which implies

$$\tau_n^{(1)} \leqslant (K/\varepsilon)\,\alpha\left(\frac{1}{n}\right). \tag{5.51}$$

Hence by Theorem 30 with $p = 1$, $p' = \infty$, we have $f(x) \epsilon \operatorname{Lip} \alpha(t)$.

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On symmetric derivatives in L^p

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Chapter I

1. It is a familiar fact that symmetric properties of functions play an important role in a number of problems. This is particularly true of problems in the theory of trigonometric series. The first symmetric derivative (Lebesgue's derivative), the second symmetric derivative (Schwarz's derivative) and their generalizations are familiar notions in the theory of trigonometric series, and the investigation of their properties is a legitimate topic in Real Variables. In this note we study symmetric derivatives associated with the metric L^p. We begin by recalling familiar facts and definitions. We consider only measurable functions.

Suppose a function $f(x)$ is defined in a neighborhood of a point x_0. If there is a polynomial $P(t) = P_{x_0}(t)$ of degree k such that

$$f(x_0+t) = P(t) + o(t^k) \quad (t \to 0), \tag{1.1}$$

we say that f has at x_0 a *k-th* (*unsymmetric*) *derivative* in the sense of Peano, and that the value of this derivative is a_k if $a_k/k!$ is the *coefficient* of t^k in $P(t)$.

We now define the symmetric derivative (sometimes called the de la Vallée-Poussin derivative) of order k. If there exists a polynomial $P(t) = P_{x_0}(t)$ of degree k such that

$$\tfrac{1}{2}\{f(x_0+t) + (-1)^k f(x_0-t)\} = P(t) + o(t^k) \quad (t \to 0), \tag{1.2}$$

then we say that f has at x_0 a *k-th symmetric derivative* and this derivative is a_k if $a_k/k!$ is the leading coefficient of $P(t)$. It is clear that if k is even then P has only even powers of t, and if k is odd, only odd powers.

If f has an unsymmetric k-th derivative at a point it also has a symmetric k-th derivative and both derivatives are equal. The converse is obviously false but the following result is known to be true (see [1]):

THEOREM A. *If $f(x)$ has a k-th symmetric derivative at each point of a set E, then f has an unsymmetric k-th derivative almost everywhere in E.*

Let now f be defined almost everywhere in some neighborhood of x_0 and belong to some L^p in that neighborhood; here $1 \leqslant p < \infty$. If there exists a polynomial $P(t) = P_{x_0}(t)$ of degree k for which

$$\left\{\frac{1}{h}\int_0^h |f(x_0+t)-P(t)|^p dt\right\}^{1/p} = o(h^k) \quad (h \to 0), \tag{1.3}$$

we say that f has at x_0 a *k-th unsymmetric derivative in the L^p mean*, or just *in L^p*. As before, the value of the derivative is a_k if $a_k/k!$ is the leading coefficient of $P(t)$. This definition was considered in [2].

The definition of the k-th symmetric derivative in L^p is now obvious. Let $f(x)$ be defined almost everywhere in some neighborhood of x_0 and belong to L^p, $1 \leqslant p < \infty$, there. If there is a polynomial $P(t) = P_{x_0}(t)$ of degree k for which

$$\left\{\frac{1}{h}\int_0^h |\tfrac{1}{2}\{f(x_0+t)+(-1)^k f(x_0-t)\}-P(t)|^p dt\right\}^{1/p} = o(h^k) \quad (h \to +0), \tag{1.4}$$

f will be said to have a *k-th symmetric derivative in L^p* at the point x_0, and the value of the derivative is a_k if $a_k/k!$ is the leading coefficient of P. Here again $P(t)$ has the same parity as k.

Remark. Strictly speaking, the requirement in the preceding definition that f should belong to L^p is unnecessarily strong, and it is enough to assume that $f(x_0+t)+(-1)^k f(x_0-t)$ belongs to L^p near $t = 0$. However, if the latter condition is satisfied for each x_0 belonging to a set E of positive measure, then in the neighborhood of almost all $x_0 \in E$ the function f itself is in L^p. Thus in theorems of "almost everywhere" type nothing is gained by weakening the assumption about the L^p integrability of f.

2. The rest of the chapter will be devoted to proving about symmetric derivatives in L^p a theorem analogous to Theorem A, namely,

THEOREM 1. *If $f(x)$ has a k-th symmetric derivative in L^p at each point of a set E, then at almost all points of E the function has an unsymmetric k-th derivative in L^p.*

In order to prove Theorem 1 we will need Theorem A above, another theorem (Theorem B, below) which is known and which we will take for granted here, and a lemma which we must prove ourselves.

THEOREM B. *Suppose that $F(x)$ has an unsymmetric r-th derivative at every point of a set E of positive measure. Then there exists a perfect set $\Pi \subset E$ of measure arbitrarily close to that of E, and a decomposition*

$$F(x) = G(x)+L(x) \tag{2.1}$$

with the following properties:

(i) *$G(x)$ has a continuous r-th derivative throughout the interval of definition of $F(x)$;*

(ii) *$L(x) = 0$ for $x \in \Pi$.*

For the proof, see either [3] or [4_{II}, p.73].

The lemma we will need is as follows:

LEMMA. *Suppose that $f(x)$ is zero on a set E of positive measure and, furthermore, for all x in E either*

$$\int_0^h |f(x+t)+f(x-t)|^p dt = o(h^a) \quad (h \to +0), \tag{2.2}$$

or

$$\int_0^h |f(x+t)-f(x-t)|^p dt = o(h^a) \quad (h \to +0) \tag{2.3}$$

holds, where a and p are positive numbers. Then for almost all points of E we have

$$\int_{-h}^h |f(x+t)|^p dt = o(h^a) \quad (h \to +0). \tag{2.4}$$

3. Proof of the lemma. Suppose for the sake of definiteness that it is condition (2.3) which holds for all x in E; in the case of condition (2.2) the proof remains the same. The hypothesis of the lemma implies that there is a subset of E, with measure arbitrarily close to that of E, on which (2.3) holds uniformly. Without loss of generality we may assume that (2.3) holds uniformly on E itself. Thus there is a function $\delta = \delta(h)$ which tends to 0 with h and such that

$$\int_0^h |f(x+t)-f(x-t)|^p dt \leqslant \delta h^a \tag{3.1}$$

for all x in E. It is sufficient to show that (2.4) holds at every point of density of E. To simplify notation we may assume that 0 is a point of density of E and we wish to show that

$$\int_{-h}^h |f(t)|^p dt = o(h^a). \tag{3.2}$$

More specifically, we will show that for h sufficiently small we have

$$\int_{-h}^h |f(t)|^p dt \leqslant 4\delta h^a. \tag{3.3}$$

At least one of the integrals $\int_0^h |f(t)|^p dt$ and $\int_{-h}^0 |f(t)|^p dt$ is not less than $\frac{1}{2}\int_{-h}^h |f(t)|^p dt$. Suppose, for example, that

$$\int_0^h |f(t)|^p dt \geqslant \tfrac{1}{2}\int_{-h}^h |f(t)|^p dt, \tag{3.4}$$

the argument in the other case being essentially the same. Denote by F the set complementary to E. For any x in the interval $[0, \frac{1}{2}h]$ we may write

$$\begin{aligned}\int_0^h |f(t)|^p dt &\leqslant \int_0^{h-x} |f(x+t)|^p dt + \int_0^{h-x} |f(x-t)|^p dt \\ &\leqslant \int_{A(x,h)} |f(x+t)|^p dt + \int_{B(x,h)} |f(x-t)|^p dt + \int_0^{h-x} |f(x+t)-f(x-t)|^p dt \\ &= \int_{C(x,h)} |f(t)|^p dt + \int_0^{h-x} |f(x+t)-f(x-t)|^p dt,\end{aligned} \tag{3.5}$$

where

$$A(x,h) = [0, h-x] \cap \{t: x-t \in F\},$$
$$B(x,h) = [0, h-x] \cap \{t: x+t \in F\},$$
$$C(x,h) = [2x-h, h] \cap \{t: 2x-t \in F\}.$$

We know that if x is in E as well as $[0, \frac{1}{2}h]$, then

$$\int_0^{h-x} |f(x+t)-f(x-t)|^p dt \leqslant \delta(h-x)^a \leqslant \delta h^a. \tag{3.6}$$

If we show that there exists an x in $E \cap [0, \frac{1}{2}h]$ for which

$$\int_{C(x,h)} |f(t)|^p dt \leqslant \tfrac{1}{2}\int_0^h |f(t)|^p dt, \tag{3.7}$$

then (3.5) and (3.7) will imply that $\int_0^h |f(t)|^p dt \leqslant 2\delta h^a$, which together with (3.4) will imply (3.3) and so complete the proof of the lemma. Thus it suffices to show that there exists an x in $E \cap [0, \frac{1}{2}h]$ for which (3.7) holds.

4. Denote by $\chi_F(v)$ the characteristic function of the set F and write

$$s(u,v) = f(u)\chi_F(v),$$

where $-h \leqslant u \leqslant h$, $-h \leqslant v \leqslant h$. Since 0 is a point of density of E, we have

$$\int_{-h}^h \chi_F(v)\, dv = \varepsilon h,$$

where $\varepsilon = \varepsilon(h)$ tends to 0 with h. Hence

$$\int_{-h}^h \int_{-h}^h |s(u,v)|^p du\, dv = \varepsilon h \int_{-h}^h |f(u)|^p du \leqslant 2\varepsilon h \int_0^h |f(u)|^p du.$$

Let L_x be the linear segment $\{(t, 2x-t),\ 2x-h \leqslant t \leqslant h\}$ and dL_x the element of length along L_x. Clearly,

$$\begin{aligned}\int_{C(x,h)} |f(t)|^p dt &= \int_{2x-h}^h \chi_F(2x-u)\,|f(u)|^p du \\ &= \int_{2x-h}^h |s(t, 2x-t)|^p dt = \int_{L_x} |s(u,v)|^p 2^{-1/2} dL_x,\end{aligned}$$

and

$$\int_0^{h/2} \Big\{\int_{L_x} |s(u,v)|^p dL_x\Big\} dx \leqslant \int_{-h}^h \int_{-h}^h |s(u,v)|^p du\, dv.$$

Hence

$$\begin{aligned}\int_{E\cap[0,h/2]} \Big\{\int_{C(x,h)} |f(t)|^p dt\Big\} dx &\leqslant \int_{-h}^h \int_{-h}^h |s(u,v)|^p du\, dv \\ &\leqslant 2h\varepsilon \int_0^h |f(t)|^p dt.\end{aligned}$$

This inequality implies that there exists an x in $E \cap [0, \frac{1}{2}h]$ for which

$$|E \cap [0, \tfrac{1}{2}h]| \int_{C(x,h)} |f(t)|^p dt \leqslant 2h\varepsilon \int_0^h |f(t)|^p dt,$$

and since

$$|E \cap [0, \tfrac{1}{2}h]| \leqslant \tfrac{1}{2}h(1-2\varepsilon),$$

the preceding inequality implies that

$$\int_{C(x,h)} |f(t)|^p dt \leqslant \frac{4\varepsilon}{1-2\varepsilon}\int_0^h |f(t)|^p dt.$$

Since ε tends to 0 with h, the factor preceding the last integral is $\leqslant \frac{1}{2}$ for h sufficiently small and the last inequality gives (3.7). This completes the proof of the lemma.

5. We now pass to the proof of Theorem 1. The hypothesis of the theorem is that, for each x in E,

$$\Big\{h^{-1}\int_0^h |\tfrac{1}{2}\{f(x+t)+(-1)^k f(x-t)\} - P_x(t)|^p dt\Big\}^{1/p} = o(h^k)$$

and so also, by Hölder's inequality,

$$h^{-1}\int_0^h \{\tfrac{1}{2}[f(x+t)+(-1)^k f(x-t)]-P_x(t)\}\,dt = o(h^k).$$

If F is the indefinite integral of f, the last equation may be written

$$(5.1)\qquad \tfrac{1}{2}\{F(x+h)+(-1)^{k+1}F(x-h)\} = \tfrac{1}{2}\{1+(-1)^{k+1}\}F(x)+\int_0^h P_x(t)\,dt+o(h^{k+1}),$$

which shows that F has a $(k+1)$-st symmetric derivative at x for all x in E.

By Theorem A, the function F has an unsymmetric $(k+1)$-st derivative in a subset $\mathscr{E}$ of E, with the same measure as E. Apply Theorem B to $F(x)$ and $\mathscr{E}$. Let us fix a perfect subset Π of $\mathscr{E}$, with measure arbitrarily close to that of $\mathscr{E}$, and consider the decomposition $F(x) = G(x) + L(x)$ given by Theorem B. Since F has a $(k+1)$-st unsymmetric derivative in Π, $F'(x)$ exists in Π, and so also $L'(x)$ exists in Π. Since $L(x)$ vanishes over Π and Π is perfect, it follows that $L'(x) = 0$ in Π. Hence, writing $G'(x) = g(x)$, $L'(x) = l(x)$, we obtain a decomposition

$$(5.2)\qquad f(x) = g(x)+l(x),$$

valid in the set where $F'(x)$ exists and equals $f(x)$ (and so almost everywhere). Here $g(x)$ has k continuous derivatives and $l(x)$ is 0 in the subset Π of E.

Since Π is perfect and $L(x)$ is 0 on Π, we immediately see that at each point x in Π the polynomial part (of degree $k+1$) of $L(x+t)$ is identically 0, and so the polynomial parts of $F(x+t)$ and $G(x+t)$ are the same. But, as we see from (5.1), the polynomial part of $\tfrac{1}{2}\{F(x+t) +(-1)^{k+1}F(x-t)\}$ is obtained by integrating with respect to t the polynomial part (of degree k) of $\tfrac{1}{2}\{f(x+t)+(-1)^k f(x-t)\}$. Hence, conversely, the polynomial part of the latter function is obtained by differentiating the polynomial part of the former. It is obvious that the polynomial part of $\tfrac{1}{2}\{g(x+t)+(-1)^k g(x-t)\}$ is obtained by differentiating the polynomial part of $\tfrac{1}{2}\{G(x+t)+(-1)^{k+1}G(x-t)\}$. From this and (5.2) we see that the function $l(x)$, which vanishes on Π, has a k-th symmetric derivative in L^p everywhere in Π and that the polynomial part of $\tfrac{1}{2}\{l(x+t) +(-1)^k l(x-t)\}$ is 0 at each point of Π. In other words,

$$\Big\{h^{-1}\int_0^h |l(x+t)+(-1)^k l(x-t)|^p\,dt\Big\}^{1/p} = o(h^k)$$

for $x \in \Pi$.

Applying the lemma of Section 2 with $\alpha = kp+1$, we see that $l(x)$ has an unsymmetric k-th derivative at almost all points of Π. In view of (5.2) the same holds for f. Since $|E-\Pi|$ can be arbitrarily small, f has a k-th unsymmetric derivative in L^p almost everywhere in E and Theorem 1 is established.

Chapter II

1. We shall now apply the main result of Chapter I to the theory of trigonometric series and begin, for simplicity, with the case of convergent series.

Suppose that a trigonometric series

$$(1.1)\qquad \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

converges at the point x_0 to the sum s. Suppose also that the series

$$(1.2)\qquad \tfrac{1}{2}a_0 x+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n,$$

obtained by the termwise integration of (1.1), converges almost everywhere in some neighborhood of x_0. Call $F(x)$ the sum of (1.2). A known result asserts that $F(x)$ has at x_0 a *symmetric approximate* derivative equal to s, that is,

$$\frac{F(x_0+h)-F(x_0-h)}{2h}\to s$$

as h tends to 0 through all points of a set having 0 as a point of density (see [4_{I}], p. 324). We shall show below (see Theorem C) that F has a stronger property, namely, F has at x_0 a symmetric derivative in L^p, equal to s, for any $p<\infty$.

Hence, if (1.1) converges in a set E of positive measure (which implies that $a_n, b_n \to 0$, so that (1.2) converges almost everywhere) then, by the result just stated and Theorem 1, the sum $F(x)$ of (1.2) has almost everywhere in E an unsymmetric first derivative in any L^p, $p<\infty$. This strengthens the familiar result of Lusin (see [4_{I}], p. 325, and [4_{II}], p. 219]) that F has almost everywhere in E an unsymmetric approximate derivative.

Remark. The hypothesis that (1.1) converges at x_0 and the sum $F(x)$ of (1.2) exists almost everywhere in some neighborhood of x_0, implies that $F(x_0+t)-F(x_0-t)$ is in L^p in some neighborhood of $t=0$, for any $p<\infty$. To see this, suppose, for simplicity, that $x_0 = 0$. Then, the convergence of (1.1) at $x=0$ implies that $a_n\to 0$. About the b_n we know nothing. The odd part $\sum a_n n^{-1}\sin nx$ of (1.2), having coefficients

$o(1/n)$, converges almost everywhere and by the theorem of Hausdorff-Young represents a function which is in L^p for every finite p. About the even part $\sum b_n n^{-1}\cos nx$ we know only that it converges almost everywhere in some neighborhood of $x=0$. Its sum, however, being even contributes nothing to $F(t)-F(-t)$, and hence $F(t)-F(-t)$ is in L^p near $t=0$.

2. We now formulate the main result of this chapter.

THEOREM 2. *Suppose that the series* (1.1) *is summable* $(C, r-1)$ $(r=1,2,\ldots)$ *at each point of a set* E *of positive measure. Then the series obtained by integrating* (1.1) *termwise* r *times converges almost everywhere to a function* $F(x)$ *which is in every* L^p, $p<\infty$, *and which has almost everywhere in* E *an unsymmetric* r*-th derivative in* L^p *equal to the* $(C, r-1)$ *sum of the series* (1.1).

According as r is even or odd, the series obtained by integrating (1.1) r times is

$$\frac{1}{2}a_0\frac{x^r}{r!}+(-1)^{r/2}\sum_1^\infty (a_n\cos nx+b_n\sin nx)n^{-r}, \tag{2.1}$$

or

$$\frac{1}{2}a_0\frac{x^r}{r!}+(-1)^{(r-1)/2}\sum_1^\infty (a_n\sin nx-b_n\cos nx)n^{-r}, \tag{2.2}$$

and has coefficients $o(1/n)$, so that it converges almost everywhere and its sum is in L^p. In view of the remarks made above about the special case $r=1$ of Theorem 2 it is immediate that the theorem is a corollary of Theorem 1 of Chapter I and the following

THEOREM C. *Suppose that the series* (1.1) *is summable* $(C, r-1)$ $(r=1,2,\ldots)$ *at the point* x_0 *to sum* s *and that the series obtained by integrating* (1.1) *termwise* r *times converges almost everywhere in some neighborhood of* x_0 *to sum* $F(x)$. *Then* F *has at* x_0 *an* r*-th symmetric derivative in* L^p, $p<\infty$, *equal to* s.

Theorem C is stated without proof in [5]. To make the proof of Theorem 2 complete we prove Theorem C here and the rest of the chapter is devoted to that proof. Part of the argument that follows is familiar (see [4_{II}], p. 66).

3. Suppose that r is even; for r odd the proof is the same. Hence $F(x)$ is given by (2.1). We may assume without loss of generality that $x_0=0$ and $a_0=0$. Since the hypothesis implies that $a_n=o(n^{r-1})$, the even part of (2.1) is in L^p and the odd part contributes nothing to $F(t)+F(-t)$. Thus we may neglect the odd part and assume that both (1.1) and (2.1) are cosine series.

Let us set

$$\gamma(t)=t^{-r}\cos t,$$

and denote the Cesàro sums of order k of the series $0+a_1+a_2+\ldots$ by s_n^k. For any sequence $\{u_n\}$ let us write $\Delta u_n=\Delta^1 u_n=u_n-u_{n+1}$, $\Delta^k u_n=\Delta(\Delta^{k-1}u_n)$. Clearly,

$$F(t)=(-1)^{r/2}t^r\sum_{n=1}^\infty a_n\gamma(nt),$$

and summation by parts gives

$$F(t)=(-1)^{r/2}t^r\sum_{n=1}^\infty s_n^{r-1}\Delta^r\gamma(nt) \tag{3.1}$$

at each point of convergence of the preceding series. We will use the classical formula

$$\Delta^k u_n v_n=\sum_{s=0}^k\binom{k}{s}\Delta^s u_n\Delta^{k-s}v_{n+s}, \tag{3.2}$$

and the fact that if $u(t)$ is a k times differentiable function and $u_n(t)=u(nt)$, then

$$\Delta^k u_n(t)=(-1)^k t^k u^{(k)}(nt+\theta kt), \tag{3.3}$$

where $0\leqslant\theta\leqslant 1$.

Let

$$P(x)=\sum_{\nu=0}^{r/2-1}(-1)^\nu\frac{x^{2\nu}}{(2\nu)!},\qquad \lambda(x)=\frac{\cos x-P(x)}{x^r}.$$

Clearly,

$$\gamma(nt)=\lambda(nt)+P(nt)(nt)^{-r},$$

so that, by (3.1),

$$F(t)=\sum_{\nu=0}^{r/2-1}\frac{D_\nu}{(2\nu)!}t^{2\nu}+t^rR(t)=P(t)+t^rR(t), \tag{3.4}$$

where

$$D_\nu=(-1)^{r/2+\nu}\sum_{n=1}^\infty s_n^{r-1}\Delta^r n^{2\nu-r},$$

$$R(t)=(-1)^{r/2}\sum_{n=1}^\infty s_n^{r-1}\Delta^r\lambda(nt).$$

Since $\Delta^r n^{2\nu-r}=O(n^{2\nu-2r})=O(n^{-2-r})$, and $s_n^{r-1}=o(n^{r-1})$, the series defining D_ν is absolutely convergent.

We want to show that the function $F(t)$, which is even, has an r-th symmetric derivative in L^p at 0, equal to 0. Since the polynomial $P(t)$ in (3.4) is of degree strictly less than r, the result will follow if we show that

$$\frac{1}{h}\int_0^h |R(t)|^p dt = o(1), \tag{3.5}$$

for all $p \geqslant 2$. It is enough to prove this for h taking the values 2^{-N}, $N = 1, 2, \ldots$

4. We write

$$\frac{1}{2^{-N}}\int_0^{2^{-N}} |R(t)|^p dt = 2^N \sum_{j=N}^{\infty} \int_{2^{-j-1}}^{2^{-j}} |R(t)|^p dt \tag{4.1}$$
$$\leqslant 2^N 2^p \sum_{j=N}^{\infty} U_j + 2^N 2^p \sum_{j=N}^{\infty} V_j,$$

where

$$U_j = \int_{2^{-j-1}}^{2^{-j}} \Big|\sum_{1}^{2^j} s_n^{r-1} \Delta^r \lambda(nt)\Big|^p dt,$$

$$V_j = \int_{2^{-j-1}}^{2^{-j}} \Big|\sum_{2^j+1}^{\infty} s_n^{r-1} \Delta^r \lambda(nt)\Big|^p dt.$$

Since $\lambda(t)$ has all derivatives, (3.3) shows that $\Delta^r \lambda(nt) = O(t^r)$, so that

$$U_j = \int_{2^{-j-1}}^{2^{-j}} \Big|O(t^r)\sum_{1}^{2^j} o(n^{r-1})\Big|^p dt = O(2^{-jrp})\, o(2^{j(rp-1)}) = o(2^{-j}),$$

and

$$2^N \sum_{j=N}^{\infty} U_j = 2^N \sum_{j=N}^{\infty} o(2^{-j}) = o(1). \tag{4.2}$$

If we prove a similar inequality for $\sum_N^{\infty} V_j$, Theorem C will be established.

5. In view of the definition of $\lambda(t)$,

$$V_j \leqslant 2^p \int_{2^{-j-1}}^{2^{-j}} \Big|\sum_{2^j+1}^{\infty} s_n^{r-1} \Delta^r \frac{\cos nt}{(nt)^r}\Big|^p dt + 2^p \int_{2^{-j-1}}^{2^{-j}} \Big|\sum_{2^j+1}^{\infty} s_n^{r-1} \Delta^r \frac{P(nt)}{(nt)^r}\Big|^p dt \tag{5.1}$$
$$= V_j' + V_j'',$$

say. Since $P(t)t^{-r}$ is a polynomial in t^{-1} beginning with term t^{-2}, an application of (3.3) shows that $\Delta^r\{P(nt)(nt)^{-r}\} = t^r O(nt)^{-2-r} = O(t^{-2} n^{-2-r})$, so that

$$V_j'' = O(2^{2jp}) \int_{2^{-j-1}}^{2^{-j}} \Big|\sum_{2^j+1}^{\infty} o(n^{-3})\Big|^p dt = o(2^{-j}). \tag{5.2}$$

On the other hand, using (3.2) we have

$$\Delta^r \frac{\cos nt}{n^r} = \sum_{s=0}^{r} \binom{r}{s} \Delta^s \cos nt\, \Delta^{r-s} \frac{1}{(n+s)^r}. \tag{5.3}$$

Observe now that

$$\Delta^s \cos nt = \pm (2\sin\tfrac{1}{2}t)^s Z, \tag{5.4}$$

where we have $Z = \cos(n+\frac{1}{2}s)t$ or $Z = \sin(n+\frac{1}{2}s)t$, according as s is even or odd. Fix s and suppose e. g. that s is even. Then the contribution of the s-th term on the right of (5.3) to the first integral in (5.1) is

$$\leqslant \mathrm{Const} \int_{2^{-j-1}}^{2^{-j}} \frac{(2\sin\frac{1}{2}t)^s}{t^r} \Big|\sum_{2^j+1}^{\infty} a_n \cos(n+\tfrac{1}{2}s)t\Big|^p dt \tag{5.5}$$
$$= O(2^{j(r-s)p}) \int_{2^{-j-2}}^{2^{-j-1}} \Big|\sum a_n \cos(2n+s)t\Big|^p dt,$$

where

$$a_n = s_n^{r-1} \Delta^{r-s} \frac{1}{(n+s)^r} = o(n^{r-1})\, O(n^{-2r+s}) = o(n^{s-r-1}).$$

The right-hand side of (5.5) increases if we replace the interval of integration $(2^{-j-2}, 2^{-j-1})$ by $(0, \pi)$. Hence, if $p \geqslant 2$ and $p' = p/(p-1)$, an application of the Hausdorff-Young theorem shows that the right-hand side of (5.5) does not exceed

$$O(2^{j(r-s)p}) \Big(\sum_{2^j+1}^{\infty} |a_n|^{p'}\Big)^{p/p'} = O(2^{j(r-s)p}) \Big\{\sum_{2^j+1}^{\infty} o(n^{(s-r-1)p'}\Big\}^{p-1}$$
$$= O(2^{j(r-s)p})\, o(2^{j[s-r-1)p'+1})^{p-1} = o(2^{-j}).$$

Collecting the results we see that

$$V_j' = o(2^{-j}),$$

which together with (5.1) and (5.2) gives $V_j = o(2^{-j})$. Hence V_j satisfies an inequality analogous to that for U_j in (4.2) and Theorem C follows.

Added in proof (27. II. 1964). 1. The following result generalizes Theorem C in the case when the trigonometric series is of power series type (i. e., is of the form $c_0 + c_1 e^{ix} + c_2 e^{2ix} + \ldots$):

THEOREM C'. *If the series* (1.1) *in Theorem* C *is of power series type, then the function* F *has at* x_0 *an* r*-th Peano unsymmetric derivative in* L^p, $p < \infty$, *equal to* s.

The proof parallels that of Theorem C. The generalization obviously adds nothing to our Theorem 2.

2. The conclusion of Theorem 1 holds if the hypothesis is replaced by the following one: at each point $x \in E$ we have (1.4) with O instead of o (the polynomial $P(t)$ may then, of course, be of degree $k-1$ or less). The proof remains unchanged if we note that the conclusion of the lemma on p. 91 remains unchanged if we replace the o in (2.3) and (2.4) by O, provided $\alpha > 1$.

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On a theorem of Mackey, Stone and v. Neumann

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The canonical commutation rules of the quantum mechanics of a system with N degrees of freedom have the following form:

$$\begin{aligned} Q_nP_m - P_mQ_n &= i\delta_{nm}, \\ P_nP_m - P_mP_n &= 0, \qquad (m, n = 1, 2, \ldots, N) \\ Q_nQ_m - Q_mQ_n &= 0, \end{aligned}$$

where Q_n, P_n $(n = 1, 2, \ldots, N)$ are self-adjoint operators in the Hilbert space H. For $\hat{x} \in E^N$, $x \in E^{N*}$ we define

$$U(\hat{x}) = \exp\Big(i\sum_{n=1}^{N}\hat{x}_nQ_n\Big), \qquad V(x) = \exp\Big(i\sum_{n=1}^{N}x_nP_n\Big).$$

The commutation rules were put by H. Weyl in the following correct form:

$$\begin{aligned} U(\hat{x}_1)\,U(\hat{x}_2) &= U(\hat{x}_1+\hat{x}_2), \\ V(x_1)\,V(x_2) &= V(x_1+x_2), \\ V(x)\,U(\hat{x}) &= U(\hat{x})\,V(x)\exp\Big(i\sum\hat{x}_nx_n\Big). \end{aligned}$$

Let us notice that $x \to \exp\big(i\sum\hat{x}_nx_n\big)$ is a character of the group E^N. We assume that the algebra generated by the operators $U(\hat{x})$ is cyclic. This assumption means that this algebra has a simple spectrum. In the language of physics we say that the operators Q_n form a complete set of commuting observables (cf. [6], p. 122). From these assumptions it follows that there exists so called *Schrödinger representation* of the operators Q and P. This means that there exists an isomorphism

$$H \xrightarrow{I} L^2(E^N)$$

such that

$$(IQ_nI^{-1}\varphi)(y) = y_n\varphi(y), \qquad (IP_nI^{-1}\varphi)(y) = -i\frac{\partial}{\partial y_n}\varphi(y)$$

for $\varphi \in C_0^\infty(E^N) \subset L^2(E^N)$.