a constant.

Let

$$M(u) = \int_0^{|u|} p(t)dt,$$

where $p(t)$ is a non-decreasing function, such that $1<\beta\leqslant up(u)/M(u)\leqslant\alpha$ holds, and let $L^*_M(D)$ be the corresponding Orlicz space defined by the bounded measurable set D of E_m. Simonenko proved that the singular operator (4.16) is defined and bounded in the space $L^*_M(D)$, i. e.

$$\|Kf\|_M\leqslant C\|f\|_M, \tag{4.17}$$

where C depends on α, β, Ω, D, only. In fact, the condition $1<\beta\leqslant\leqslant up(u)/M(u)\leqslant\alpha$ is equivalent to $M(u)\sim[\beta,\alpha]$, where $M(u)$ is a convex function. It follows that if we assume $\varphi(u)\sim[\beta,\alpha]$, then by our Lemma 5, we also have $\Phi_1(u)\sim[\beta,\alpha]$, where $\Phi_1(u) = \int_0^u\{\varphi(t)/t\}dt$ $(u>0)$. Thus we obtain a generalization of Simonenko's theorem:

THEOREM 28. *Let* $\varphi(u)\sim[\beta,\alpha]$, *where* $1<\beta\leqslant\alpha<\infty$. *Then the singular operator* (4.16) *is defined and bounded in the space* $L^{**}_\varphi(D) = L^{**}_{\Phi_1}(D)\supset L^*_\varphi$, $L^{**}_\varphi\neq L^*_\varphi$, *where* $\Omega(P;\theta)$ *satisfies the above defined conditions.*

5. Linear functionals defined by functions satisfying generalized Lipschitz conditions. Let U denote an operation which transforms every integrable function $f(x)$ to a sequence of Fourier constants $y = \{a_1, b_1, \ldots, a_n, b_n, \ldots\}$ where a_i, b_i are the i-th Fourier cosine coefficient, i-th Fourier sine coefficient of $f(x)$, respectively.

THEOREM 29. (a) *Let* $\alpha(t) \sim \langle 1/p-\frac{1}{2}, \infty\rangle$, $0 < p \leqslant 2$ [13], *and let* $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$. *Then the operation* U *maps the space* $\mathrm{Lip}\,\alpha(t)$ *linear in the space* $R_{\psi(n),p}$ *where* $\psi(t) = \alpha(t)t^{1/2-1/p}$ (*this implies* $\psi(t) \sim \langle 0, \infty])$.

In other words, the operation U *is said to be linear* [14] *if*

$$U(f_1+f_2) = U(f_1)+U(f_2) \tag{5.1}$$

and

$$\|Uf\|_{R_{\psi,p}} \leqslant K\|f\|_{\mathrm{Lip}\,\alpha(t)}, \tag{5.2}$$

where

$$\|Uf\|_{R_{\psi,p}} = \mathfrak{R}_{\psi,p}[y] = \frac{\{\sum_{k=n}^{\infty}|a_k|^p+|b_k|^p\}^{1/p}}{\psi(1/n)}, \tag{5.3}$$

$$\|f\|_{\mathrm{Lip}\,\alpha(t)} = \operatorname*{Ess\,max}_{h,x} \frac{|f(x+h)-f(x)|}{\alpha(h)}. \tag{5.4}$$

(b) *Let* $\alpha(t) \sim \langle 0, 1/p-\frac{1}{2}\rangle$, $0 < p < 2$, *and let* $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$. *Then the operation* U *maps the space* $\mathrm{Lip}\,\alpha(t)$ *linear in the space* $M_{\psi(n),p}$, *where now* $\psi(t) = \alpha(1/t)t^{1/p-1/2}$.

(c) *If* $p \geqslant 2$, $\alpha(t) \sim \langle 0, \infty\rangle$, *then the operation* U *maps the space* $\mathrm{Lip}\,\alpha(t)$ *linear in* $R_{\alpha(n),p}$.

The result includes Satz 1 in [9] as an important particular case with $\alpha(t) = t^{\alpha}$.

For the proof of (a), let us observe that

$$|f(x+h)-f(x-h)| \leqslant K\alpha(h)\|f\|_{\mathrm{Lip}\,\alpha(t)}. \tag{5.5}$$

Setting $h = \pi/4n$, and in virtue of $\alpha(x) \sim [1/p-\frac{1}{2}+\varepsilon, N]$ for some $N < \infty$ and

$$\{f(x+h)-f(x-h)\} \sim -2\sum_{1}^{\infty} a_n \sin nh \sin nx + 2\sum_{1}^{\infty} b_n \sin nh \cos nx, \tag{5.6}$$

we infer from Parseval's relation and Hölder's inequality that

$$\sum_{k=n}^{2n-1}(a_k^2+b_k^2) \leqslant K\|f\|^2_{\mathrm{Lip}\,\alpha(t)}\left\{\alpha\left(\frac{\pi}{4n}\right)\right\}^2 \leqslant K\left\{\alpha\left(\frac{1}{n}\right)\right\}^2\|f\|^2, \tag{5.7}$$

$$\sum_{k=n}^{2n-1}(|a_k|^p+|b_k|^p) \leqslant \Big\{\sum_{k=n}^{2n-1}(a_k^2+b_k^2)\Big\}^{p/2}(2n)^{1-p/2} \leqslant K\left\{\alpha\left(\frac{1}{n}\right)\right\}^p n^{1-p/2}\|f\|^p. \tag{5.8}$$

[13] This theorem is a generalization of Theorem 1 in Lorentz's paper [9]. The power p in Lorentz's work is restricted as to satisfy $1 < p < 2$ only, but this has been generalized to $0 < p \leqslant 2$ in N. K. Bary [1], p. 208.

[14] See [2], p. 54.

From the hypothesis $\alpha(x) \sim \langle 1/p-\frac{1}{2}, \infty\rangle$; it follows that there exists a positive constant ε, such that

$$\left(\frac{1}{p}-\frac{1}{2}+\varepsilon\right)\frac{\alpha(u)}{u} \leqslant \alpha'(u),$$

where $\alpha'(u)$ exists almost everywhere. On the other hand, it follows then from (5.8) that

$$\begin{aligned}\Big\{\sum_{k=n}^{\infty}(|a_k|^p+|b_k|^p)\Big\}^{1/p} &= \Big\{\sum_{j=0}^{\infty}\sum_{k=2^jn}^{2^{j+1}n-1}(|a_k|^p+|b_k|^p)\Big\}^{1/p} \\ &\leqslant K\Big\{\sum_{j=0}^{\infty}2^{(1-p/2)j}[\alpha(2^{-j}/n)]^p\Big\}^{1/p} n^{(1/p-1/2)}\|f\| \\ &\leqslant Kn^{(1/p-1/2)}\Big\{\int_0^{\infty}2^{(1-p/2)t}\alpha^p(2^{-t}/n)\,dt\Big\}^{1/p}\|f\|,\end{aligned} \tag{5.9}$$

where

$$\int_0^{\infty}2^{(1-p/2)t}\alpha^p(2^{-t}/n)\,dt = \frac{n^{(p/2-1)}}{\log 2}\int_0^{1/n}u^{-(2-p/2)}\alpha^p(u)\,du = Kn^{(p/2-1)}I, \tag{5.10}$$

say. Since

$$\begin{aligned}I = \int_0^{1/n}u^{-(2-p/2)}\alpha^p(u)\,du &= \left(\frac{p}{2}-1\right)^{-1}n^{(1-p/2)}\alpha^p\left(\frac{1}{n}\right)+ \\ &\quad+\left(\frac{1}{p}-\frac{1}{2}\right)^{-1}\int_0^{1/n}u^{(p/2-1)}\alpha^{(p-1)}(u)\,\alpha'(u)\,du,\end{aligned} \tag{5.11}$$

and in virtue of

$$\alpha'(u) \geqslant \left(\frac{1}{p}-\frac{1}{2}+\varepsilon\right)\frac{\alpha(u)}{u},$$

we find

$$(1+\varepsilon'-1)\int_0^{1/n}u^{-(2-p/2)}\alpha^p(u)\,du \leqslant \left(1-\frac{p}{2}\right)^{-1}n^{(1-p/2)}\alpha^p\left(\frac{1}{n}\right), \tag{5.12}$$

where $\varepsilon' = \varepsilon(1/p-\frac{1}{2})^{-1}$. From (5.9), (5.10), (5.11) and (5.12), we obtain

$$\begin{aligned}\psi\left(\frac{1}{n}\right)\|vf\|_{R_{\psi,p}} = \psi\left(\frac{1}{n}\right)\mathfrak{R}_{\psi,p}[vf] &= \Big\{\sum_{k=n}^{\infty}(|a_k|^p+|b_k|^p)\Big\}^{1/p} \\ &\leqslant K_{\varepsilon,p}n^{(1/p-1/2)}\alpha\left(\frac{1}{n}\right)\|f\|_{\mathrm{Lip}\,\alpha(t)},\end{aligned} \tag{5.13}$$

which is the required result (5.2).

Corollary 1. *Setting* $p=2$, *we get, for* $\alpha(t)\sim\langle 0,\infty\rangle$,

$$\left\{\sum_{k=n}^{\infty}(a_k^2+b_k^2)\right\}^{1/2}=O\left\{\alpha\left(\frac{1}{n}\right)\right\}. \tag{5.14}$$

Corollary 2. *If* $p=1$, *then we have* $\alpha(t)\sim\langle\frac{1}{2},\infty\rangle$,

$$\sum_{n}^{\infty}|a_k|+|b_k|=O\left\{n^{1/2}\alpha\left(\frac{1}{n}\right)\right\}. \tag{5.15}$$

Proof of (b). Let $2^k\leqslant n<2^{k+1}$. Then, by (5.8), we obtain

$$\mathfrak{M}_{\psi,p}[Uf]\psi(n)=\left\{\sum_{i=1}^{n}(|a_i|^p+|b_i|^p)\right\}^{1/p}, \tag{5.16}$$

$$\begin{aligned}\{\psi(n)\}^p\mathfrak{M}^p_{\psi,p}[Uf]&\leqslant\sum_{i=0}^{k}\sum_{i=2^j}^{2^{j+1}-1}(|a_i|^p+|b_i|^p)\\ &\leqslant K\left\{\sum_{j=0}^{k}2^{j(1-p/2)}[\alpha(2^{-j})]^p\right\}\|f\|^p_{\mathrm{Lip}\,\alpha(t)}\\ &\leqslant K\|f\|^p\int_0^k 2^{(1-p/2)t}\{\alpha(2^{-t})\}^p dt\\ &=-K(\|f\|^p/\log 2)\int_1^{2^{-k}}u^{-(2-p/2)}\{\alpha(u)\}^p du\\ &\leqslant\frac{K\|f\|}{\log 2}\int_{1/n}^{1}u^{-(2-p/2)}\{\alpha(u)\}^p du=K\|f\|^p I,\end{aligned} \tag{5.17}$$

say, where

$$\begin{aligned}I&=\int_{1/n}^{1}u^{-(2-p/2)}\{\alpha(u)\}^p du\\ &=u^{-(1-p/2)}\{\alpha(u)\}^p|_{1/n}^{1}+\left(2-\frac{p}{2}\right)\int_{1/n}^{1}u^{-(2-p/2)}\{\alpha(u)\}^p du-\\ &\quad -p\int_{1/n}^{1}u^{-(1-p/2)}\{\alpha(u)\}^{p-1}\alpha'(u)\,du.\end{aligned} \tag{5.18}$$

Since $\alpha(u)\sim\langle 0,1/p-\frac{1}{2}\rangle$ implies $\alpha(u)\sim[\varepsilon,1/p-\frac{1}{2}-\varepsilon]$ for some $\varepsilon>0$, it follows that $\varepsilon\alpha(u)\leqslant u\alpha'(u)\leqslant(1/p-\frac{1}{2}-\varepsilon)\alpha(u)$. Substituting

this in (5.18), we find

$$\left\{\left(1-\frac{p}{2}\right)-p\left(\frac{1}{p}-\frac{1}{2}-\varepsilon\right)\right\}I\leqslant n^{(1-p/2)}\left\{\alpha\left(\frac{1}{n}\right)\right\}^p, \tag{5.19}$$

$$I\leqslant K_{p,\varepsilon}n^{(1-p/2)}\left\{\alpha\left(\frac{1}{n}\right)\right\}^p. \tag{5.20}$$

This means

$$\|Uf\|_{M_{\psi,p}}\leqslant K_{p,\varepsilon}\|f\|_{\mathrm{Lip}\,\alpha(t)}, \tag{5.21}$$

provided that $\psi(t)=\alpha(1/t)t^{1/p-1/2}$.

Proof of (c). In a similar way as in [9], p. 135, we see that by (a) with $p=2$,

$$\left\{\sum_{k=n}^{\infty}(a_k^2+b_k^2)\right\}^{1/2}\leqslant K\|f\|\psi\left(\frac{1}{n}\right)=K\|f\|\alpha\left(\frac{1}{n}\right), \tag{5.22}$$

where $\alpha(t)\sim\langle 0,\infty\rangle$. Then, for $p\geqslant 2$,

$$\left\{\sum_{k=n}^{\infty}(|a_k|^p+|b_k|^p)\right\}^{1/p}\leqslant\sup_{k\geqslant n}\max\{|a_k|,|b_k|\}^{(p-2)/p}\left\{\sum_{k=n}^{\infty}(a_k^2+b_k^2)\right\}^{1/p}. \tag{5.23}$$

From (5.7) it is easy to see that $\max_n\{|a_n|,|b_n|\}/\alpha(1/n)\leqslant K\|f\|_{\mathrm{Lip}\,\alpha(t)}$. It follows that $R_{\alpha(t),\infty}\leqslant K\|f\|_{\mathrm{Lip}\,\alpha(t)}$. Hence

$$\left\{\sum_{k=n}^{\infty}(|a_k|^p+|b_k|^p)\right\}^{1/p}\leqslant K\alpha\left(\frac{1}{n}\right)\|f\|, \tag{5.24}$$

and the proof is completed.

Theorem 30. *Let* $\psi(x)=\alpha(x)x^{-1/p'}$, $\alpha(x)\sim\langle 1/p',1+1/p'\rangle$, *where* $1\leqslant p\leqslant\infty$, $1/p+1/p'=1$, *and let* $T=U^{-1}$ *be an operation which ts the inverse of* U *defined above. Then the functional* T *maps the space* $R_{\alpha(n),p}$ *linear in the space* $\mathrm{Lip}\,\psi(x)$.

This theorem includes Satz 2 in [9] as a particular case when $\alpha(x)=x^{\alpha}$, $1/p'<\alpha<1+1/p'$. In fact, by Hölder's inequality,

$$\begin{aligned}\sum_{k=n}^{2n-1}(|a_k|+|b_k|)&\leqslant\left\{\sum_{k=n}^{2n-1}(|a_k|^p+|b_k|^p)\right\}^{1/p}(2n)^{1/p'},\\ &\leqslant 2^{1/p'}\|y\|_{R_{\alpha(t),p}}n^{1/p'}\alpha\left(\frac{1}{n}\right)=Kn^{1/p'}\alpha\left(\frac{1}{n}\right)\|y\|_{R_{\alpha(t),p}}\end{aligned} \tag{5.25}$$

and

(5.26) $$|f(x+h)-f(x-h)| \leqslant 2\sum_{j=1}^{n}\sum_{k=2^{j-1}}^{2^j-1}(|a_k|+|b_k|)k|h| + 2\sum_{k=2^n}^{\infty}(|a_k|+|b_k|)\ ({}^{15}).$$

It follows that

(5.27) $$\tau_n^{(1)} = \sum_{j=n}^{\infty}\sum_{k=2^j}^{2^{j+1}-1}(|a_k|+|b_k|) \leqslant K\|y\|\sum_{j=n}^{\infty}(2^j)^{1/p'}\alpha\left(\frac{1}{2^j}\right) \leqslant K\|y\|\int_n^{\infty}2^{t/p'}\alpha(2^{-t})\,dt.$$

The last integral on the right is

(5.28) $$\int_n^{\infty}2^{t/p'}\alpha(2^{-t})\,dt = -\left(\frac{p'}{\log 2}\right)2^{n/p'}\alpha(2^{-n})+p'\int_n^{\infty}2^{t/p'}\alpha'(2^{-t})2^{-t}dt \geqslant -\frac{p'}{\log 2}2^{n/p'}\alpha(2^{-n})+(1+p'\varepsilon)\int_n^{\infty}2^{t/p'}\alpha(2^{-t})\,dt,$$

for some $\varepsilon>0$, and is equivalent to

(5.29) $$\int_n^{\infty}2^{t/p'}\alpha(2^{-t})\,dt \leqslant \frac{1}{\varepsilon\log 2}2^{n/p'}\alpha(2^{-n}).$$

On the other hand, by (5.25), we have

(5.30) $$\sum_{j=1}^{n}\sum_{k=2^{j-1}}^{2^j-1}(|a_k|+|b_k|)k \leqslant K\|y\|\sum_{j=1}^{n}2^j2^{(j-1)/p'}\alpha(2^{-(j-1)}) \leqslant K\|y\|\int_0^n 2^{(1+1/p')t}\alpha(2^{-t})\,dt.$$

(15) This means that if the right-hand member is finite, then the inequality holds p. p. For if $f(x) \sim \frac{1}{2}a_0+\sum_1^{\infty}a_n\cos nx+b_n\sin nx$ and if $\sum_1^{\infty}|a_n|+|b_n|<\infty$, then $f(x) = \frac{1}{2}a_0+\sum_1^{\infty}a_n\cos nx+b_n\sin x$ p. p. To see this, let us denote by $\varphi(x) = f(x)-\frac{1}{2}a_0-\sum_1^{\infty}a\cos nx+b\sin nx$. Then the Fourier series of $f(x)$ converges uniformly and may be termwise integrated. It follows then that the Fourier constants of $\varphi(x)$ are all zero. In virtue of Parseval's relation, we see that $\int_0^{2\pi}\varphi^2(x)\,dx = 0$. Hence $\varphi(x) = 0$ p. p.

Since $\alpha(x) \sim \langle 1/p', 1+1/p'\rangle$, it follows that $x\alpha'(x) \leqslant (1+1/p'-\varepsilon)\alpha(x)$ for some $\varepsilon>0$, and

(5.31) $$\int_0^n 2^{(1+1/p')t}\alpha(2^{-t})\,dt = \frac{1}{(1+1/p')\log 2}\{2^{(1+1/p')n}\alpha(2^{-n})-\alpha(1)\}+\frac{1}{(1+1/p')}\int_0^n 2^{(1+1/p')t}\alpha'(2^{-t})2^{-t}dt \leqslant \frac{1}{(1+1/p')\log 2}2^{(1+1/p')n}\alpha(2^{-n})+\left(1-\frac{\varepsilon}{(1+1/p')}\right)\int_0^n 2^{(1+1/p')t}\alpha(2^{-t})\,dt,$$

which yields

(5.32) $$\int_0^n 2^{(1+1/p')t}\alpha(2^{-t})\,dt \leqslant \frac{1}{\varepsilon\log 2}2^{(1+1/p')n}\alpha(2^{-n}).$$

Substituting (5.29) and (5.32) in (5.27) and (5.30), respectively, and then in (5.26) together with $1/2^{n+1}<|h|\leqslant 1/2^n$, we obtain

(5.33) $$|f(x+h)-f(x-h)| \leqslant K\|y\|\{|h|2^{(1+1/p')n}\alpha(2^{-n})+2^{n/p'}\alpha(2^{-n})\} \leqslant K\|y\|h^{-1/p'}\alpha(h) = K\|y\|\psi(h).$$

The result is a generalization of Satz 2 in [9]. In particular, if $p=1$, $p'=\infty$, then $\psi(t)=\alpha(t)$ and $\|f\|_{\mathrm{Lip}\,\alpha(t)} \leqslant K\|y\|_{R_{\alpha(t),1}}$, where $\alpha(t)\sim\langle 0,1\rangle$.

The following result is a generalization of Satz 3 in [9]:

THEOREM 31. *Suppose that*

(5.34) $$\left\{\sum_n^{\infty}(|a_k|^p+|b_k|^p)\right\}^{1/p} \leqslant K\alpha\left(\frac{1}{n}\right)\|y\|_{R_{\alpha(n),p}},$$

where $1\leqslant p\leqslant 2$, $\alpha(t)\sim\langle 0,1\rangle$, $y=\{a_1,b_1,\ldots,a_n,b_n,\ldots\}$. *Then the set* $\{y\}$ *of Fourier constants maps* $R_{\alpha(n),p}$ *linear in the space* $\mathrm{Lip}\{\alpha(t),p'\}$, *where* $1/p+1/p'=1$.

Proof. By Hausdorff-Young's theorem, we have

(5.35) $$\left\{\int_{-\pi}^{\pi}|f(t+h)-f(t-h)|^{p'}dt\right\}^{1/p'} \leqslant K\left\{\sum_{k=1}^{\infty}(|a_k|^p+|b_k|^p)|\sin kh|^p\right\}^{1/p} \leqslant K\sum_{\nu=0}^{n-1}\left\{\sum_{k=2^\nu}^{2^{\nu+1}-1}k^p(|a_k|^p+|b_k|^p)\right\}^{1/p}|h|+K\left\{\sum_{k=2^n}^{\infty}(|a_k|^p+|b_k|^p)\right\}^{1/p}.$$

Since

(5.36) $$\sum_{k=n}^{2n-1}k^p(|a_k|^p+|b_k|^p) \leqslant (2n)^p\sum_{k=n}^{\infty}(|a_k|^p+|b_k|^p) \leqslant 2^p\|y\|^pn^p\left[\alpha\left(\frac{1}{n}\right)\right]^p,$$

and

$$(5.37)\qquad \int_0^{n-1} 2^t\alpha(2^{-t})\,dt \leqslant \frac{1}{\log 2}\int_0^n \alpha(2^{-t})\,d2^t$$

$$= \frac{1}{\log 2}[2^t\alpha(2^{-t})]_0^n + \int_0^n 2^t 2^{-t}\alpha'(2^{-t})\,dt$$

$$\leqslant \frac{1}{\log 2}2^n\alpha(2^{-n}) + (1-\varepsilon)\int_0^n 2^t\alpha(2^{-t})\,dt,$$

which implies

$$(5.38)\qquad \int_0^{n-1} 2^t\alpha(2^{-t})\,dt \leqslant \frac{1}{\varepsilon\log 2}2^n\alpha(2^{-n}).$$

It follows that

$$(5.39)\qquad \Big\{\int_{-\pi}^{\pi}|f(t+h)-f(t-h)|^{p'}dt\Big\}^{1/p'} \leqslant K\|y\|\Big\{|h|\sum_{\nu=0}^{n-1}2^\nu\alpha(2^{-\nu})+\alpha(2^{-n})\Big\}$$

$$\leqslant K\|y\|_{R_{\alpha(n)},p}\,\alpha(h),$$

where $1/2^{n+1} < |h| \leqslant 1/2^n$. Hence

$$(5.40)\qquad \|f\|_{\mathrm{Lip}\,\{\alpha(t),p\}} \leqslant K\|y\|_{R_{\alpha(n)},p}.$$

THEOREM 32. *Let $f(x) \sim \sum_{k=1}^{\infty} A_{n_k}(x)$ be a lacunary trigonometric series. Then a necessary and sufficient condition that $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$, where $\alpha(t) \sim \langle 0,1\rangle$, is that $a_n = 0\{\alpha(1/n)\}$ and $b_n = 0\{\alpha(1/n)\}$, as $n\to\infty$.*

Proof. To show that the condition is sufficient write $n_{k+1}/n_k \geqslant \lambda > 1$, and

$$(5.41)\qquad \tau_n^{(1)} = \sum_n^\infty |a_k|+|b_k| \leqslant K\sum_{n_k\geqslant n}^{k=\infty}\alpha\left(\frac{1}{n_k}\right) \leqslant K\int_1^\infty \alpha(n^{-1}\lambda^{-t})\,dt.$$

Since $\alpha(t) \sim \langle 0,1\rangle$, there exists a positive contant $\varepsilon > 0$ such that

$$(5.42)\qquad \frac{\alpha(n^{-1}\lambda^{-t})}{(n^{-1}\lambda^{-t})^\varepsilon} \leqslant \frac{\alpha(n^{-1})}{(n^{-1})^\varepsilon},\qquad t\geqslant 1.$$

It follows that

$$(5.43)\qquad \alpha(n^{-1}\lambda^{-t}) \leqslant \lambda^{-\varepsilon t}\alpha\left(\frac{1}{n}\right)$$

and

$$(5.44)\qquad \tau_n^{(1)} \leqslant K\int_1^\infty \lambda^{-\varepsilon t}\alpha\left(\frac{1}{n}\right)dt = K\alpha\left(\frac{1}{n}\right).$$

Hence by Theorem 30, $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$.

To show that the condition is also necessary suppose now that $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$. From Theorem 29, with $p=2$, we see that

$$(5.45)\qquad \tau_n^{(2)} = \Big\{\sum_n^\infty a_k^2+b_k^2\Big\}^{1/2} = O\left\{\alpha\left(\frac{1}{n}\right)\right\},\quad \text{as}\quad n\to\infty,$$

which implies that $a_n = O\{\alpha(1/n)\}$, $b_n = O\{\alpha(1/n)\}$, as $n\to\infty$. Hence the result. Theorem 31 is a generalization of Satz 6 in [9].

THEOREM 33 ([16]). *If $a_n \searrow 0$, $b_n\searrow 0$, as $n\to\infty$, and if $\alpha(x)\sim\langle 0,1\rangle$,*

$$(5.46)\qquad f(x) = \sum_1^\infty a_n\cos nx,\qquad g(x) = \sum_1^\infty b_n\sin nx,$$

then $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$ if and only if

$$a_n = O\left\{\frac{1}{n}\,\alpha\left(\frac{1}{n}\right)\right\},$$

and $g(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$ if and only if

$$b_n = O\left\{\frac{1}{n}\,\alpha\left(\frac{1}{n}\right)\right\}.$$

We only consider the function $f(x)$. The proof for $g(x)$ follows in a similar way. In fact, if we put $h=\pi/n$, $f(x)\,\epsilon\,\mathrm{Lip}\,\alpha(t)$, then

$$(5.47)\qquad |f(x)-f(x+h)| < \alpha(h).$$

This implies that

$$(5.48)\qquad 2\sum_{k=[n/2]}^n a_k\sin^2(k/2n) \leqslant K\alpha\left(\frac{1}{n}\right)$$

and

$$(5.49)\qquad \sum_{k=[n/2]}^n a_k \leqslant K\alpha\left(\frac{1}{n}\right),\qquad a_n\leqslant K\left(\frac{1}{n}\right)\alpha\left(\frac{1}{n}\right).$$

Conversely, if

$$a_n = O\left\{\frac{1}{n}\,\alpha\left(\frac{1}{n}\right)\right\},$$

then there exists $\varepsilon > 0$ such that

$$(5.50)\qquad \tau_n^{(1)} = \sum_n^\infty |a_k|+|b_k| \leqslant K\int_n^\infty t^{-1}\alpha(t^{-1})\,dt$$

$$= K\alpha\left(\frac{1}{n}\right) + K\int_n^\infty \alpha'(t^{-1})t^{-2}\,dt \leqslant K\alpha\left(\frac{1}{n}\right) + K(1-\varepsilon)\int_n^\infty t^{-1}\alpha(t^{-1})\,dt,$$

([16]) Cf. [1], p. 678/9.

which implies

$$\tau_n^{(1)} \leqslant (K/\varepsilon)\, a\left(\frac{1}{n}\right). \tag{5.51}$$

Hence by Theorem 30 with $p = 1$, $p' = \infty$, we have $f(x) \in \operatorname{Lip} a(t)$.

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On symmetric derivatives in L^p

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Chapter I

1. It is a familiar fact that symmetric properties of functions play an important role in a number of problems. This is particularly true of problems in the theory of trigonometric series. The first symmetric derivative (Lebesgue's derivative), the second symmetric derivative (Schwarz's derivative) and their generalizations are familiar notions in the theory of trigonometric series, and the investigation of their properties is a legitimate topic in Real Variables. In this note we study symmetric derivatives associated with the metric L^p. We begin by recalling familiar facts and definitions. We consider only measurable functions.

Suppose a function $f(x)$ is defined in a neighborhood of a point x_0. If there is a polynomial $P(t) = P_{x_0}(t)$ of degree k such that

$$f(x_0+t) = P(t) + o(t^k) \quad (t \to 0), \tag{1.1}$$

we say that f has at x_0 a *k-th* (*unsymmetric*) *derivative* in the sense of Peano, and that the value of this derivative is a_k if $a_k/k!$ is the *coefficient* of t^k in $P(t)$.

We now define the symmetric derivative (sometimes called the de la Vallée-Poussin derivative) of order k. If there exists a polynomial $P(t) = P_{x_0}(t)$ of degree k such that

$$\tfrac{1}{2}\{f(x_0+t) + (-1)^k f(x_0-t)\} = P(t) + o(t^k) \quad (t \to 0), \tag{1.2}$$

then we say that f has at x_0 a *k-th symmetric derivative* and this derivative is a_k if $a_k/k!$ is the leading coefficient of $P(t)$. It is clear that if k is even then P has only even powers of t, and if k is odd, only odd powers.

If f has an unsymmetric k-th derivative at a point it also has a symmetric k-th derivative and both derivatives are equal. The converse is obviously false but the following result is known to be true (see [1]):

THEOREM A. *If $f(x)$ has a k-th symmetric derivative at each point of a set E, then f has an unsymmetric k-th derivative almost everywhere in E.*