On double integrals and Fourier series

by R. TABERSKI (Poznań)

In this paper we present some results concerning the Riemann-Stieltjes, Lebesgue and Titchmarsh integrals on a given rectangle, and Fourier series of two variables.

1. We shall introduce fundamental definitions and notation used in the sequel.

Let $\varphi(x, y)$ be a function defined in the rectangle $R = [a, b; c, d]$, and let

$$(1) \quad P = \left\{ \begin{array}{l} a = x_1 < x_2 < \dots < x_i < \dots < x_m < x_{m+1} = b \\ c = y_1 < y_2 < \dots < y_j < \dots < y_n < y_{n+1} = d \end{array} \right\}$$

be a partition of R . Write

$$\Delta\varphi(x_i, y_j) = \varphi(x_i, y_j) - \varphi(x_{i+1}, y_j) - \varphi(x_i, y_{j+1}) + \varphi(x_{i+1}, y_{j+1}).$$

If $\Delta\varphi(x_i, y_j) \geq 0$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) for any partition P , we say that $\varphi(x, y)$ satisfies the *condition Ω* in R .

The function $\varphi(x, y)$ is said to be of *finite variation* in R if the sums

$$V(P) = \sum_{i=1}^m \sum_{j=1}^n |\Delta\varphi(x_i, y_j)|$$

remain uniformly bounded for all possible P . The upper bound V of all $V(P)$, called the *variation* of φ in R , will be denoted by

$$V = \int_a^b \int_c^d |d\varphi(x, y)|.$$

Of course, φ is of finite variation in R if it satisfies the condition Ω in R ; in this case

$$\int_a^b \int_c^d |d\varphi(x, y)| = \varphi(a, c) - \varphi(b, c) - \varphi(a, d) + \varphi(b, d).$$

It is also well-known ([9], pp. 470, 471) that any function $\varphi(x, y)$ of finite variation in R can be represented in the form

$$(2) \quad \varphi(x, y) = -\varphi(a, c) + \varphi(x, c) + \varphi(a, y) + V_1^+(x, y) - V_1^-(x, y)$$

or

$$(2') \quad \varphi(x, y) = -\varphi(b, d) + \varphi(x, d) + \varphi(b, y) + V_2^+(x, y) - V_2^-(x, y),$$

where the functions $V_i^+(x, y)$ and $V_i^-(x, y)$ are non-negative, non-decreasing for $i = 1$ and non-increasing for $i = 2$ with respect to each variable separately and satisfy the condition Ω in R . Moreover,

$$V_1^+(x, c) = V_1^-(x, c) = V_1^+(a, y) = V_1^-(a, y) = 0$$

and

$$V_2^+(x, d) = V_2^-(x, d) = V_2^+(b, y) = V_2^-(b, y) = 0$$

for $(x, y) \in R$. The functions $V_i^\pm(x, y)$ ($i = 1, 2$) are continuous in R if $\varphi(x, y)$ is continuous therein.

Let $f(x, y)$ and $g(x, y)$ be two functions defined in the rectangle $R = [a, b; c, d]$. Take a normal sequence of partitions

$$(3) \quad P_k = \left\{ \begin{array}{l} a = x_1^{(k)} < x_2^{(k)} < \dots < x_i^{(k)} < \dots < x_{m_k}^{(k)} < x_{m_k+1}^{(k)} = b \\ c = y_1^{(k)} < y_2^{(k)} < \dots < y_j^{(k)} < \dots < y_{n_k}^{(k)} < y_{n_k+1}^{(k)} = d \end{array} \right\}$$

and points $(\xi_{i,j}^{(k)}, \eta_{i,j}^{(k)})$ and $(\xi_i^{(k)}, \eta_j^{(k)})$ in rectangles $[x_i^{(k)}, x_{i+1}^{(k)}; y_j^{(k)}, y_{j+1}^{(k)}]$ ($i = 1, 2, \dots, m_k; j = 1, 2, \dots, n_k$). Write

$$S(P_k) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f(\xi_{i,j}^{(k)}, \eta_{i,j}^{(k)}) \Delta g(x_i^{(k)}, y_j^{(k)})$$

and

$$\bar{S}(P_k) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f(\xi_i^{(k)}, \eta_j^{(k)}) \Delta g(x_i^{(k)}, y_j^{(k)}).$$

If a finite limit

$$\lim_{k \rightarrow \infty} S(P_k) = I \quad [\lim_{k \rightarrow \infty} \bar{S}(P_k) = \bar{I}]$$

exists for any normal sequence $\{P_k\}$ and arbitrary points $(\xi_{i,j}^{(k)}, \eta_{i,j}^{(k)})$ $[(\xi_i^{(k)}, \eta_j^{(k)})]$, we say that f is *S-integrable* [$\bar{S}$ -integrable] with respect to g on R ; the number I [$\bar{I}$] is called the *Riemann-Stieltjes integral* of f and denoted by

$$(S) \int_a^b \int_c^d f(x, y) dg(x, y) \quad \left[(\bar{S}) \int_a^b \int_c^d f(x, y) dg(x, y) \right].$$

The Lebesgue integral of the Lebesgue-integrable (L-integrable) function $f(x, y)$ on R will be denoted by

$$(L) \int_a^b \int_c^d f(x, y) dx dy \quad \text{or} \quad (L) \int\int_R f(x, y) dx dy.$$

An analogical terminology will be used in the one-dimensional case. For example, we say that $f(x)$ is S-integrable with respect to $g(x)$ in the interval $\langle a, b \rangle$ if the simple Stieltjes integral

$$(S) \int_a^b f(x) dg(x)$$

is finite. Evidently, the concepts of S and $\bar{S}$ -integrability now coincide.

Finally, we define a double integral of Titchmarsh's type (cf. [1], pp. 585-587).

Let a function $f(x, y)$ be plane-measurable with respect to a Lebesgue measure μ in the rectangle $R = [a, b; c, d]$ and let

$$\mu\{(x, y) \in R: |f(x, y)| > n\} = o(1/n) \quad \text{as } n \rightarrow \infty.$$

Write

$$[f(x, y)]_n = \begin{cases} f(x, y) & \text{when } |f(x, y)| \leq n, \\ 0 & \text{when } |f(x, y)| > n. \end{cases}$$

If the Lebesgue integrals

$$(L) \int_a^s \int_c^t [f(x, y)]_n dx dy, \quad (s, t) \in R,$$

tend uniformly to a limit $F(s, t)$ (as $n \rightarrow \infty$) in R , we say that f is T-integrable on R ; we denote the value $F(b, d)$, termed the T-integral of f , by

$$(T) \int_a^b \int_c^d f(x, y) dx dy.$$

We assume that the last integral equals zero in the case $b = a$ or $d = c$. Also, by definition,

$$(T) \int_b^a \int_c^d f(x, y) dx dy = -(T) \int_a^b \int_c^d f(x, y) dx dy = (T) \int_a^b \int_d^c f(x, y) dx dy$$

and

$$(T) \int_b^a \int_d^c f(x, y) dx dy = (T) \int_a^b \int_c^d f(x, y) dx dy.$$

All L-integrable functions are T-integrable; the converse is false (see [2], § 3). A T-integral is homogeneous and additive.

For the sake of brevity, we shall write $f \in L(R)$ or $f \in T(R)$ etc. if f is L-integrable or T-integrable etc. on R , respectively. If $f(x, y)$ is L-inte-

grable [T-integrable] with p th power on the square $Q = [-\pi, \pi; -\pi, \pi]$ and 2π -periodic in each variable, we write $f \in L_{2\pi}^p$ [$f \in T_{2\pi}^p$]. The symbols $L_{2\pi}$ and $T_{2\pi}$ will be used instead of $L_{2\pi}^1$ and $T_{2\pi}^1$. Also (S), ($\bar{S}$), (L), (T) before integrals will sometimes be omitted.

Let $\{u_{m,n}\}$ and $\{v_{m,n}\}$, $m, n = 1, 2, \dots$, be two double sequences. We shall write

$$u_{m,n} = O(v_{m,n}) \quad \text{as} \quad (m, n) \rightarrow \infty,$$

if the quotient $u_{m,n}/v_{m,n}$ is bounded for all pairs (m, n) .

2. We shall now give some results concerning the S or $\bar{S}$ -integrability, and the relations between integrals on the rectangle $R = [a, b; c, d]$. Start with the well-known theorem ([3], pp. 515-516).

2.1. *A continuous function $f(x, y)$ is S-integrable on R with respect to a function $g(x, y)$ of finite variation in R .*

The following two-dimensional analogue of integration by parts can also easily be deduced (cf. [3], p. 609).

2.2. *Let $f(x, y)$ be $\bar{S}$ -integrable with respect to $g(x, y)$ on rectangle R , in the interval $\langle a, b \rangle$ for $y = c$, $y = d$, and in the interval $\langle c, d \rangle$ for $x = a$, $x = b$. Then $g(x, y)$ is $\bar{S}$ -integrable with respect to $f(x, y)$ on R and*

$$(4) \quad (\bar{S}) \int_a^b \int_c^d g(x, y) df(x, y) = (\bar{S}) \int_a^b \int_c^d f(x, y) dg(x, y) + \int_a^b f(x, c) dg(x, c) - \\ - \int_a^b f(x, d) dg(x, d) + \int_c^d f(a, y) dg(a, y) - \int_c^d f(b, y) dg(b, y) + D,$$

where

$$D = f(a, c)g(a, c) - f(b, c)g(b, c) - f(a, d)g(a, d) + f(b, d)g(b, d).$$

In particular, we have

2.3. *Suppose that $f(x, y)$ is continuous and $g(x, y)$ is of finite variation in R , and that $f(x, y)$ is S-integrable with respect to $g(x, y)$ in the intervals $\langle a, b \rangle$ (for $y = c, y = d$) and $\langle c, d \rangle$ (for $x = a, x = b$). Then $g(x, y)$ is $\bar{S}$ -integrable with respect to $f(x, y)$ on R .*

By 2.1, 2.2 and representation (2) [(2')], we obtain (cf. [3], p. 516).

2.4. *Let $f(x, y)$ be continuous in R and let $g(x, y)$ together with $f(x, y)$ have finite variations in R . If, moreover, $g(x, c)$ [$g(x, d)$] and $g(a, y)$ [$g(b, y)$] are of finite variation or continuous in the intervals $\langle a, b \rangle$ and $\langle c, d \rangle$, respectively, then $g(x, y)$ is S-integrable with respect to $f(x, y)$ on R ; in particular, $g(x, y)$ is Riemann-integrable on R (take $f(x, y) = xy$).*

From 2.4 and a suitable test of integrability the next theorem follows immediately (see [6], p. 233).

2.5. Suppose that $f(x, y)$ is L -integrable on R and $g(x, y)$ satisfies the same assumptions as in 2.4. Then $g(x, y)$ is S -integrable with respect to

$$F(x, y) = (L) \int_a^x \int_c^y f(s, t) ds dt \quad \text{on} \quad R, \quad fg \in L(R)$$

and

$$(S) \int_a^b \int_c^d g(x, y) dF(x, y) = (L) \int_a^b \int_c^d f(x, y) g(x, y) dx dy.$$

This result remains true for $g(x, y)$ continuous in R . Applying 2.5 and 2.2, we get (see [2], § 3)

2.6. Let $f(x, y)$ be T -integrable on R , and let

$$F(x, y) = (T) \int_a^x \int_c^y f(s, t) ds dt \quad \text{for} \quad (x, y) \in R.$$

Suppose that $g(x, y)$ is of finite variation in R and that the variations $\int_a^b |dg(x, v)|$, $\int_c^d |dg(u, y)|$ are uniformly bounded for all $(u, v) \in R$. Then $g(x, y)$ is $\bar{S}$ -integrable with respect to $F(x, y)$ on R , $fg \in T(R)$ and

$$(5) \quad (\bar{S}) \int_a^b \int_c^d g(x, y) dF(x, y) = (T) \int_a^b \int_c^d f(x, y) g(x, y) dx dy.$$

In particular our assertion holds if the function $g(x, y)$ of finite variation in R is monotonic with respect to each variable separately in R .

Denote by $C(R)$ the Banach space of all functions $f(x, y)$ continuous in R , with norm $\|f\| = \max_R |f(x, y)|$.

Arguing as in [5], pp. 167-170, we can state the following analogue of F. Riesz's theorem.

2.7. A necessary and sufficient condition for $\Phi(f)$ to be a linear functional in $C(R)$ is that

$$\Phi(f) = (S) \int_a^b \int_c^d f(x, y) dg(x, y),$$

where $g(x, y)$ is of finite variation in R .

3. In this section we present two-dimensional analogues of the theorems given in [4], pp. 79-81, and [7], pp. 211-213.

For the sake of brevity we shall write $\int_a^x \int_c^y d\gamma(s, t)$ instead of $\gamma(a, c) - \gamma(x, c) - \gamma(a, y) + \gamma(x, y)$.

3.1. Let $\alpha(x, y)$ and $\beta(x, y)$ be defined in the rectangle $R = [a, b; c, d]$ and let $\int_a^x \int_c^y d\beta(s, t) > 0$ for $a < x \leq b$, $c < y \leq d$. Write

$$M_1 = \inf_{(x,y)} \frac{\int_a^x \int_c^y d\alpha(s, t)}{\int_a^x \int_c^y d\beta(s, t)}, \quad M_2 = \sup_{(x,y)} \frac{\int_a^x \int_c^y d\alpha(s, t)}{\int_a^x \int_c^y d\beta(s, t)}$$

for $a < x \leq b$, $c < y \leq d$. Suppose that $\varphi(x, y)$ is non-negative, non-increasing with respect to each variable separately, satisfies the condition Ω in R and is $\bar{S}$ -integrable with respect to $\alpha(x, y)$ and $\beta(x, y)$ on this rectangle. Then

$$(6) \quad M_1 I(\beta) \leq I(\alpha) \leq M_2 I(\beta),$$

where

$$I(\gamma) = (\bar{S}) \int_a^b \int_c^d \varphi(x, y) d\gamma(x, y).$$

Proof. Given partition (1) let

$$(7) \quad \bar{S}(P; \alpha) = \sum_{i=1}^m \sum_{j=1}^n \varphi(\xi_i, \eta_j) \Delta \alpha(x_i, y_j),$$

where

$$\xi_1 = x_1 = a, \quad \eta_1 = y_1 = c, \quad \xi_m = x_{m+1} = b, \quad \eta_n = y_{n+1} = d, \\ (\xi_i, \eta_j) \in [x_i, x_{i+1}; y_j, y_{j+1}] \quad (i = 2, 3, \dots, m-1; j = 2, 3, \dots, n-1).$$

By Abel's transformation ([7], p. 210, (2)),

$$\begin{aligned} \bar{S}(P; \alpha) = & \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \int_a^{x_{i+1}} \int_c^{y_{j+1}} d\alpha(s, t) \cdot \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} d\varphi(s, t) + \\ & + \sum_{i=1}^{m-1} \int_a^{x_{i+1}} \int_c^d d\alpha(s, t) \cdot \int_{\xi_{i+1}}^{\xi_i} d\varphi(s, d) + \sum_{j=1}^{n-1} \int_a^b \int_c^{y_{j+1}} d\alpha(s, t) \cdot \int_{\eta_{j+1}}^{\eta_j} d\varphi(b, t) + \\ & + \int_a^b \int_c^d d\alpha(s, t) \cdot \varphi(b, d); \end{aligned}$$

whence

$$(8) \quad M_1 \bar{S}(P; \beta) \leq \bar{S}(P; \alpha) \leq M_2 \bar{S}(P; \beta).$$

If $\{P_k\}$ denotes the normal sequence of partitions (3), $\bar{S}(P_k; \alpha)$ and $\bar{S}(P_k; \beta)$ are of form (7) with suitable points $(\xi_i^{(k)}, \eta_j^{(k)})$, we have

$$(9) \quad \begin{aligned} \lim_{k \rightarrow \infty} \bar{S}(P_k; \alpha) &= (\bar{S}) \int_a^b \int_c^d \varphi(x, y) d\alpha(x, y), \\ \lim_{k \rightarrow \infty} \bar{S}(P_k; \beta) &= (\bar{S}) \int_a^b \int_c^d \varphi(x, y) d\beta(x, y). \end{aligned}$$

Hence, by (9) and (8), inequality (6) follows.

The following corollary is useful.

3.2. Let $f(x, y)$ and $g(x, y)$ be T-integrable on $R = [a, b; c, d]$,

$$\alpha(x, y) = (T) \int_a^x \int_c^y f(s, t) ds dt, \quad \beta(x, y) = (T) \int_a^x \int_c^y g(s, t) ds dt > 0$$

for $a < x \leq b$, $c < y \leq d$. Suppose that $\varphi(x, y)$, non-negative and non-increasing in each variable, satisfies the condition Ω in R . Then, by 2.6 and 3.1,

$$M_1(T) \int_a^b \int_c^d g \varphi dx dy \leq (T) \int_a^b \int_c^d f \varphi dx dy \leq M_2(T) \int_a^b \int_c^d g \varphi dx dy,$$

where M_1, M_2 denote the infimum and the supremum of the quotient

$$(T) \int_a^x \int_c^y f(s, t) ds dt / (T) \int_a^x \int_c^y g(s, t) ds dt$$

for $a < x \leq b$, $c < y \leq d$ (cf. [7], p. 211).

Taking $g(x, y) \equiv 1$ in 3.2, we get the estimate (cf. [7], pp. 213-215)

$$(10) \quad \left| (T) \int_a^b \int_c^d f(x, y) \varphi(x, y) dx dy \right| \leq M \int_a^b \int_c^d \varphi(x, y) dx dy,$$

with

$$(11) \quad M = \sup_{\substack{a < x \leq b \\ c < y \leq d}} \left| \frac{1}{(x-a)(y-c)} (T) \int_a^x \int_c^y f(s, t) ds dt \right|.$$

This is an analogue of Natanson's lemma ([6], p. 245).

Now a generalization of (10) will be given.

3.3. Let $\varphi(x, y)$ satisfy the same assumptions as $g(x, y)$ in 2.6. If $f(x, y)$ is T-integrable on R , then

$$(12) \quad \begin{aligned} \left| (T) \int_a^b \int_c^d f(x, y) \varphi(x, y) dx dy \right| \\ \leq M \int_a^b \int_c^d [A(x, y) + B(x) + C(x) + |\varphi(b, d)|] dx dy, \end{aligned}$$

where M is defined by (11) and

$$A(x, y) = \begin{cases} \int_x^b \int_y^d |d\varphi(s, t)| & \text{for } a \leq x < b, c \leq y < d, \\ 0 & \text{otherwise;} \end{cases}$$

$$B(x) = \begin{cases} \int_x^b |d\varphi(s, d)| & \text{for } a \leq x < b, \\ 0 & \text{for } x = b; \end{cases}$$

$$C(x) = \begin{cases} \int_y^d |d\varphi(b, t)| & \text{for } c \leq y < d, \\ 0 & \text{for } y = d. \end{cases}$$

Proof. Write

$$I = (T) \int_a^b \int_c^d f(x, y) \varphi(x, y) dx dy,$$

and

$$F(x, y) = (T) \int_a^x \int_c^y f(s, t) ds dt \quad \text{for } (x, y) \in R.$$

Applying (5) and (4), we have

$$I = (\bar{S}) \int_a^b \int_c^d \varphi(x, y) dF(x, y) = (S) \int_a^b \int_c^d F(x, y) d\varphi(x, y) - \\ - \int_a^b F(x, d) d\varphi(x, d) - \int_c^d F(b, y) d\varphi(b, y) + F(b, d) \varphi(b, d),$$

whence

$$|I| \leq \int_a^b \int_c^d |F(x, y)| dA(x, y) - \int_a^b |F(x, d)| dB(x) - \int_c^d |F(b, y)| dC(x) + \\ + |F(b, d)| |\varphi(b, d)| = I_1 - I_2 - I_3 + K.$$

In view of (11) and (4)

$$I_1 \leq M \int_a^b \int_c^d (x-a)(y-c) dA(x, y) = M \int_a^b \int_c^d A(x, y) dx dy.$$

Similarly, (11) and integration by parts leads to

$$-I_2 \leq M(d-c) \int_a^b (x-a) d(-B(x)) = M \int_a^b \int_c^d B(x) dx dy, \\ -I_3 \leq M(b-a) \int_c^d (y-c) d(-C(x)) = M \int_a^b \int_c^d C(x) dx dy.$$

Summing up the results, we obtain inequality (12).

Also, three variants of (12) can be established by taking

$$M = \sup_{\substack{a \leq x \leq b \\ c \leq y \leq d}} \left| \frac{1}{(b-x)(y-c)} (T) \int_x^b \int_c^y f(s, t) ds dt \right|, \quad \text{etc.}$$

instead of M defined by (11).

4. Now we shall give two theorems of Lebesgue's type (cf. [6], pp. 240-242).

4.1. Suppose that the functions $g_n(x, y)$ of finite variations in $R = [a, b; c, d]$ are uniformly bounded in R for $n = 1, 2, \dots$. Moreover, let the variations

$$\int_a^b \int_c^d |dg_n(x, y)|, \quad \int_a^b |dg_n(x, d)|, \quad \int_c^d |dg_n(b, y)|$$

remain uniformly bounded for $n = 1, 2, \dots$, and let

$$\lim_{n \rightarrow \infty} (L) \int_a^x \int_c^y g_n(x, y) dx dy = 0$$

for each $(x, y) \in R$. Then

$$\lim_{n \rightarrow \infty} (T) \int_a^b \int_c^d f(x, y) g_n(x, y) dx dy = 0$$

for any f T-integrable on R .

Proof. Given $\varepsilon > 0$ there is a function f_* L-integrable on R such that

$$\left| (T) \int_a^x \int_c^y f(s, t) ds dt - (L) \int_a^x \int_c^y f_*(s, t) ds dt \right| < \varepsilon$$

uniformly, with respect to (x, y) , in R . Write

$$\begin{aligned} F(x, y) &= (T) \int_a^x \int_c^y f(s, t) ds dt, & F_*(x, y) &= (L) \int_a^x \int_c^y f_*(s, t) ds dt, \\ H_n &= (T) \int_a^b \int_c^d f(s, t) g_n(s, t) ds dt, & I_n &= (L) \int_a^b \int_c^d f_*(s, t) g_n(s, t) ds dt. \end{aligned}$$

In view of 2.6 and 2.2,

$$|H_n - I_n| = \left| (\bar{S}) \int_a^b \int_c^d g_n(x, y) d[F(x, y) - F_*(x, y)] \right| \leq K\varepsilon,$$

where K is a positive constant.

Since $I_n \rightarrow 0$ as $n \rightarrow \infty$, our result is established.

It is easy to see that if $g_n(x, y)$ are uniformly bounded and monotonic in each variable separately, and if they satisfy the condition Ω in R , the variation-restrictions on g_n follow automatically.

The index n in 4.1 can be replaced by the parameter $\sigma = (\xi, \eta)$, where (ξ, η) belong to a set E on the plane, with an accumulation point $\tau = (\xi_0, \eta_0)$. In this case, we consider $\sigma \rightarrow \tau$ instead of $n \rightarrow \infty$.

This parameter will be introduced in the formulation of the next theorems.

4.2. Suppose that the functions $K(s, t; \sigma)$ are non-negative, 2π -periodic, even and non-increasing in the square $Q_0 = [0, \pi; 0, \pi]$ with respect to each of the variables s and t separately, and that they satisfy the condition Ω in Q_0 for every fixed $\sigma \in E$. Let δ, x_0, c, d be fixed, $0 < \delta < \pi$, $-\pi + \delta \leq x_0 \leq \pi - \delta$, $-\pi \leq c < d \leq \pi$, and let the values $K(\delta/2, 0; \sigma)$ remain uniformly bounded for all $\sigma \in E$. Then, if the relation

$$\lim_{\delta \rightarrow \tau} \int_{x_0+\delta}^{\pi} \int_c^d K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

holds uniformly in x, y , $x_0 - \delta/2 \leq x \leq x_0 + \delta/2$, $-\pi \leq y \leq \pi$, we have

$$(13) \quad \lim_{\delta \rightarrow \tau} (T) \int_{x_0+\delta}^{\pi} \int_c^d f(s, t) K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

uniformly with respect to x, y for any function f T-integrable on $[x_0 + \delta, \pi; c, d]$.

Relation (13) for continuous or essentially bounded f can be established similarly to that of [6], p. 241. In the general case ($f \in T$) we argue as in 4.1.

Analogical results hold for the integrals

$$\int_{-\pi}^{x_0-\delta} \int_c^d, \quad \int_a^b \int_{y_0+\delta}^{\pi}, \quad \int_a^b \int_{-\pi}^{y_0-\delta}$$

of the function $f(s, t)K(s-x, t-y; \sigma)$.

5. Suppose that the functions $\varphi(x, y; \sigma)$ are non-negative, and non-increasing with respect to x and y separately, and that they satisfy the condition Ω in the rectangle $R = [a, b; c, d]$ for any fixed $\sigma \in E$. Denote by $T^*(R)$ the class of all functions $f \in T(R)$ such that

$$\lim_{\substack{h \rightarrow 0+ \\ k \rightarrow 0+}} \frac{1}{hk} \int_a^{a+h} \int_c^{c+k} f(x, y) dx dy = f(a, c).$$

We shall now give a theorem of Romanovski's type concerning the convergence of singular integrals

$$I(\sigma; f) = (T) \int_a^b \int_c^d f(x, y) \varphi(x, y; \sigma) dx dy$$

as $\sigma \rightarrow \tau$, for $f \in T^*(R)$. Of course, these integrals exist by 2.6.

5.1. *If*

$$\lim_{\sigma \rightarrow \tau} \int_a^a \int_c^\gamma \varphi(x, y; \sigma) dx dy = 1 \quad (\sigma \in E)$$

and

$$\overline{\lim}_{\sigma \rightarrow \tau} \varphi(a, c; \sigma) < \infty, \quad \overline{\lim}_{\sigma \rightarrow \tau} \varphi(a, \gamma; \sigma) < \infty \quad (\sigma \in E),$$

with any fixed a, γ ($a < a \leq b$, $c < \gamma \leq d$), then

$$\lim_{\sigma \rightarrow \tau} I(\sigma; f) = f(a, c) \quad (\sigma \in E)$$

for every $f \in T^*(R)$.

Proof. Obviously, it is sufficient to show that

$$U(\sigma; f) = (T) \int_a^b \int_c^d g(x, y) \varphi(x, y; \sigma) dx dy \rightarrow 0 \quad \text{as } \sigma \rightarrow \tau,$$

where $g(x, y) = f(x, y) - f(a, c)$.

Since $f \in T^*(R)$, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{1}{hk} (T) \int_a^{a+h} \int_c^{c+k} g(x, y) dx dy \right| < \varepsilon$$

for $0 < h, k \leq \delta$; whence, by (10),

$$(14) \quad \left| (T) \int_a^{a+\delta} \int_c^{c+\delta} g(x, y) \varphi(x, y; \sigma) dx dy \right| \leq \varepsilon \int_a^b \int_c^d \varphi(x, y; \sigma) dx dy.$$

Write

$$U(\sigma; f) = \left(\int_a^{a+\delta} \int_c^{c+\delta} + \int_{a+\delta}^b \int_c^d + \int_a^{a+\delta} \int_{c+\delta}^d \right) g \varphi dx dy = A_\sigma + B_\sigma + C_\sigma.$$

In view of (14),

$$|A_\sigma| < 2\varepsilon$$

for σ sufficiently near to τ .

Applying 4.1, we get

$$\lim_{\sigma \rightarrow \tau} B_\sigma = 0 = \lim_{\sigma \rightarrow \tau} C_\sigma,$$

and our result is established.

We now formulate a theorem of Faddeev's type ([6], p. 246) concerning the convergence of singular integrals

$$J(\sigma; f) = (L) \int_a^b \int_c^d f(x, y) \psi(x, y; \sigma) dx dy ,$$

where $\psi(x, y; \sigma)$ are bounded (measurable) for any fixed $\sigma \in E$, and together with $f(x, y)$ subject to further restrictions.

Denote by $L^*(R)$ the class of all $f \in L(R)$ for which

$$\lim_{\substack{h \rightarrow 0+ \\ k \rightarrow 0+}} \frac{1}{hk} \int_a^{a+h} \int_c^{c+k} |f(x, y) - f(a, c)| dx dy = 0 .$$

5.2. Suppose that

$$\lim_{\sigma \rightarrow \tau} \int_a^a \int_c^y \psi(x, y; \sigma) dx dy = 1$$

and $\psi(x, y; \sigma)$ remain uniformly bounded in the domain $[a, b; c, d] - [a, a; c, \gamma]$ for every fixed a, γ ($a < a \leq b, c < \gamma \leq d$), $\sigma \in E$. If, moreover, there exist functions $\mu(x, y; \sigma)$ non-negative, non-increasing in x and y , satisfying the condition Ω in R , and such that

$$|\psi(x, y; \sigma)| \leq \mu(x, y; \sigma)$$

for $(x, y) \in R, \sigma \in E$, and

$$\overline{\lim}_{\sigma \rightarrow \tau} \int_a^b \int_c^d \mu(x, y; \sigma) dx dy < \infty \quad (\sigma \in E) ,$$

then

$$\lim_{\sigma \rightarrow \tau} J(\sigma; f) = f(a, c) \quad (\sigma \in E)$$

for every $f \in L^*(R)$.

6. In the theory of summability of double Fourier series the integrals

$$V(x, y; \sigma, f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) K(s-x, t-y, \sigma) ds dt \quad (\sigma \in E)$$

play a fundamental role, where the functions $K(s, t; \sigma)$ are bounded (measurable), even and 2π -periodic in each variable x, y separately, and

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(s, t; \sigma) ds dt = 1 \quad (\sigma \in E) .$$

These integrals, for $f \in T_{2\pi}$, can be rewritten in the form

$$V(x, y; \sigma, f) = (T) \int_0^\pi \int_0^\pi g(s, t) K(s, t; \sigma) ds dt,$$

where

$$g(s, t) = f(x+s, y+t) + f(x-s, y+t) + f(x+s, y-t) + f(x-s, y-t).$$

Applying 5.1 and 5.2, we get the following theorem.

6.1. *If $4K(s, t; \sigma)$ satisfies the same conditions as $\varphi(s, t; \sigma)$ [$\psi(s, t; \sigma)$] in 5.1 [5.2], with $a = c = 0$, $b = d = \pi$, and if*

$$\lim_{\substack{h \rightarrow 0+ \\ k \rightarrow 0+}} \frac{1}{hk} (T) \int_0^h \int_0^k g(s, t) ds dt = 4f(x, y) \quad (f \in T_{2\pi})$$

$$\left[\lim_{\substack{h \rightarrow 0+ \\ k \rightarrow 0+}} \frac{1}{hk} (L) \int_0^h \int_0^k |g(s, t) - 4f(x, y)| ds dt = 0 \quad (f \in L_{2\pi}) \right],$$

we have

$$\lim_{\sigma \rightarrow \tau} V(x, y; \sigma, f) = f(x, y).$$

The above-mentioned and the following statements refer to the restricted summability of double Fourier series.

6.2. *Let $K(s, t; \sigma)$ satisfy all the assumptions of 4.2 and let $K(\delta/2, 0; \sigma)$, $K(0, \delta/2; \sigma)$ be uniformly bounded in E , for any fixed $\delta > 0$ as small as we please. Suppose that the relations*

$$\lim_{\sigma \rightarrow \tau} \int_{x_0+\delta}^\pi \int_{-\pi}^\pi K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

and

$$\lim_{\sigma \rightarrow \tau} \int_{-\pi}^{x_0-\delta} \int_{-\pi}^\pi K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

hold uniformly in x, y , $x_0 - \delta/2 \leq x \leq x_0 + \delta/2$, $-\pi \leq y \leq \pi$, for every fixed δ , $0 < \delta \leq \pi \pm x_0$. Suppose that the relations

$$\lim_{\sigma \rightarrow \tau} \int_{-\pi}^\pi \int_{y_0+\delta}^\pi K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

and

$$\lim_{\sigma \rightarrow \tau} \int_{-\pi}^\pi \int_{-\pi}^{y_0-\delta} K(s-x, t-y; \sigma) ds dt = 0 \quad (\sigma \in E)$$

hold uniformly in x, y , $-\pi \leq x \leq \pi$, $y_0 - \delta/2 \leq y \leq y_0 + \delta/2$ ($y_0 = \text{const}$) for every fixed δ , $0 < \delta \leq \pi \pm y_0$. Denote by Z the set of points $(x, y; \sigma)$, $\sigma \in E$, such that the product $(x - x_0)(y - y_0)K(0, 0; \sigma)$ remains bounded. Then the relation

$$\lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{1}{hk} (T) \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(s, t) ds dt = f(x_0, y_0) \quad (f \in T_{2\pi})$$

implies

$$\lim V(x, y; \sigma, f) = f(x_0, y_0) \quad (\sigma \in E)$$

as $(x, y; \sigma) \rightarrow (x_0, y_0; \tau)$ on Z .

The proof is similar to that of 3.1, [8]. Namely, given $\varepsilon > 0$ we choose a $\delta > 0$ such that

$$\left| \frac{1}{hk} (T) \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} [f(s, t) - f(x_0, y_0)] ds dt \right| < \varepsilon$$

for $|h| \leq \delta$, $|k| \leq \delta$. Write

$$\begin{aligned} & \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(s, t) - f(x_0, y_0)] K(s - x, t - y; \sigma) ds dt \\ &= \left(\iint_G + \iint_{\Gamma} \right) [f(s, t) - f(x_0, y_0)] K(s - x, t - y; \sigma) ds dt = I_1 + I_2, \end{aligned}$$

where

$$G = [x_0 - \delta, x_0 + \delta; y_0 - \delta, y_0 + \delta], \quad \Gamma = [-\pi, \pi; -\pi, \pi] - G.$$

Applying 3.2, 3.3 and 4.2, we get

$$|I_1| + |I_2| \leq C\varepsilon, \quad C = \text{const},$$

for $(x, y; \sigma)$ sufficiently near to $(x_0, y_0; \tau)$. The proof is thus finished. An analogical statement of Faddeev's type can also be given.

We note that theorem (3.1) of [10], p. 309 ($m = 2$) follows at once from 6.1 and 6.2 (see also [9], p. 497).

7. Write

$$\Delta_{h,k}^{(r)} f(x, y) = \sum_{\mu=0}^r \sum_{\nu=0}^r (-1)^{2r-\mu-\nu} \binom{r}{\mu} \binom{r}{\nu} f(x + \mu h, y + \nu k),$$

$$\Delta_h^{(r,1)} f(x, y) = \sum_{\mu=0}^r (-1)^{r-\mu} \binom{r}{\mu} f(x + \mu h, y),$$

$$\Delta_k^{(r,2)} f(x, y) = \sum_{\nu=0}^r (-1)^{r-\nu} \binom{r}{\nu} f(x, y + \nu k).$$

The class of all functions $f \in L^2_{2\pi}$ such that

$$\left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{h,k}^{(r)} f(x, y)|^2 dx dy \right\}^{1/2} \leq M h^{\alpha} k^{\beta} \quad (0 < \alpha, \beta \leq 1)$$

for sufficiently small $h, k > 0$, M being a constant depending only on f , will be denoted by $\Lambda^{(r)}(\alpha, \beta)$.

The class of all $f \in \Lambda^{(r)}(\alpha, \beta)$ for which

$$\left\{ \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \Delta_h^{(r,1)} f(x, y) dy \right)^2 dx \right\}^{1/2} \leq M_1 h^{\alpha'} \quad (0 < \alpha' \leq 1)$$

and

$$\left\{ \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \Delta_k^{(r,2)} f(x, y) dx \right)^2 dy \right\}^{1/2} \leq M_2 k^{\beta'} \quad (0 < \beta' \leq 1)$$

as $h, k > 0$ small enough, M_1, M_2 are positive constants, will be denoted by $\Lambda^{(r)}(\alpha, \beta; \alpha', \beta')$.

Analogously we define the classes $H^{(r)}(\alpha, \beta)$ and $H^{(r)}(\alpha, \beta; \alpha', \beta')$ of certain 2π -periodic functions $f(x, y)$ essentially bounded in the square $Q = [-\pi, \pi; -\pi, \pi]$; we take the metrics $\text{ess sup}_Q |\varphi(x, y)|$ and $\text{ess sup}_{-\pi \leq z \leq \pi} |\psi(z)|$

in the above inequalities instead of $\left\{ \iint_Q \varphi^2(x, y) dx dy \right\}^{1/2}$ and $\left\{ \int_{-\pi}^{\pi} \psi^2(z) dz \right\}^{1/2}$.

In the sequel, only $r = 1$ and $r = 2$ will be considered.

We shall now give some relations between the above-mentioned classes of functions and the classes of Fourier coefficients corresponding to them.

7.1. Let $0 < \alpha, \beta, \alpha', \beta' < 1$. Suppose that $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$ ($m, n \geq 1$) are of the order $O(m^{-(\alpha+1/2)} n^{-(\beta+1/2)})$ as $(m, n) \rightarrow \infty$ and that $a_{m,0}, b_{m,0}$ and $a_{0,n}, c_{0,n}$ are $O(m^{-(\alpha'+1/2)})$ as $m \rightarrow \infty$ and $O(n^{-(\beta'+1/2)})$ as $n \rightarrow \infty$, respectively. Then there is a function $f \in \Lambda^{(1)}(\alpha, \beta; \alpha', \beta')$ having the double Fourier series ([9], pp. 435-446)

$$(15) \quad \sum_{m,n=0}^{\infty} \lambda_{m,n} (a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny).$$

Proof. By the Riesz-Fischer theorem, there is an $f \in L^2_{2\pi}$ with Fourier series (15).

It is easy to verify that

$$\begin{aligned} \Delta_{2h,2k}^{(1)} f(x-h, y-k) \\ = f(x-h, y-k) - f(x+h, y-k) - f(x-h, y+k) + f(x+h, y+k) \end{aligned}$$

has the Fourier series

$$4 \sum_{m,n=1}^{\infty} (a_{m,n} \sin mx \sin ny - b_{m,n} \cos mx \sin ny - \\ - c_{m,n} \sin mx \cos ny + d_{m,n} \cos mx \cos ny) \sin mh \sin nk.$$

In view of Parseval's identity ([9], p. 509),

$$(16) \quad \sum_{m,n=1}^{\infty} \varrho_{m,n}^2 (\sin mh \sin nk)^2 = \frac{1}{(4\pi)^2} \iint_Q \{ \Delta_{2h,2k}^{(1)} f(x-h, y-k) \}^2 dx dy,$$

where

$$\varrho_{m,n} = (a_{m,n}^2 + b_{m,n}^2 + c_{m,n}^2 + d_{m,n}^2)^{1/2}.$$

The left-hand side of (16) does not exceed

$$\sum_{m=1}^{2^p-1} \sum_{n=1}^{2^q-1} \varrho_{m,n}^2 (\sin mh \sin nk)^2 + \sum_{m=2^p}^{\infty} \sum_{n=2^q}^{\infty} \varrho_{m,n}^2 + \\ + \sum_{m=2^p}^{\infty} \sum_{n=1}^{2^q-1} \varrho_{m,n}^2 \sin^2 nk + \sum_{m=1}^{2^p-1} \sum_{n=2^q}^{\infty} \varrho_{m,n}^2 \sin^2 mh = S_1 + S_2 + S_3 + S_4.$$

By hypothesis

$$(17) \quad \sum_{m=\mu}^{\infty} \sum_{n=\nu}^{\infty} \varrho_{m,n}^2 \leq A \frac{1}{\mu^{2\alpha} \nu^{2\beta}}, \quad A = \text{const};$$

whence

$$\sum_{m=\mu}^{2^{\mu}-1} \sum_{n=\nu}^{2^{\nu}-1} (mn \varrho_{m,n})^2 \leq 16A \mu^{2(1-\alpha)} \nu^{2(1-\beta)}.$$

Therefore

$$S_1 \leq 16A (hk)^2 \sum_{i=1}^p \sum_{j=1}^q 2^{2(1-\alpha)(i-1)} 2^{2(1-\beta)(j-1)}.$$

Suppose that h, k are positive and choose integers p, q such that

$$(18) \quad 2^{-p} < h \leq 2^{-p+1}, \quad 2^{-q} < k \leq 2^{-q+1}.$$

Then

$$S_1 = 16A (hk)^2 \frac{2^{2(1-\alpha)p} - 1}{2^{2(1-\alpha)} - 1} \cdot \frac{2^{2(1-\beta)q} - 1}{2^{2(1-\beta)} - 1} \leq A_1 h^{2\alpha} k^{2\beta}, \quad A_1 = \text{const}.$$

By (17) and (18),

$$S_2 \leq A \cdot 2^{-2\alpha p} \cdot 2^{-2\beta q} \leq A h^{2\alpha} k^{2\beta}.$$

Further, we have

$$\begin{aligned} S_3 &\leq k^2 \sum_{n=1}^{2^q-1} \left\{ n^2 \sum_{m=2^p}^{\infty} \varrho_{m,n}^2 \right\} \leq B k^2 \sum_{n=1}^{2^q-1} \frac{1}{n^{2\beta-1}} \cdot \frac{1}{2^{2\alpha p}} \\ &\leq B_1 k^2 \cdot \frac{1}{2^{2\alpha p}} \cdot 2^{2(1-\beta)q} < 4 B_1 h_-^{2\alpha} k^{2\beta}, \end{aligned}$$

where B and B_1 are positive constants. A similar inequality holds for S_4 .

Summing up the results, we get $f \in \Lambda^{(1)}(\alpha, \beta)$.

The functions

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) dy \quad \text{and} \quad \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) dx$$

have the Fourier series

$$\frac{a_{0,0}}{2} + \sum_{m=1}^{\infty} (a_{m,0} \cos mx + b_{m,0} \sin mx)$$

and

$$\frac{a_{0,0}}{2} + \sum_{n=1}^{\infty} (a_{0,n} \cos ny + c_{0,n} \sin ny),$$

respectively. Hence, $f \in \Lambda^{(1)}(\alpha, \beta; \alpha', \beta')$ (see [1], pp. 208-210).

Also, we can prove that if the numbers $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$ ($m, n \geq 0$) satisfy the assumptions of 7.1 for $\alpha = \beta = \alpha' = \beta' = 1$, then there exists a continuous $f \in \Lambda^{(2)}(1, 1; 1, 1)$ with Fourier series (15).

Under the additional restriction on the coefficients, the following inverse result holds.

7.2. Let $0 < \alpha, \beta, \alpha', \beta' < 1$. Suppose that (15) is the Fourier series of $f \in \Lambda^{(1)}(\alpha, \beta)$ and that $a_{m,n}^2 + b_{m,n}^2 + c_{m,n}^2 + d_{m,n}^2$ is non-increasing in $m, n \geq 1$, separately. Then $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$ ($m, n \geq 1$) are $O(m^{-(\alpha+\frac{1}{2})} n^{-(\beta+\frac{1}{2})})$ as $(m, n) \rightarrow \infty$. If, moreover, $f \in \Lambda^{(1)}(\alpha, \beta; \alpha', \beta')$ and if $a_{m,0}^2 + b_{m,0}^2$ and $a_{0,n}^2 + c_{0,n}^2$ are non-increasing in m and n , then $a_{m,0}, b_{m,0}, a_{0,n}, c_{0,n}$ are $O(m^{-(\alpha'+\frac{1}{2})})$ as $m \rightarrow \infty$ and $O(n^{-(\beta'+\frac{1}{2})})$ as $n \rightarrow \infty$, respectively.

We prove the first part only (cf. [1], pp. 678-679). By formula (16),

$$\sum_{m=p}^q \sum_{n=r}^s \varrho_{m,n}^2 (\sin mh \sin nk)^2 \leq C |h|^{2\alpha} |k|^{2\beta},$$

where

$$p = [(q+1)/2], \quad r = [(s+1)/2], \quad C = \text{const.}$$

Set $h = \pi/2q, k = \pi/2s$. Since

$$\sin^2 m \frac{\pi}{2q} \geq \frac{1}{2}, \quad \sin^2 n \frac{\pi}{2s} \geq \frac{1}{2}$$

for $p \leq m \leq q$ and $r \leq n \leq s$, we have

$$\frac{1}{2^2} \sum_{m=p}^q \sum_{n=r}^s \varrho_{m,n}^2 \leq C \left(\frac{\pi}{2q} \right)^{2\alpha} \left(\frac{\pi}{2s} \right)^{2\beta}.$$

From the last inequality our result follows immediately.

We remark that the class $\Lambda^{(1)}(1,1)$ in 7.2 can be replaced by $\Lambda^{(2)}(1,1)$.

Similar results hold for the classes $H^{(r)}(\alpha, \beta)$ and $H^{(r)}(\alpha, \beta; \alpha', \beta')$ ($r = 1, 2$). Namely,

7.3. If $f(x, y)$ is expansible in series (15) with coefficients $a_{m,n}$, $b_{m,n}$, $c_{m,n}$, $d_{m,n}$ ($m, n \geq 1$) of the order $O(m^{-(\alpha+1)}n^{-(\beta+1)})$ ($0 < \alpha, \beta < 1$) [$O(m^{-2}n^{-2})$], then $f \in H^{(1)}(\alpha, \beta)$ [$f \in H^{(2)}(1, 1)$]. If, moreover, $a_{m,0}$, $b_{m,0}$, $a_{0,n}$, $c_{0,n}$, are $O(m^{-(\alpha'+1)})$ [$O(m^{-2}n^{-2})$], $O(n^{-(\beta'+1)})$ [$O(m^{-2}n^{-2})$], respectively, where $0 < \alpha', \beta' < 1$, then $f \in H^{(1)}(\alpha, \beta; \alpha', \beta')$ [$f \in H^{(2)}(1, 1; 1, 1)$] (cf. [1], pp. 208-210).

7.4. Let $f \in H^{(1)}(\alpha, \beta)$ ($0 < \alpha, \beta \leq 1$) [$f \in H^{(2)}(1, 1)$] and let

$$f(x, y) = \sum_{m,n=0}^{\infty} \lambda_{m,n} a_{m,n} \cos mx \cos ny$$

for all (x, y) , where $a_{m,n}$ are non-negative and non-increasing in $m, n \geq 1$. Then

$$a_{m,n} = O(m^{-(\alpha+1)}n^{-(\beta+1)}) \quad [a_{m,n} = O(m^{-2}n^{-2})] \quad \text{as} \quad (m, n) \rightarrow \infty.$$

We obtain this statement at once from the identity

$$\Delta_{2h,2k}^{(2)} f(x-2h, y-2k) = 16 \sum_{m,n=1}^{\infty} a_{m,n} \cos mx \cos ny (\sin mh \sin nk)^2,$$

taking $x = y = 0$ and arguing as in [1], p. 678.

7.5. Suppose that the function $f \in H^{(1)}(\alpha, \beta)$ ($0 < \alpha, \beta \leq 1$) [$f \in H^{(2)}(1, 1)$] has the Fourier series

$$\sum_{m,n=1}^{\infty} d_{m,n} \sin mx \sin ny$$

with coefficients $d_{m,n} \geq 0$ such that $d_{m,n}/mn$ is non-increasing in $m \geq 1$ and $n \geq 1$, separately. Then

$$d_{m,n} = O(m^{-(\alpha+1)}n^{-(\beta+1)}) \quad [d_{m,n} = O(m^{-2}n^{-2})] \quad \text{as} \quad (m, n) \rightarrow \infty.$$

To prove 7.5 we observe that $\Delta_{2h,2k}^{(2)} f(x-2h, y-2k)$ has the Fourier series

$$16 \sum_{m,n=1}^{\infty} d_{m,n} \sin mx \sin ny (\sin mh \sin nk)^2.$$

Hence ([9], pp. 514-516)

$$\int_0^{2h} \int_0^{2k} \Delta_{2h,2k}^{(2)} f(x-2h, y-2k) dx dy = 64 \sum_{m,n=1}^{\infty} \frac{d_{m,n}}{mn} (\sin mh \sin nk)^4.$$

Taking $h = \pi/2m$, $k = \pi/2n$ and reasoning as in [1], pp. 678-679, we shall establish our result.

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