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Total and partial differentiability in L^p

by

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1. A function $f(x) = f(x_1, x_2, \ldots, x_n)$ defined in the neighborhood of a point $x^0 = (x_1^0, \ldots, x_n^0)$ and of the class L^p there, $1 \leqslant p \leqslant \infty$, is said to have a k-th differential in L^p at x^0 if there is a polynomial $P(t) = P(t_1, \ldots, t_n)$ of degree k (or less) such that

$$\left\{\frac{1}{\varrho^n}\int_{|t|\leqslant\varrho} |f(x^0+t)-P(t)|^p dt\right\}^{1/p} = o(\varrho^k) \qquad (\varrho \to 0).$$

If $p = \infty$, the expression on the left is to be interpreted, of course, as $\operatorname{ess\,sup}|f(x^0+t)-P(t)|$ for $|t| \leqslant \varrho$. The definition has been introduced in [1]. The domain of integration $|t| \leqslant \varrho$ can clearly be replaced by a cube containing the origin and of side tending to 0.

The main result of the present paper is the following

THEOREM 1. *Let $f(x) = f(x_1, \ldots, x_n)$ belong to L^p, $1 \leqslant p \leqslant \infty$, over the unit cube*

$$(Q_0) \qquad 0 \leqslant x_j \leqslant 1 \quad (j = 1, 2, \ldots, n),$$

and suppose that at each point x of a set $E \subset Q_0$ the function f has a k-th differential in L^p. Let m be a fixed integer satisfying $1 \leqslant m < n$. Then at almost all points $x \in E$ the function f has a k-th differential in L^p with respect to the variable $x' = (x_1, x_2, \ldots, x_m)$.

The sets and functions that occur in the proof below are all Lebesgue measurable, even if it is not stated explicitly (the proofs of measurability, when needed, are routine). The cubes will be always closed cubes. We may restrict our argument to the case $1 \leqslant p < \infty$, since if $p = \infty$ it is not difficult to see that the function f^* which coincides with f at the points of set Z where $f(x)$ is the derivative of its indefinite integral and elsewhere is defined by the condition $f(x_0) = \limsup f(x)$ for x tending to x_0 through Z, satisfies the relation $f^*(x_0+t)-P(t) = o(|t|^k)$, and it is enough to observe that the m-dimensional measure of the intersection of the complement of T with almost all subspaces $x_{m+1} = \text{const}, \ldots, x_n = \text{const}$, is 0.

2. We begin by simplifying our problem. By a known result (see [1], p. 186, Theorem 9), if $f(x)$ has a k-th differential in L^p at each point of a set E of positive measure, then given any $\varepsilon > 0$ we can find a perfect subset E_1 of E with $|E-E_1| < \varepsilon$ and a decomposition

$$f(x) = g(x)+h(x)$$

with the following properties:

1° $g(x)\in C^k$,

2° $g(x) = f(x)$ on E_1.

Theorem 1 will therefore be established if we show that $h(x)$ has a k-th differential in L^p at almost all points of E_1. Since, by 2°, $h(x) = 0$ on E_1, the polynomial $P(t) = P_x(t)$ associated with h is identically 0 at every point of density of E_1, and so almost everywhere in E_1 we have

$$\left\{\frac{1}{\varrho^n}\int\limits_{|t|\leqslant\varrho}|h(x+t)|^p dt\right\}^{1/p} = o(\varrho^k).$$

Thus replacing $|h(x)|^p$ by $f(x)$, E_1 by E and setting $\alpha = pk$ we reduce Theorem 1 to the following

THEOREM 1′. *Set $x' = (x_1, \ldots, x_m)$, $x'' = (x_{m+1}, \ldots, x_n)$ and let $f(x) = f(x_1, \ldots, x_n) = f(x', x'')$ be non-negative and integrable over the unit cube Q_0: $0 \leqslant x_j \leqslant 1$, $j = 1, 2, \ldots, n$. Let α be a positive number and let Q and I denote respectively arbitrary n-dimensional and m-dimensional cubes with edge h. If at each point $x = (x', x'')$ of a set $E\in Q_0$ we have*

$$\int\limits_Q f(\xi)\,d\xi = o(h^{n+\alpha}) \qquad (h\to 0)$$

for cubes Q containing x, then at almost all points $x = (x', x'')$ of E

$$\int\limits_I f(\xi', x'')\,d\xi' = o(h^{m+\alpha}) \qquad (h\to 0)$$

for cubes I containing x'.

3. Theorem 1′ is easily deducible from the following result which is of independent interest; the notation here is the same as in Theorem 1.

THEOREM 2. *There is a positive constant A depending only on the dimension n and having the following property. Let $f(x) = f(x_1, \ldots, x_n)$ defined in a cube Q^0 be non-negative and integrable. Denote by U the set of points $x\in Q^0$ such that there is a cube $Q\ni x$ with*

$$\int\limits_Q f(\xi)\,d\xi \geqslant (\tfrac{1}{2}h)^{n+\alpha},$$

and by V the set points $x = (x', x'')\in Q^0$ such that for some $I\ni x'$ we have

$$\int\limits_I f(\xi', x'')\,d\xi' \geqslant h^{m+\alpha}.$$

Then

$$|U| \geqslant A|V|.$$

We first show that Theorem 1′ is actually a simple consequence of Theorem 2.

Suppose that the assertion of Theorem 1′ is not true. This means that there is a positive constant C and a subset E_1 of E of positive measure such that

$$\limsup_{h\to 0} h^{-(m+\alpha)}\int\limits_I f(\xi', x'')\,d\xi' > C.$$

By multiplying f by a constant we may assume that $C = 1$.

Now we can find a number $h_0 > 0$ and a subset E_2 of E_1 of positive measure such that for each $x\in E_2$ and any cube Q containing x and of edge $\leqslant h_0$ we have

$$\int\limits_Q f(\xi)\,d\xi < (\tfrac{1}{2}h)^{n+\alpha}.$$

Let $x^0\in E_2$ be a point of density of E_2 and let Q^0 be a cube with center x^0 and edge $h_1 \leqslant h_0$. Let U and V be the sets of Theorem 2 for this cube Q^0. If h_1 is small enough, the density of E_2 in Q^0 exceeds $\frac{1}{2}$. Since, obviously, $V\supset E_2\cap Q^0$, the inequality $|U|\geqslant A|V|$ implies that the density of U in Q^0 exceeds $\frac{1}{2}A$. But this is impossible if h_1 is small enough since U is contained in the intersection of Q^0 with the complement of the set E_2 and x^0 is a point of density of E_2. This contradiction proves Theorem 1′.

4. The proof of Theorem 2 rests on two lemmas.

LEMMA 1. *Let $f(x) = f(x_1, \ldots, x_n)$ be non-negative and integrable over a cube Q^0 of edge h_0 and suppose that*

$$h_0^{-(n+\alpha)}\int\limits_{Q^0} f(x)\,dx < 2^{-(n+\alpha)},$$

where $\alpha > 0$. Then there is a sequence of non-overlapping cubes $Q_1, Q_2, \ldots$ contained in Q^0 with edges respectively $h_1, h_2, \ldots$ such that

$$2^{-(n+\alpha)} \leqslant h_k^{-(n+\alpha)}\int\limits_{Q_k} f(x)\,dx < 1 \qquad (k = 1, 2, \ldots) \tag{1}$$

and $f(x) = 0$ almost everywhere in the complement of $\bigcup Q_k$.

Proof. We subdivide Q^0 into 2^n equal cubes. If for any of these partial cubes, call them Q', we have

$$(h')^{-(n+\alpha)}\int\limits_{Q'} f(x)\,dx \geqslant 2^{-(n+\alpha)}, \tag{2}$$

we let all such Q' be members of $\{Q_k\}$. Clearly we have for each such Q'

$$(h')^{-(n+\alpha)}\int\limits_{Q'} f\,dx \leqslant (\tfrac{1}{2}h_0)^{-(n+\alpha)}\int\limits_{Q^0} f\,dx < 2^{n+\alpha}2^{-(n+\alpha)} = 1.$$

Each of the cubes Q' for which (2) does not hold we split into 2^n equal cubes Q'' and include in $\{Q_k\}$ those Q'' for which

$$(h'')^{-(n+\alpha)} \int_{Q''} f(x)\,dx \geqslant 2^{-(n+\alpha)},$$

and so on. It is clear that the sequence $\{Q_k\}$ defined in this way satisfies (1). Each of the points of Q which is not in $\{Q_k\}$ is included in a sequence of cubes Q with edge h tending to 0 and such that

$$h^{-(n+\alpha)} \int_Q f\,dx < 2^{-(n+\alpha)}.$$

Hence $f = 0$ almost everywhere in the complement of $\bigcup Q_k$.

5. LEMMA 2. *Let $f(x) = f(x_1, \ldots, x_n)$ be non-negative and integrable over a cube Q^0 and suppose that Q^0 contains a sequence of non-overlapping cubes $Q_1, Q_2, \ldots$ such that*

$$\int_{Q_k} f(x)\,dx \geqslant h_k^{n+\alpha} \qquad (k = 1, 2, \ldots),$$

where h_k is the edge of Q_k and α is a fixed positive number. Suppose, moreover, that $f = 0$ in $Q^0 - \bigcup Q_k$. Then, if H is the set of points $x \in Q^0$ such that for some $Q \supset x$ we have

(3) $$\int_Q f(x)\,dx \geqslant h^{n+\alpha},$$

where h is the edge of Q, the measure of H satisfies the inequality

$$|H| \leqslant C \sum_k \Big\{\int_{Q_k} f(x)\,dx\Big\}^{n/(n+\alpha)},$$

where C is a constant depending on the dimension n only.

The set H contains $\bigcup Q_k$. Let us consider all the cubes Q for which we have (3). For each such Q consider those among the cubes Q_k, if any such exist, which have points in common with Q, and denote by $\bar{Q}$ the smallest cube concentric with Q and containing all the cubes Q_k having points in common with Q. It is not difficult to see that

(4) $$|\bar{Q}| \leqslant 3^n \Big\{|Q| + \sum_{Q_k \cap Q \neq 0} |Q_k|\Big\}.$$

The cubes $\bar{Q}$ cover the set H. It is a familiar fact that we can then find among the $\bar{Q}$ a finite number of *disjoint* cubes, call them $\bar{Q}^1, \bar{Q}^2, \ldots, \bar{Q}^r$, such that

(5) $$\sum_1^r |\bar{Q}^s| \geqslant \varepsilon |H|,$$

where ε is a positive number depending on the dimension n only (see, e. g., [2_{II}], p. 309, Lemma). Let Q^s denote the cube generating $\bar{Q}^s$. By (5) and (4),

$$|H| \leqslant \varepsilon^{-1} \sum |\bar{Q}^s| \leqslant \varepsilon^{-1} \cdot 3^n \sum_s \Big\{|Q^s| + \sum_{Q_k \cap Q^s \neq 0} |Q_k|\Big\} \leqslant \varepsilon^{-1} \cdot 3^n \Big\{\sum_s |Q^s| + \sum_k |Q_k|\Big\},$$

since no Q_k can have points in common with more than one Q^s. For any Q satisfying (3) we have

$$|Q| = h^n \leqslant \Big(\int_Q f\,dx\Big)^{n/(n+\alpha)},$$

which gives

(6) $$|H| \leqslant \varepsilon^{-1} \cdot 3^n \Big\{\sum_s \Big(\int_{Q^s} f\,dx\Big)^{n/(n+\alpha)} + \sum_k \Big(\int_{Q_k} f\,dx\Big)^{n/(n+\alpha)}\Big\}.$$

Since the function f vanishes outside $\bigcup Q_k$, it follows that

$$\Big(\int_{Q^s} f\,dx\Big)^{n/(n+\alpha)} = \Big(\sum_{Q_k \cap Q^s \neq 0} \int_{Q_k} f\,dx\Big)^{n/(n+\alpha)} \leqslant \sum_{Q_k \cap Q^s \neq 0} \Big(\int_{Q_k} f\,dx\Big)^{n/(n+\alpha)}.$$

This combined with (6) gives the conclusion of Lemma 2 with $C = 2 \cdot 3^n \varepsilon^{-1}$.

6. We will now conclude the proof of Theorem 2.

The function f of Theorem 2 is non-negative and integrable. We may assume that

$$\frac{1}{h_0^{n+\alpha}} \int_{Q^0} f\,dx < 2^{-(n+\alpha)},$$

where h_0 is the edge of Q^0. By lemma 1, there is a sequence $Q_1, Q_2, \ldots$ of non-overlapping cubes in Q^0 for which the inequalities (1) hold; moreover, $f = 0$ almost everywhere in $Q^0 - \bigcup Q_k$.

Let I_k be the projection of Q_k on the space of x', and J_k the projection of Q_k on the space of x''; thus $Q_k = I_k + J_k$. Reserving the notation $|F|$ for the n-dimensional measure of a set F, we shall denote by $|F|_m$ the m-dimensional measure of F. Let $f_1(x)$ be the function defined by the conditions

$$f_1(x) = f(x) + h_k^\alpha \quad \text{for} \quad x \in Q_k \ (k = 1, 2, \ldots);$$

$$f_1(x) = 0 \quad \text{in} \quad Q^0 - \bigcup Q_k.$$

The function $f_1(x) = f_1(x', x'')$ has the property that for any $x'' \in J_k$ we have

(6) $$\frac{1}{h_k^{\alpha+m}} \int_{I_k} f_1(x', x'')\,dx' \geqslant 1.$$

Let U, V be the sets of Theorem 2 and let V_1 be the set analogous to V but corresponding to the function f_1. Clearly,

$$(7)\qquad |U| \geqslant \sum |Q_k|, \qquad |V| \leqslant |V_1|.$$

Let $V_1(x'')$ be the intersection of V_1 with the x'' subspace (i. e., with the set of points (x', x'') where x'' is fixed and x' is arbitrary). For any given x'' we set $\varepsilon_k(x'')$ equal to 1 or to 0 according as the subspace x'' does or does not meet the cube Q_k.

In view of the condition (6), an application of Lemma 2 to the m-dimensional space gives

$$|V_1(x'')|_m \leqslant C \sum_k \varepsilon_k(x'') \Big\{ \int_{I_k} f(x', x'')\,dx' \Big\}^{m/(m+a)}.$$

Hence, denoting by J^0 the projection of Q^0 on the x'' space,

$$(8)\qquad |V_1| = \int_{J^0} |V_1(x'')|_m\,dx'' \leqslant C \sum_k \int_{J^0} \varepsilon_k(x'') \Big\{ \int_{I_k} f_1(x', x'')\,dx' \Big\}^{m/(m+a)} dx''$$

$$= C \sum_k \int_{J_k} \Big\{ \int_{I_k} f_1(x', x'')\,dx' \Big\}^{m/(m+a)} dx''$$

since, for k fixed, $\varepsilon_k(x'') = 1$ if and only if $x'' \in J_k$.

By Hölder's inequality, the repeated integral in the last sum does not exceed

$$|J_k|_{n-m}^{a/(m+a)} \Big\{ \int_{Q_k} f_1(x)\,dx \Big\}^{m/(m+a)} = h_k^{a(n-m)/(m+a)} \Big\{ \int_{Q_k} (f + h_k^a)\,dx \Big\}^{m/(m+a)}$$

$$\leqslant h_k^{a(n-m)/(m+a)} \Big\{ \Big(\int_{Q_k} f\,dx \Big)^{m/(m+a)} + h_k^{(a+n)m/(m+a)} \Big\}$$

$$\leqslant h_k^{a(n-m)/(m+a)} \{ 2h_k^{(a+n)m/(m+a)} \} = 2h_k^n.$$

Hence, in view of (8) and (7),

$$|V| \leqslant |V_1| \leqslant 2C \sum_k h_k^n = 2C \sum_k |Q_k| \leqslant 2C\,|U|,$$

i. e., $|V| \leqslant A\,|U|$ with $A = 2C$, and Theorem 2 is established.

7. We conclude by briefly describing a generalization of Theorem 1.

Let $u = k+\gamma$, where k is a non-negative integer and $0 < \gamma < 1$. Using the terminology of [1], we say that the function $f(x) = f(x_1, \ldots, x_n)$ defined in the neighborhood of a point x^0 satisfies condition $T_u^p(x_0)$ if there is a polynomial $P(t)$ of degree $\leqslant k$ such that

$$\Big\{ \varrho^{-n} \int_{|t| \leqslant \varrho} |f(x^0+t) - P(t)|^p\,dt \Big\}^{1/p} = O(\varrho^u)$$

as $\varrho \to 0$. Condition $t_u^p(x^0)$ is defined by replacing here O by o. (The limiting case $\gamma = 0$, covered by Theorem 1, has somewhat special properties insofar as it is known that conditions T_u^p and t_u^p are equivalent almost everywhere; see [1].)

THEOREM 3. *Suppose that $f(x)$ belongs to L^p, $1 \leqslant p \leqslant \infty$, in the unit cube Q_0 and suppose that at each point x^0 of a set $E \subset Q_0$ the function f satisfies condition $T_u^p(x^0)$, where $u = k+\gamma$, $k = 0, 1, \ldots$, $0 < \gamma < 1$. Then almost everywhere in E the function f satisfies condition $T_u^p(x^0)$ with respect to the variables $x_1, x_2, \ldots, x_m$, where $m < n$. The corresponding result holds for the condition t_u^p.*

The proof closely parallels that of Theorem 1 if we use the fact (see [1], Theorem 9) that the general case can be reduced to the case when f satisfies the additional condition of vanishing in the set E.

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