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Applications of elementary topological methods to existence problems for bounded solutions of systems of ordinary differential equations

by J. Lewowicz (Montevideo)

1. Introduction. This paper contains essentially two results, of which we give elementary topological proofs. The first one refers to a non-autonomous system with the trivial solution $x = 0$; the second one concerns the autonomous case, although the nature of the conclusions and of the arguments used allows us to state it as a theorem on dynamical systems.

These results generalize those obtained recently by N. Onuchic [2] and P. Mendelson [1] as applications of the method of T. Ważewski [4] and of the theorem of A. Pliś [3] (plus results of algebraic topology). We thus find two new examples of problems in which it is advantageous to replace the latter methods by an elementary topological approach.

I am indebted to Professor J. L. Massera for his generous assistance and guidance. His valuable suggestions have had a significant influence on the final statements and proofs of the theorems.

2. We shall assume that the systems considered in this paper satisfy conditions which ensure existence and uniqueness of the solutions and their continuous dependence on the initial conditions.

Consider the system

$$(1)\qquad \dot{x} = f(t, x), \quad x \in R^n,\ t \in R^+ = [0, \infty), \quad f\colon R^+ \times R^n \to R^n.$$

Assume that $f(t, 0) = 0$, $t \in R^+$.

We shall denote with $x(t, t_0, x_0)$ the solution of (1) through the initial point (t_0, x_0), i.e. $x(t_0, t_0, x_0) = x_0$, and with $\alpha(t_0, x_0)$, $\beta(t_0, x_0)$ the endpoints of the maximum interval on which $x(t, t_0, x_0)$ is defined.

Let ω be an open set in $R^+ \times R^n$ containing $R^+ \times \{0\}$, and let H be the hyperplane $\{0\} \times R^n$. We shall denote by $\partial\omega$ the boundary of ω.

A point $(t_0, x_0) \in \partial\omega - H$ is a *point of ingress* (T. Ważewski) (with respect to ω and (1)) if there exists $\delta > 0$, such that $\big(t, x(t, t_0, x_0)\big) \in \omega$, $t_0 < t < t_0 + \delta$; it is a *point of strict ingress*, if it is one of ingress and if, for every $\delta > 0$, $(t, x(t, t_0, x_0)) \notin \overline{\omega}$ for some t, $t_0 - \delta < t < t_0$. The *points of egress* and *strict egress* are defined in a similar way. The sets of points of egress, strict egress, ingress, and strict ingress, shall be denoted by E, E^*, I, I^*, respectively. In the applications of the method of Ważewski the assumption $E = E^*$ plays a fundamental rôle while we shall assume instead $I = I^*$. In this connection, it is to be remarked that *if no solution has an arc on $\partial\omega$, then $E = E^*$ if, and only if, $I = I^*$.*

THEOREM 1. *Let $\omega_0 \subset \omega \cap H$ be a bounded open set which contains the origin. Suppose that for every $n = 1, 2, \ldots$ there is an arc Γ_n contained (except for one endpoint) in $\omega \cap \{(t, x)\colon t > n\}$ that connects $x = 0$ with a point of I. Assume moreover that $I = I^*$. Then, there exists $(0, x^*) \in \overline{\omega}_0$, $x^* \neq 0$, such that $(t, x(t, 0, x^*)) \in \overline{\omega}$, $0 \leqslant t < \beta(0, x^*)$.*

Remark. There exists such a point x^* with either $x^* \in \partial\omega_0$ or $\beta(0, x^*) < \infty$.

Proof. Let Γ_n $(n = 1, 2, \ldots)$ be an arc which satisfies the conditions mentioned in the statement of the theorem. Consider the following disjoint subsets of Γ_n:

$$A_n = \{(t_0, x_0)\colon (t, x(t, t_0, x_0)) \in \omega,\ 0 \leqslant t \leqslant t_0,\ (0, x(0, t_0, x_0)) \in \omega_0\},$$

$$B_n = \{(t_0, x_0)\colon (t, x(t, t_0, x_0)) \in \omega,\ 0 < t \leqslant t_0,\ (0, x(0, t_0, x_0)) \in \partial\omega_0\},$$

$$C_n = \{(t_0, x_0)\colon x(t, t_0, x_0) \text{ exists on } [0, t_0] \text{ and either } (0, x(0, t_0, x_0)) \in H - \overline{\omega}_0, \text{ or, } (\bar{t}, x(\bar{t}, t_0, x_0)) \notin \overline{\omega} \text{ for some } \bar{t},\ 0 < \bar{t} < t_0\},$$

$$D_n = \{(t_0, x_0)\colon x(t, t_0, x_0) \text{ is not defined on the closed interval } [0, t_0]\}.$$

From the continuous dependence of the solutions on the initial conditions we easily infer that A_n is open in Γ_n, and it is not void because it contains the end point of Γ_n with $x = 0$. Similarly C_n is open, and obviously $C_n \cup D_n$ contains the other endpoint of Γ_n (which belongs to I^*).

Consider the set $C_n \cup D_n$, and suppose first that for some n it is not open. Then there exists a point $(t_0, x_0) \in D_n$ and a sequence (t_{0m}, x_{0m}) in Γ_n, such that $\lim (t_{0m}, x_{0m}) = (t_0, x_0)$, $(t_{0m}, x_{0m}) \notin C_n \cup D_n$. Then $x(0, t_{0m}, x_{0m})$ is defined for every $m = 1, 2, \ldots$, $(0, x(0, t_{0m}, x_{0m})) \in \overline{\omega}_0$, and $(t, x(t, t_{0m}, x_{0m})) \in \overline{\omega}$, $0 \leqslant t \leqslant t_{0m}$. Let (m_k) be a subsequence of (m), such that $(0, x(0, t_{0m_k}, x_{0m_k}))$ converges to, say, $(0, x^*) \in \overline{\omega}_0$. Continuous dependence on the initial conditions obviously implies that $\beta(0, x^*) \leqslant t_0$ (hence $x^* \neq 0$) and that $(t, x(t, 0, x^*)) \in \overline{\omega}$, $0 \leqslant t < \beta(0, x^*)$ so that in this case the statement and the contention of the Remark are proved.

Assume now that $C_n \cup D_n$ is open for every n. We claim that $A_n \cup B_n \cup C_n \cup D_n = \Gamma_n$. Indeed, if $(t_0, x_0) \notin A_n \cup B_n \cup D_n \cup \partial\omega$, then $x(t, t_0, x_0)$ is defined on $0 \leqslant t \leqslant t_0$, and, either $(0, x(0, t_0, x_0)) \in H - \overline{\omega}_0$ — in which case $(t_0, x_0) \in C_n$ — or $(t^*, x(t^*, t_0, x_0)) \in \partial\omega$ for some t^*, $0 < t^* < t_0$.

Since $(t_0, x_0) \in \omega$, there exists t^{**}, $t^* \leqslant t^{**} < t_0$ such that $(t^{**}, x(t^{**}, t_0, x_0)) \in \partial\omega$, $(t, x(t, t_0, x_0)) \in \omega$, $t^{**} < t \leqslant t_0$, and therefore $(t^{**}, x(t^{**}, t_0, x_0)) \in I^*$. Then, $(\bar{t}, x(\bar{t}, t_0, x_0)) \notin \bar{\omega}$ for some $\bar{t}$, $0 < \bar{t} < t_0$, and $(t_0, x_0) \in C_n$.

Since Γ_n is a connected set, B_n is not void, and consequently it is possible to find x_n, $(0, x_n) \in \partial\omega_0$ such that $(t, x(t, 0, x_n)) \in \omega$, $0 < t \leqslant n$. If x^* is a point of accumulation of the sequence (x_n), then $(t, x(t, 0, x^*)) \in \bar{\omega}$, $0 \leqslant t < \beta(0, x^*)$ as follows easily from the continuity of $x(t, t_0, x_0)$.

Remarks. (a) If the projection of ω into H is bounded, we can assert the existence of $x^* \in \partial\omega_0$, such that the solution issuing from $(0, x^*)$ remains in $\bar{\omega}$ in the future.

(b) If ω is defined as $\{(t, x)\colon \|x\|^2 < \varphi(t)\}$, where $\varphi(t)$ is a positive real function twice differentiable for $t \geqslant 0$, the usual way to check the assumption $E = E^*$ consists in proving that if (t, x), $t > 0$ is such that

$$\|x\|^2 = \varphi(t), \qquad 2[x, f(t, x)] - \varphi^{\cdot}(t) = 0\,,$$

then

$$2\|f(t, x)\|^2 + 2[x, f_t(t, x)] + 2[x, J(t, x) f(t, x)] - \varphi^{\cdot\cdot}(t) > 0\,,$$

where $J(t, x)$ is the Jacobian matrix $J(t, x) = \left(\dfrac{\partial f_i(t, x)}{\partial x_j}\right)$ $(i, j = 1, 2, \ldots, n)$ and where we have denoted by $[u, v]$ the inner product in R^n. In this case we also obviously have $I = I^*$.

(c) Onuchic obtained a result less general than Theorem 1 using the hypothesis $E = E^*$, ω being the cylinder $\|x\| < 1$; he assumes, furthermore, that $\partial\omega - E$ contains a generator of this cylinder.

(d) Mendelson's result is also less general than Theorem 1; he considers an autonomous system with a critical point at $x = 0$, and under the assumption $E = E^*$, he proves the existence of a non-trivial semi-trajectory contained in the cylinder $\omega = \{(t, x)\colon x \in \omega_0\}$, $\omega_0 \subset R^n$ being an open topological n-cell.

3. We observe, furthermore, that in Mendelson's paper the assumption $E = E^*$, which is fundamental in his proof, is in fact superfluous; as the following theorem shows, Mendelson's conclusion can be even strengthened, without using this assumption.

Let M be a metric space and $f(p, t)$ a family of transformations of the space, $-\infty < t < \infty$, such that the pair $(M, f(p, t))$ is a dynamical system.

THEOREM 2. *If $\omega \subset M$ is a domain with compact and non-void boundary which contains trajectory arcs of arbitrarily large time-length, then there exists a semitrajectory issuing from $\partial\omega$, which is contained in $\bar{\omega}$.*

Proof. Let $\omega_n = \{p\colon f(p, t) \in \omega,\ 0 \leqslant t \leqslant n\}$; $n = 1, 2, \ldots$ By the assumptions ω_n is evidently open and non-void, $\omega_{n+1} \subset \omega_n$. If $\omega_n = \omega$

for every n, then, for each $p \in \partial\omega$ the positive semitrajectory must be in $\overline{\omega}$. If $\omega_n \neq \omega$ for sufficiently large n, since ω is connected and ω_n not empty, there exists $p_n \in \omega \cap \partial\omega_n$. Then $f(p_n, t) \in \overline{\omega}$ for every t, $0 \leqslant t \leqslant n$, and since ω_n is open, $p_n \notin \omega_n$, and there exists t_n, $0 \leqslant t_n \leqslant n$, such that $q_n = f(p_n, t_n) \in \partial\omega$. We may choose a subsequence (n_k) of (n), so that $q_{n_k} \to q^*$ and either $-t_{n_k} \to -\infty$, or $n_k - t_{n_k} \to +\infty$; then either $f(q^*, t) \in \overline{\omega}$ when $t \leqslant 0$, or when $t \geqslant 0$.

THEOREM 3. *Consider the periodic system*

$$\dot{x} = f(t, x), \quad x \in R^n, \ t \in R, \quad f\colon R \times R^n \to R^n,$$
$$f(t+\tau, x) = f(t, x), \quad \tau > 0, \ -\infty < t < \infty .$$

If $\omega \subset R^n$ is a domain with compact and not void boundary, which contains trajectory arcs of arbitrarily large time-length, then there exists a semitrajectory issuing from $\partial\omega$ which is contained in $\overline{\omega}$.

Proof. The previous argument applies, with $x(t, 0, p)$ in the place of $f(p, t)$ and choosing (n_k) such that (t_{n_k}) converges modulo τ.

References

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INSTITUTO DE MATEMÁTICA Y ESTADÍSTICA
UNIVERSIDAD DE LA REPÚBLICA
MONTEVIDEO, URUGUAY

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