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Reçu par la Rédaction le 11. 5. 1964

ACTA ARITHMETICA
XI (1965)

On the zeros of L-functions

by

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Introduction

1. Let $L(s,\chi)$ be any L-function of Dirichlet with a character χ to modulus $D>2$. Using an unproved hypothesis in 1945, Linnik proved (see [10], §17) that for any $\lambda\epsilon[0,\log D]$ and $t_0\epsilon[-\log^3 D,\log^3 D]$ the number of zeros of $L(s,\chi)$ lying in the rectangle $(1-\lambda/\log D\leqslant\sigma\leqslant 1$, $t_0\leqslant t\leqslant t_0+1)$ in the plane of the complex variable $s=\sigma+it$ does not exceed $e^{c_0\lambda}$, where c_0 (and later on $c, c', c_1, c_2, \ldots$) stands for an appropriate absolute constant >0(¹). In 1944 Linnik [9] proved by a very complicated method that the number of functions $L(s,\chi)$ having at least one zero in the rectangle $\{1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t|\leqslant\min(\lambda^{100},\log^3 D)\}$ does not exceed $e^{c_2\lambda}$. Ten years later Rodosskiĭ ([12], pp. 333-341) gave a simpler proof, but merely for the rectangles $(1-\lambda/\log D\leqslant\sigma\leqslant 1$, $|t|\leqslant e^{\lambda}/\log D)$. In 1961 Turán [13] proved by his new method a slightly more general result: The number of zeros of the function $Z(s)=\prod_{\chi}L(s,\chi)$ in the rectangle $(1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t-t_0|\leqslant e^{\lambda}/\log D)$ with $|t_0|<D^{1/2}$ does not exceed $e^{c_3\lambda}$.

The height of the rectangle considered by Turán or Rodosskiĭ for a large D and $\lambda<\log\log\log D$ (for example) is very small. In order to eliminate this restriction I have combined Turán's method with some ideas taken from Linnik's paper [10]. By these means I have succeeded in proving the following

Theorem. (i) *For any $T\geqslant D$ and $\lambda\epsilon[0,\log T]$ the number of zeros of the function $L(s,\chi)$ in the rectangle*

$$(1-\lambda/\log T\leqslant\sigma\leqslant 1,\ |t|\leqslant T) \tag{1}$$

does not exceed $e^{c\lambda}$

(ii) *The same is true for the function $Z(s)=\prod_{\chi}L(s,\chi)$.*

(¹) Linnik's proof is based on the following hypothesis: Any circle of radius $1/\log D$ with the centre in the rectangle $(1-\log\log D/\log D<\sigma<1,\ |t|<\log^3 D)$ contains no more than c_1 zeros of $L(s,\chi)$. He promised (see [10], pp. 111 and 118) to publish another proof for the case in which this hypothesis does not hold. Twenty years have elapsed since, but no proof of this kind has been published yet.

The theorem and the proof hold as well for the product $\prod_{\chi}\zeta(s,\chi)$ of Hecke's L-functions on any algebraic field K with characters χ mod$\mathfrak{f}$ and $D = |\Delta|N\mathfrak{f}$ (Δ being the discriminant), except that now the constant c depends on the degree of K; see further §§ 8 and 10.

The theorem may be applied to get an estimate for the sum $\sum_{p\leqslant x}\chi(p)$ of the characters of primes (see § 11). Once proved, the theorem provides a rather short way to Linnik's estimate $p_1 = D^{O(1)}$ of the least prime $p \equiv l(\mathrm{mod} D)$. In § 12 by means of the theorem we shall prove a formula (45) for the number $\pi(x; D, l)$ of primes $p \equiv l(\mathrm{mod} D)$, $p \leqslant x$, which gives positive information for all $x \geqslant D^{c_4}$ and represents the usual estimate as $x \to \infty$. For small x, however, we cannot prove anything better than the inequality

(2) $$\pi(x; D, l) > c_5(\varepsilon)x/hD^{\varepsilon}\log x \quad (x \geqslant D^{c_4},\ D > D_0(\varepsilon)),$$

where ε stands for any positive constant and h denotes the number of reduced classes mod$D$; it is understood that D and l have no common divisor > 1. A similar inequality but for $x > D^{c'\log(c/\varepsilon)}$, $0 < \varepsilon \leqslant c$, $D > D_0(\varepsilon)$ was proved in my previous paper [1] by a more complicated method.

As another application of the theorem in § 13 we shall prove the existence of an absolute constant $\theta < 1$ such that if $D > D_0$, then for any $x \geqslant D^{c_6}$ in the interval $(x, x+x^{\theta})$ there is a prime $p \equiv l(\mathrm{mod} D)$. This improves the theorem of [1], where the interval is (x, xD^{ε}) and where the restrictions $D > D_0(\varepsilon)$, $x > D^{c'\log(c/\varepsilon)}$ are used.

The corresponding results hold as well for primes p which are norms of prime ideals of any class $\mathfrak{H}$ mod$\mathfrak{f}$ in any algebraic field K. In this case the constants c_4, $c_5(\varepsilon)$ and θ depend on the degree of K (cf. §§ 11-13).

In different paragraphs the constants $c, c_1, \ldots$ may have different meanings. The constants implied in $\ll$ and O are supposed to be independent of any parameters (as D, T, x, ε^{-1}) which may increase indefinitely.

The proof of the theorem will be developed by two stages. We begin in § 5 by proving a weaker theorem for a single function $L(s,\chi)$; the result will then be improved in § 7. The changes occurring in the proof for the function $\prod_{\chi}L(s,\chi)$ will be considered in § 9. For a sketch of the proof concerning Hecke functions, see §§ 4, 8, 10.

The last section of this paper (§§ 14-16) will contain the proof of an analogous theorem for L-functions of a semigroup $\mathfrak{G}$, used in my previous papers [6] and [7]. By means of that theorem we shall improve the results of those papers about the distribution of the generators of $\mathfrak{G}$ (see §§ 14, 17 and 18).

The results of the present paper have been announced in [8].

Preliminaries

2. The proof of the theorem rests on the following properties of the function $L(s,\chi)$:

(i) *In the region*

(3) $$\sigma \geqslant 1 - c_0/\log T(2+|t|) \geqslant \tfrac{3}{4}$$

for all characters χ to modulus D ($D \leqslant T$) we have $L(s,\chi) \neq 0$, with at most one exception corresponding to a function $L(s,\chi')$ with a real non-principal character χ'; this function $L(s,\chi')$ may have in (3) a single real "exceptional" zero $\beta' < 1$.

(ii) *If $\nu = \nu(r; \chi, t_0)$ denote the number of the zeros of $L(s,\chi)$ in $|s-1-it_0| \leqslant r$ $(c_0/\log T(2+|t_0|) \leqslant r \leqslant 2)$, then*

(4) $$\nu \ll r\log T(2+|t_0|).$$

(iii) *We have uniformly in $-\frac{1}{2} \leqslant \sigma \leqslant 2$*

(5) $$\frac{L'}{L}(s,\chi) = \sum_{|s-\varrho|<1}\frac{1}{s-\varrho} - \frac{E_0}{s-1} + O\left\{\frac{1}{|s|} + \log T(2+|t_0|)\right\}$$

where ϱ runs through the zeros of $L(s,\chi)$ and $E_0 = 1$ if χ is the principal character χ_0, and $= 0$ otherwise.

For $T = D$ the proofs are given in [11], pp. 130, 331, 225. Being true for $T = D$, the relations evidently hold for any $T \geqslant D$.

3. Lemma 1. *Let $T \geqslant D \geqslant 2$, $T^{-2} \leqslant \varepsilon \leqslant 1$, $T^{B} \leqslant x \leqslant T^{3B}$, where $4 < B \ll 1$ and let I be the interval $[x, xe^{\varepsilon}] = [x, x+x']$ $(\varepsilon x < x' < 2\varepsilon x)$. Denoting by $\pi(I; D, l)$ the number of primes $p \equiv l$ (mod$D$), $p \in I$, we have*

(6) $$\pi(I; D, l) \ll x'/h\log x.$$

Proof. Let $\mathfrak{G}$ be the semigroup of all natural integers a prime to D and let a_m $(m = 1, 2, \ldots, N)$ be all the numbers $a \equiv l$ (mod$D$), $a \in I$. Then

$$N = x'/D + O(1)$$

and for any d, prime to D, we have

$$\sum_{\substack{a_m \\ d|a_m}} 1 = N/d + O(1).$$

Hence, in the notation of [6], Lemma 12 (representing the sieve method of A. Selberg)

$$f(d) = d, \quad R_d \ll 1$$

and (cf. [6], § 12)

$$S_z > \sum_{\sqrt{z}<a<z} \frac{1}{a} > c_1 \frac{h}{D} \log z.$$

Putting $z = x^{1/8}$ we have $S_z > c_2 h D^{-1} \log x$, whence

$$N/S_z \ll (x'/D+1)D/h\log x \ll x'/h\log x.$$

The numbers $|\lambda_a|$ being $\leqslant 1$ (cf. [3], (38)), we deduce

$$\sum_{\substack{a_1\leqslant z\\ a_2\leqslant z}} |\lambda_{a_1}\lambda_{a_2} R_{a_1a_2/(a_1,a_2)}| \ll \Big(\sum_{a\leqslant z} 1\Big)^2 \leqslant z^2 = x^{1/4} < x'/h\log x.$$

Hence (6) follows by the arguments of [6], § 14.

Corollary. *If $B > 6$, then*

$$\sum_{\substack{m\in I\\ m\equiv l (\mathrm{mod}\, D)}} \frac{\Lambda(m)}{m} \ll \frac{\varepsilon}{h}. \tag{7}$$

Proof. By (6) the left-hand side of (7) is evidently

$$< \frac{\log x}{x}\{\pi(I;D,l)+O(\sqrt{x}\log^2 x)\} \ll \frac{\log x}{x}\cdot\frac{x'}{h\log x} \ll \frac{\varepsilon}{h}.$$

4. Lemma 2. *Let $\mathfrak{H}$ denote any class of ideals* mod $\mathfrak{f}$ *in the algebraic field K with the discriminant Δ and the class-number h, and let $D = |\Delta| N\mathfrak{f}$ ($N\mathfrak{f}$ being the norm of $\mathfrak{f}$). If $T \geqslant D$, $T^{-2} \leqslant \varepsilon \leqslant 1$, $T^B \leqslant x \leqslant T^{3B}$ (where B stands for a sufficiently large constant), I denotes the interval $[x, xe^\varepsilon] = [x, x+x']$ and $\pi(I, \mathfrak{H})$ the number of prime ideals $\mathfrak{p}\in\mathfrak{H}$ such that $N\mathfrak{p}\in I$, then*

$$\pi(I,\mathfrak{H}) < c_1 x'/h\log x. \tag{8}$$

In this paragraph B, c_1 and other constants may depend on the degree of K but not on Δ, $N\mathfrak{f}$ or other parameters.

The proof rests on [3], Lemma 3 (Selberg's sieve for ideals). In that Lemma let $\mathfrak{a}_m$ ($m = 1, 2, \ldots, X$) be all the ideals $\mathfrak{a}\in\mathfrak{H}$ with $N\mathfrak{a}\in I$ and let Q be the empty set. By [3], Lemma 1, the number $\nu(t,\mathfrak{H})$ of ideals $\mathfrak{a}\in\mathfrak{H}$ with $N\mathfrak{a}\leqslant t$ for any $t\geqslant 1$ satisfies

$$\nu(t;\mathfrak{H}) = \mu t + O(D^{2/3}t^{1-c}),$$

where

$$\mu = h^{-1}\operatorname*{Res}_{s=1}\zeta(s,\chi_0),$$

$\zeta(s,\chi_0)$ being the Hecke L-function with the principal character mod $\mathfrak{f}$. Hence

$$X = \nu(x+x',\mathfrak{H}) - \nu(x,\mathfrak{H}) = \mu x' + O(D^{2/3}x^{1-c}).$$

For any ideal $\mathfrak{b}$ prime to $\mathfrak{f}$ let $\nu(x,\mathfrak{H},\mathfrak{b})$ denote the number of ideals $\mathfrak{a}\in\mathfrak{H}$ such that $N\mathfrak{a}\leqslant x$ and $\mathfrak{b}|\mathfrak{a}$. By [3], (11),

$$\nu(x;\mathfrak{H},\mathfrak{b}) = \mu x/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-c}\right),$$

whence

$$\nu(x+x';\mathfrak{H},\mathfrak{b}) - \nu(x;\mathfrak{H},\mathfrak{b}) = \mu x'/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-c}\right)$$

and thus

$$\sum_{\substack{\mathfrak{a}_m\\ \mathfrak{b}|\mathfrak{a}_m}} 1 = \mu x'/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-c}\right) = X/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-c}\right).$$

Hence, in the notation of [3], Lemma 3,

$$f(\mathfrak{b}) = N\mathfrak{b} \quad\text{and}\quad R_{\mathfrak{b}} \ll D^{2/3}(x/N\mathfrak{b})^{1-c}.$$

Using $z \geqslant x^{1/4}$ and arguing as in [3], § 4 we get the inequality

$$S_z > c_2 h\mu\log z > c_3 h\mu \log x,$$

whence

$$X/S_z \ll x'/h\log x.$$

It remains to prove the same estimate for the term

$$W = \sum_{\substack{\mathfrak{c}_1,\mathfrak{c}_2\\ N\mathfrak{c}_1,N\mathfrak{c}_2\leqslant z}} |\lambda_{\mathfrak{c}_1}\lambda_{\mathfrak{c}_2} R_{\mathfrak{c}_1\mathfrak{c}_2/(\mathfrak{c}_1,\mathfrak{c}_2)}| \ll D^{2/3}x^{1-c} \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2\\ N\mathfrak{a}_1,N\mathfrak{a}_2\leqslant z}} \left(\frac{N(\mathfrak{a}_1,\mathfrak{a}_2)}{N\mathfrak{a}_1 N\mathfrak{a}_2}\right)^{1-c}.$$

By [3], § 5 the last sum does not exceed $c_4 D^{2/3}(h\mu)^2 z^{2c}$. Putting

$$z^{2c} = \frac{x^c(x'/x)}{D^{4/3}(h\mu)^2 h\log x}, \tag{9}$$

we get the desired estimate for W. Since $x'/x\in[T^{-2},1]$, for a sufficiently large $B \ll 1$ we have, by (9), $z\geqslant x^{1/4}$, $z < x^{2/3}$. Hence inequality (8) follows (cf. [3], § 6).

Corollary. *Let $\Lambda(\mathfrak{a}) = \log N\mathfrak{p}$ if $\mathfrak{a}$ is a power of a prime ideal $\mathfrak{p}$, and $= 0$ otherwise. Then*

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{H}\\ N\mathfrak{a}\in I}} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} \ll \frac{\varepsilon}{h}. \tag{10}$$

Proof of the theorem for a single function $L(s,\chi)$

5. Suppose that the function $L(s,\chi)$ with $\chi\neq\chi_0$ has a zero $\varrho_0\,\epsilon\, Q\,(1-\lambda/\log T\leqslant\sigma\leqslant 1,\ |t-t_0|\leqslant\lambda/2\log T)$, where $c_0\leqslant\lambda\leqslant\log T$ and $t_0\ll T$. Then by [6], § 15 (with T in place of D) for any real τ with $|\tau-\tau_0|\leqslant\lambda/2\log T$ we have

$$\Big|\sum_{\varrho} f(\varrho-1-i\tau)\Big| > e^{-c_1\lambda}, \tag{11}$$

where

$$f(s)=\big((e^{3As}-e^{As})/2As\big)^k,\qquad A=\lambda^{-1}\log T \tag{12}$$

and k stands for a suitable integer $\epsilon\,[2+c_2\lambda,\,c_3\lambda]$ $(c_2>2)$. Let $N(\lambda;u,T_0)$ denote the number of zeros of the function $L(s,\chi)$ lying in the rectangle

$$R(1-\lambda/\log T\leqslant\sigma\leqslant 1,\ T_0\leqslant t\leqslant T_0+u/\log T)\quad (T_0\ll T;\ 2\leqslant u\ll T^2). \tag{13}$$

Considering a series of squares $Q=Q_{t_0}$ with a variable $t_0=T_0+(m+\frac{1}{2})\lambda/\log T$ $(m=0,1,\ldots,[u/\lambda])$ for which there is a zero $\varrho\,\epsilon\, Q_{t_0}$, in the corresponding inequalities (11) we use those particular values of $\tau=\tau(t_0)$ which differ from T_0 by positive multiples of $c'/\log T$, c' being a sufficiently small constant $<c_0$. (This restriction concerning τ is necessary merely during the first stage of the proof.)

Let k_1 be that value or one of those values of k which appear in (12) with the largest frequency; the numbers τ corresponding to $k=k_1$ will be denoted by

$$\tau_j=T_0+w_j\qquad(1\leqslant j\leqslant V).$$

Then by (4) (since $k<c_3\lambda$)

$$N(\lambda;u,T_0)\ll\lambda^2V. \tag{14}$$

Writing

$$R(n)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{e^{3As}-e^{As}}{2As}\right)^k e^{-s\log n}\,ds\qquad(n\geqslant 1) \tag{15}$$

we have by [13], Lemma I (or [6], § 17)

$$|R(n)|\leqslant\begin{cases} e^{c_4k}/A & \text{if}\quad e^{kA}<n<e^{3kA},\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

Let $\Lambda(n)=\log p$ if n is a power of a prime p, and $=0$ otherwise. Using (15), (12) and (3), (4), we deduce:

$$-\sum_n\frac{\chi(n)\Lambda(n)}{n^{1+i\tau}}R(n)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}f(s)\frac{L'}{L}(s+1+i\tau,\chi)\,ds \tag{17}$$

$$=\sum_{\varrho}f(\varrho-1-i\tau)+\frac{1}{2\pi i}\int_{-3/2-i\infty}^{-3/2+i\infty}=\sum_{\varrho}f(\varrho-1-i\tau)+O(e^{-(3/2)Ak}\log D).$$

Hence, writing

$$k_1\lambda^{-1}=B\qquad(c_2<B<c_3) \tag{18}$$

we have, by (11), (12) and (16)

$$\Big|\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i\tau_j}}+O(T^{-3B/2}\log D)\Big|>e^{-c_1\lambda}. \tag{19}$$

In proving the theorem we may suppose that $\lambda\leqslant\lambda_0=c_2c_1^{-1}\log T$ (since for $\lambda>\lambda_0$ the theorem holds by (4)). Then, by (19) and (18)

$$\Big|\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i\tau_j}}\Big|>\frac{1}{2}e^{-c_1\lambda}.$$

Let φ_j denote the argument of the last sum. Then

$$\sum_{1\leqslant j\leqslant V}e^{-i\varphi_j}\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i\tau_j}}>\frac{1}{2}Ve^{-c_1\lambda},$$

whence, by Schwarz's inequality,

$$V<e^{c_5\lambda}\sum_{1\leqslant j\leqslant V}\Big|\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i(T_0+w_j)}}\Big|^2 \tag{20}$$

$$=e^{c_5\lambda}\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+iT_0}}\sum_{T^B<n<T^{3B}}\frac{\bar\chi(m)\Lambda(m)\bar R(m)}{m^{1-iT_0}}\sum_{1\leqslant j\leqslant V}\left(\frac{m}{n}\right)^{iw_j}$$

$$\leqslant\frac{e^{c_6\lambda}}{\log^2T}\sum_{T^B<n\leqslant m<T^{3B}}\frac{\Lambda(n)\Lambda(m)}{nm}\Big|\sum_{1\leqslant j\leqslant V}\left(\frac{m}{n}\right)^{iw_j}\Big|$$

(cf. (16), (12)).

Now let w_j in (20) run over *all* the multiples of $c'/\log T$, not exceeding $u/\log T$. By the addition of new terms the right-hand side may increase, but we can easily estimate it as a sum of a geometrical progression.

For any fixed $n\,\epsilon\,(T^B,T^{3B})$ let us write

$$\frac{m}{n}=e^{\mu}\qquad(0\leqslant\mu<2B\log T) \tag{21}$$

and let us introduce the intervals

$$\mathscr{M}_l\left\{l\frac{\log T}{u}\leqslant\mu<(l+1)\frac{\log T}{u}\right\}\qquad(l=0,1,\ldots;\ l\ll u).$$

The last inner sum $\sum_j$ in (20) corresponding to $\mathscr{M}_0$ is evidently

$$\ll u. \tag{22}$$

For any other $\mathscr{M}_l$ we use the estimate

$$\sum_{N_0<n<N} e^{in\varphi} \ll 1/\min(\varphi, 2\pi-\varphi) \qquad (0<\varphi<2\pi)$$

(cf. [11], p. 189). Since now in (20) $w_j = jc'/\log T$ $(1 \leqslant j \leqslant u)$ and the value of c' is at our disposal, we can take for granted that $w_1 \mu < (c'\log T)2B\log T$ does not exceed π. Then for any $\mu \in \mathscr{M}_l$ $(l \neq 0)$

$$\sum_j e^{i\mu w_j} \ll \left(\frac{c'}{\log T}\cdot l\,\frac{\log T}{u}\right)^{-1} \ll \frac{u}{l}. \tag{23}$$

The numbers m, which by (21) correspond to the same $\mathscr{M}_l$, are in the interval

$$M_l\left\{x_l,\, x_l\exp\left(\frac{\log T}{u}\right)\right\}$$

where

$$c_7\frac{\log T}{T^2} < \frac{\log T}{u} \leqslant \frac{1}{2}\log T.$$

If $u^{-1}\log T > 1$, then we divide M_l into subintervals I (x, xe^ε) satisfying the conditions of Lemma 1 (with $D = 2$). By (7)

$$\sum_{m\in I}\frac{\Lambda(m)}{m} \ll \varepsilon.$$

Summing over all I we deduce

$$\sum_{m\in M_l}\frac{\Lambda(m)}{m} \ll \frac{\log T}{u}.$$

(If $u^{-1}\log T \leqslant 1$, then we apply Lemma 1 directly to M_l and get the same result.) Hence, by (22), (23)

$$\sum_{m\in M_l}\frac{\Lambda(m)}{m}\left|\sum_j\left(\frac{m}{n}\right)^{iw_j}\right| \ll \begin{cases}\log T & \text{if}\quad l=0,\\ l^{-1}\log T & \text{if}\quad 1\leqslant l\leqslant u\end{cases}$$

and thus

$$\sum_{n\leqslant m<T^{3B}}\frac{\Lambda(m)}{m}\left|\sum_j\left(\frac{m}{n}\right)^{iw_j}\right| \ll \log T\log u,$$

$$\sum_{T^B<n\leqslant m<T^{3B}}\frac{\Lambda(n)\Lambda(m)}{nm}\left|\sum_j\left(\frac{m}{n}\right)^{iw_j}\right| \ll \log T\log u\sum_{T^B<n<T^{3B}}\frac{\Lambda(n)}{n} \ll \log^2 T\log u.$$

This combined with (20) proves the estimate

$$V < e^{c_8\lambda}\log u \tag{24}$$

for the rectangle (13) and the function $L(s,\chi)$ with $\chi \neq \chi_0$.

6. If $\chi = \chi_0$, then in the neighbourhood of $s=1$ Turán's method (by which we have proved (11)) does not work, since now on the right-hand side of (17) appears a term $-f(-i\tau)$ corresponding to the pole at $s=1$. However, if $|\tau| \geqslant E\lambda/\log T$ where $E \ll 1$ is large enough, then $f(-i\tau)$ is much smaller than the sum in (11) and we may go on as before.

Associating with each square Q_{t_0} a number $\tau = \tau(t_0)$ (cf. § 5) now we leave out those $\leqslant 2E+2$ squares Q_{t_0} which are in the rectangle $(1-\lambda/\log T \leqslant \sigma \leqslant 1,\ |t| \leqslant E\lambda/\log T)$. By this we lose no more than $2+2E$ numbers w_j. For the quantity V' (say) of all the other numbers w_j we can prove (24) (with V' in place of V). Then for appropriate $c_9 > c_8$

$$V < 2+2E+V' < 2+2E+e^{c_8\lambda}\log u < e^{c_9\lambda}\log u,$$

which is an inequality of the same type as (24).

Therefore we may suppose that (24) holds for any χ.

7. Let us now consider the rectangle (1), which is a particular case of (13) corresponding to $T_0 = -T$, $u = 2T\log T$. By (24) the number V of points w_j for the rectangle (1) satisfies

$$V < e^{c_1\lambda}\log T.$$

It is our aim to eliminate the factor $\log T$. In doing this we may take for granted that

$$V > e^{4c_1\lambda} \tag{25}$$

(otherwise there is nothing to prove); hence

$$V < \log^2 T. \tag{26}$$

Now let n and m_0 be any fixed integers such that $n \in (T^B, T^{3B})$ and $m_0 \in [n, T^{3B})$. Supposing that

$$m_0\exp(V^{-1/4}\log T) \leqslant T^{3B}, \tag{27}$$

we introduce the intervals

$$M = [m_0,\, m_0\exp(V^{-1/4}\log T)] \quad\text{and}\quad \mathscr{I} = [\mu_1, \mu_2],$$

where

$$\mu_1 = \log m_0, \qquad \mu_2 = \mu_1 + V^{-1/4}\log T.$$

For any $m \in M$ with $\Lambda(m) \neq 0$ let

$$S_m = \sum_{1 \leqslant j \leqslant V} e^{iw_j(\log m - \log n)}, \tag{28}$$

the numbers w_j being those which actually occur in (20) by the process described at the beginning of § 5. Writing

$$g(\mu) = \sum_{1 \leqslant j \leqslant V} e^{iw_j(\mu - \log n)}$$

we have

$$\int_{\mu_1}^{\mu_2} |g(\mu)|^2 d\mu = \int_{\mu_1}^{\mu_2} \sum_{j,j'} e^{i(w_j - w_{j'})(\mu - \log n)} d\mu = \int_{\mu_1 - \log n}^{\mu_2 - \log n} \sum_{j,j'} e^{i(w_j - w_{j'})\varphi} d\varphi \tag{29}$$

$$\ll (\mu_2 - \mu_1)V + \sum_{j'} \sum_{j > j'} \frac{1}{w_j - w_{j'}} \ll V^{3/4} \log T + V e^{c_1\lambda} \log T \ll V e^{c_1\lambda} \log T,$$

since by (24)

$$\sum_{j(>j')} \frac{1}{w_j - w_{j'}} \ll e^{c_1\lambda} \log T \int_1^{\infty} \frac{\log u}{u^2} du \ll e^{c_1\lambda} \log T.$$

From (29) and (25) we deduce that the measure of that set of points $\mu \in [\mu_1, \mu_2]$ at which $|g(\mu)| > V^{7/8}$ does not exceed

$$Y \ll V^{-3/4} e^{c_1\lambda} \log T < V^{-1/2} \log T. \tag{30}$$

Let us call these μ the "exceptional" ones.

If $m \in M$ is an integer with $\Lambda(m) \neq 0$ for which there is a non-exceptional $\mu = \mu_m \in \mathscr{I}$ such that $|\log m - \mu_m| < 2/TV^{1/4}$, then by (28) (where $0 < w_j \leqslant 2T$)

$$S_m = \sum_{1 \leqslant j \leqslant V} \exp\{iw_j(\mu_m + 2\theta/TV^{1/4} - \log n)\} = \sum_j e^{iw_j(\mu_m - \log n)} + O\Big(\sum_j V^{-1/4}\Big)$$

($|\theta| < 1$), whence

$$|S_m| < c_2 V^{7/8}. \tag{31}$$

Any pair of integers n, m for which (31) holds will be called a *normal* one; all the others — *exceptional* ones.

Let us divide the interval $\mathscr{I}$ into $[T \log T]$ equal parts $\mathscr{I}_1, \mathscr{I}_2, \ldots$ of the length $c_3/TV^{1/4}$ $(1 \leqslant c_3 < 2)$. Suppose that there is an interval $\mathscr{I}_i$ containing merely exceptional points and denote by M_i the corresponding part of M. Since by Lemma 1 and (26)

$$\sum_{m \in M_i} \frac{\Lambda(m)}{m} \ll 1/TV^{1/4},$$

we have by (30)

$$\sum_{\substack{m \in M \\ n,m \text{ exc.}}} \frac{\Lambda(m)}{m} \ll Y \ll V^{-1/2} \log T,$$

$$\sum_{\substack{m \in M \\ n,m \text{ exc.}}} \frac{\Lambda(m)}{m} \Big| \sum_{1 \leqslant j \leqslant V} \Big(\frac{m}{n}\Big)^{iw_j} \Big| \ll V^{1/2} \log T. \tag{32}$$

Now let us consider the case in which m_0 does not satisfy (27) and consequently

$$\log(T^{3B}/m_0) < V^{-1/4} \log T.$$

Writing $M' = [m_0, T^{3B})$ we have by Lemma 1

$$\sum_{m \in M'} \frac{\Lambda(m)}{m} \Big| \sum_{1 \leqslant j \leqslant V} \Big(\frac{m}{n}\Big)^{iw_j} \Big| \leqslant V \sum_{m \in M'} \frac{\Lambda(m)}{m} \ll V^{3/4} \log T.$$

Since the number of intervals M does not exceed $2BV^{1/4}$, from (32) and (31) we deduce that

$$\sum_{\substack{m \in [n, T^{3B}) \\ n,m \text{ exc.}}} \frac{\Lambda(m)}{m} \Big| \sum_{1 \leqslant j \leqslant V} \Big(\frac{m}{n}\Big)^{iw_j} \Big| \ll V^{3/4} \log T$$

and thus

$$\sum_{\substack{T^B < n \leqslant m < T^{3B} \\ n,m \text{ exc.}}} \frac{\Lambda(n)\Lambda(m)}{nm} \Big| \sum_{1 \leqslant j \leqslant V} \Big(\frac{m}{n}\Big)^{iw_j} \Big| \ll V^{3/4} \log^2 T.$$

The corresponding sum over all normal pairs n, m is evidently $\ll V^{7/8} \log^2 T$. Hence, by (20),

$$V < e^{c_4\lambda} V^{7/8}$$

and thus

$$V < e^{8c_4\lambda}.$$

From this estimate and (14) we deduce that the number of zeros of $L(s, \chi)$ lying in the rectangle (1) does not exceed $e^{c_5\lambda}$. This proves the first part of the theorem.

8. In this paragraph let $\zeta(s, \chi)$ be the Hecke L-function on the algebraic field K with characters χ mod$\mathfrak{f}$ and let $T \geqslant D = |\Delta| N\mathfrak{f} \geqslant 2$. The properties analogous to those of § 2 being true by [2] (pp. 87, 95), we can repeat the deductions of §§ 5-7. In place of (20) we now get the inequality

$$V < \frac{e^{c_1\lambda}}{\log^2 T} \sum_{\substack{\mathfrak{a}, \mathfrak{b} \\ T^B < N\mathfrak{a} \leqslant N\mathfrak{b} < T^{3B}}} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{b})}{N\mathfrak{a} N\mathfrak{b}} \Big| \sum_{1 \leqslant j \leqslant V} \Big(\frac{N\mathfrak{b}}{N\mathfrak{a}}\Big)^{iw_j} \Big|.$$

Using the estimate

$$\sum_{N\mathfrak{a}\in I} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} \ll \varepsilon,$$

which holds by (10), we may go on as before. Finally we prove that the number of zeros of $\zeta(s,\chi)$ lying in the rectangle (1) does not exceed $e^{c\lambda}$.

Proof of the theorem for the function $Z(s) = \prod_{\chi} L(s,\chi)$

9. Let us now consider the function $Z(s)$ and the rectangle (13), which we divide into squares Q_{t_0} as in § 5. By Turán's theorem (see [13], § I, or [6], (65)) there are no more than $e^{c_1\lambda}$ functions $L(s,\chi)$ having a zero in any Q_{t_0}, and by § 7 no function $L(s,\chi)$ can have zeros in more than $e^{c_2\lambda}$ squares Q. Starting with the square Q nearest to the line $t = T_0$ we associate with it some definite function $L(s,\chi)$ which has a zero ϵQ. If there is no such function, then we leave this Q out of account and pass to the next one, etc. Also we leave out any square Q in which there are zeros merely of functions already associated with former squares. In this way we get a set of squares, say $S(Q_{t_1}, Q_{t_2}, \ldots)$. For every $Q_{t_0}\epsilon S$ we choose a number $\tau = \tau(t_0)\epsilon[t_0-\lambda/2\log T, t_0+\lambda/2\log T]$ which differs from T_0 by a multiple of $c'/\log T$ ($c' < c_0 \leqslant \lambda$). Then for each $\tau(t_0)$ we write inequality (11) (where ϱ runs through the zeros of that particular $L(s,\chi)$ which has been associated with Q_{t_0}) and we mark the corresponding exponent k in (12). Let k' be that value of k ($\epsilon[c_3\lambda, c_4\lambda]$) which has the largest frequency. The numbers τ corresponding to $k = k'$ will be denoted by T_0+w_j ($1 \leqslant j \leqslant V$). Under these circumstances the number N (say) of the zeros of $Z(s)$ lying in the rectangle (13) satisfies

$$N \ll V\lambda^2 e^{(c_1+c_2)\lambda}. \tag{33}$$

Let $L(s,\chi_j)$ denote the function associated with the same square as the number w_j. Then, by the arguments of § 5,

$$\Big|\sum_{T^B<n<T^{3B}} \frac{\chi_j(n)\Lambda(n)R(n)}{n^{1+i(T_0+w_j)}}\Big| > e^{-c_5\lambda},$$

whence

$$V < \frac{e^{c_6\lambda}}{\log^2 T} \sum_{T^B<n\leqslant m<T^{3B}} \frac{\Lambda(n)\Lambda(m)}{nm} \Big|\sum_{1\leqslant j\leqslant V} \frac{\chi_j(n)}{\chi_j(m)}\Big(\frac{m}{n}\Big)^{iw_j}\Big| \tag{34}$$
$$\leqslant \frac{e^{c_6\lambda}}{\log^2 T} \sum_{T^B<n<T^{3B}} \frac{\Lambda(n)}{n} \sum_{l(\mathrm{mod}\,D)} \sum_{\substack{n\leqslant m<T^{3B}\\ m\equiv l(\mathrm{mod}\,D)}} \frac{\Lambda(m)}{m} \Big|\sum_{1\leqslant j\leqslant V} \Big(\frac{m}{n}\Big)^{iw_j}\Big|.$$

Using (34) and (7) in the same manner as in § 5 we prove first that

$$V < e^{c_7\lambda}\log u \qquad (2 \leqslant u \leqslant T^2). \tag{35}$$

Having done this we pass to the rectangle (1). Using (34), (35) and (7) we follow the method of § 7, except that now we first evaluate sums over $m \equiv l \pmod D$ with a fixed l, and then we sum the results over the reduced set of residues mod D. In this way we get the estimate $V < e^{c_8\lambda}$ which, combined with (33), proves the theorem for the function $Z(s)$.

10. The proof of an analogous theorem for the product $\prod_{\chi}\zeta(s,\chi)$ of Hecke L-functions rests on (10), on the result proved in § 8 and on the estimate $\ll e^{c_1\lambda}$ for the number of L-functions having a zero in a square Q_{t_0} (which follow from [6], (65) and [3], (8)). These results enable us to use the method of § 9. In place of (34) now we have the inequality

$$V \leqslant \frac{e^{c_2\lambda}}{\log^2 T} \sum_{\substack{\mathfrak{a}\\ T^B<N\mathfrak{a}<T^{3B}}} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} \sum_{\mathfrak{H}} \sum_{\substack{\mathfrak{b}\epsilon\mathfrak{H}\\ N\mathfrak{a}\leqslant N\mathfrak{b}<T^{3B}}} \frac{\Lambda(\mathfrak{b})}{N\mathfrak{b}} \Big|\sum_{1\leqslant j\leqslant V} \Big(\frac{N\mathfrak{b}}{N\mathfrak{a}}\Big)^{iw_j}\Big|.$$

Arithmetical applications

11. As an application of the first part of the theorem we shall prove an estimate for the sum over primes of a complex character χ mod D. The proof rests on the inequality (cf. [11], p. 376)

$$\sum_{n\leqslant x} \chi(n)\Lambda(n) + \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} \frac{x^s}{s}\cdot\frac{L'}{L}(s,\chi)\,ds \tag{36}$$
$$\ll \frac{x^\eta}{T(\eta-1)} + \frac{x\log^2 x}{T} + \log x \qquad (x>1,\ T>1,\ 1<\eta<2)$$

in which we use $x \geqslant D^{c_1}$, $\eta = 1+1/\log x$ and

$$T = De^{c_2\sqrt{\log x}},$$

where c_1 and c_2 are large enough. Then the right-hand side of (36) is $\ll T^{-1}x\log^2 x$.

Replace the contour of integration by the broken lines C_1, C_2, C_3 satisfying the following conditions: (i) the distance between C_1, C_2, C_3 and, respectively, the lines $t = T$, $\sigma = \frac{1}{2}$, $t = -T$ does not exceed $1/\log T$; (ii) the length of C_1, C_2, C_3 does not exceed, respectively, 1, $4T$, 1; (iii) any zero of $L(s,\chi)$ is at a distance $> c_3/\log^2 T$ from the contour $C = C_1+C_2+C_3$ (cf. [4], § 8).

At every point of C we have, by (5), $L'/L(s,\chi) \ll \log^3 T$, whence the integral along C is

$$\ll \frac{x}{T}\log^3 x + x^{5/8}\log^4 x \ll \frac{x}{T}\log^3 x.$$

Denoting by $G = G_\chi$ the region between C and $\sigma = 1$, we have by (36)

$$\sum_{n\leqslant x}\chi(n)\Lambda(n) = -\sum_{\varrho\epsilon G}\frac{x^\varrho}{\varrho} + O\left(\frac{x\log^3 x}{T}\right), \tag{37}$$

where ϱ runs through the zeros of $L(s,\chi)$.

Let the constants c and c_0 have the same meaning as in the theorem and in § 2. We may suppose that $(\log x)/\log T > 2c$. Writing $\varrho = 1-\delta+i\gamma$, $\delta = \lambda/\log T$ $(\lambda = \lambda_\varrho)$ and using Abel's identity ([11], p. 371) we deduce:

$$\begin{aligned}\sum_{\varrho\epsilon G}\frac{x^\varrho}{\varrho} &\ll x\sum_{\varrho\epsilon G}x^{-\delta} = x\sum_{\varrho\epsilon G}\exp\left(-\lambda_\varrho\frac{\log x}{\log T}\right) \\ &\ll x\int_{c_0}^{\log T}\frac{\log x}{\log T}\exp\left\{\lambda\left(c-\frac{\log x}{\log T}\right)\right\}d\lambda + xT^{c-\frac{\log x}{\log T}} \\ &\ll x\exp\left(-\frac{1}{2}c_0\frac{\log x}{\log T}\right).\end{aligned} \tag{38}$$

For a sufficiently large c_2 the remaining term in (37) is of a lower order of magnitude than the right-hand side of (38). Hence, writing

$$\varepsilon = \exp\left(-\frac{1}{2}c_0\frac{\log x}{\log D + c_2\sqrt{\log x}}\right) \tag{39}$$

we have

$$\sum_{n\leqslant x}\chi(n)\Lambda(n) \ll \varepsilon x. \tag{40}$$

Dividing the sum on the left into parts $n \leqslant \varepsilon x$ and $\varepsilon x < n \leqslant x$, and denoting by P the number of primes $p \leqslant x$ not dividing D, we deduce from (40) that

$$\sum_{p\leqslant x}\chi(p) \ll P\left\{\frac{1}{\log D+\sqrt{\log x}} + \exp\left(-\frac{1}{2}c_0\frac{\log x}{\log D + c_2\sqrt{\log x}}\right)\right\}. \tag{41}$$

This result is of interest only for "small" $x\,\epsilon\,(D^{c_1}, x_0)$ where $x_0 = \exp(\log^3 D)$. If $x \geqslant x_0$ or $T > \exp(\log^{3/2} D)$, then by a known theorem (cf. [11], p. 295) the constant c_0 in (3) (which is also the lower limit of the integral in (38)) increases with T giving a stronger estimate.

In any algebraic field an analogue of (41) holds for the sum over prime ideals of a complex character χ mod$\mathfrak{f}$.

For the possibly existing exceptional zero we are not in a position to prove a non-trivial estimate of the sum $\sum_{p\leqslant x}\chi(p)$ in the case of a real $\chi \neq \chi_0$ and a small x.

12. In the present paragraph we shall apply the second part of the theorem in order to get an asymptotic estimate of the function $\pi(x; D, l)$ which is of interest for small x. To this end we multiply (36) by $\bar{\chi}(l)$ and sum over all χ. Dividing by h we prove that

$$\sum_{\substack{n\leqslant x\\ n\equiv l(\mathrm{mod}\,D)}}\Lambda(n) = \frac{x}{h} - \frac{1}{h}\sum_{\varrho\epsilon G,\chi_\varrho}\bar{\chi}_\varrho(l)\frac{x^\varrho}{\varrho} + O\left(\frac{x\log^3 x}{T}\right), \tag{42}$$

where G denotes the sum of all the regions G_χ defined in the previous paragraph, and χ_ϱ (for any particulas $\varrho\,\epsilon\,G$) runs over all characters $\chi \bmod D$ such that $L(\varrho,\chi) = 0$. If the exceptional zero β' (see § 2) does not exist, then by § 11

$$\psi(x; D, l) = \sum_{x\geqslant n\equiv l(\mathrm{mod}\,D)}\Lambda(n) = \frac{x}{h}\big(1+O(\varepsilon)\big),$$

ε being defined by (39). Hence, if $x \geqslant D^{c_1}$, where c_1 is large enough, then

$$\pi(x; D, l) = \frac{x}{h\log x}\left\{1+\theta c_3\frac{\log(h/\varepsilon)}{\log x}+O(\varepsilon)\right\} \quad (0<\theta<1). \tag{43}$$

If the exceptional zero $\beta' = 1-\delta'$ does exist, then we use

$$T = D\delta'^{-2}e^{c_2\sqrt{\log x}}.$$

Now the principal term in (42) is

$$\frac{x}{h}q_l, \quad \text{where} \quad q_l = 1-\chi'(l)\frac{x^{-\delta'}}{1-\delta'} > c_4\delta'\log D, \tag{44}$$

and $q_l \to 1$ as $x \to \infty$. The sum over the zeros $\varrho\,\epsilon\,G$, $\varrho \neq \beta'$, may be estimated as in (38), except that now for the lower limit of the integral we may take $\max\{c_0, c_5\log(1/\delta'\log T)\}$ (see [11], p. 349, or [6], Lemma 25 with T in place of D); then

$$\begin{aligned}\frac{1}{h}\sum_{\varrho} &\ll \frac{x}{h}\exp\left\{-\frac{1}{2}\frac{\log x}{\log T}\max\left(c_0, c_5\log\frac{1}{\delta'\log T}\right)\right\} \\ &= \frac{x}{h}\min\left\{(\delta'\log T)^{c_6\frac{\log x}{\log T}}, e^{-\frac{1}{2}c_0\frac{\log x}{\log T}}\right\}.\end{aligned}$$

Of the two terms in $\min\{\ldots\}$ the first is the least for $x \in [D^{c_1}, \exp(\log^2 D)]$. And if c_1 is large enough, it is much smaller than (44).

For $x > \exp(\log^2 D)$ we use the estimate

$$\frac{1}{h}\sum_{\varrho} \ll \frac{x}{h}\exp\left(-\frac{1}{2}c_0\frac{\log x}{\log T}\right) \ll \frac{x}{h}D^{-c_7}.$$

This term has a smaller order of magnitude than (44), as $D \to \infty$, since $\delta' > c_8 D^{-c_7/2}$ (cf. [11], p. 144). The remaining term in (42) being $\ll (x/h)\varepsilon_0$ with

$$\varepsilon_0 = \delta'^2 e^{-c_2\sqrt{\log x}},$$

we have by (44) and (42)

$$\psi(x; D, l) = \frac{x}{h}\big(q_l + O(\varepsilon_1 + \varepsilon_0)\big),$$

where

$$\varepsilon_1 = \Big((\delta'\log T)^{-c_6} + e^{\frac{1}{2}c_0}\Big)^{-\frac{\log x}{\log T}}.$$

Hence we can deduce that

$$(45)\quad \pi(x; D, l) = \frac{x}{h\log x}\left\{q_l + \theta c_9\frac{\log(h/\varepsilon_1)}{\log x} + O(\varepsilon_1)\right\} \qquad (0 < \theta < 1).$$

q_l being much larger than ε_1 and the second term in the brackets being > 0, (45) gives positive information about the value of $\pi(x; D, l)$ for $x \geqslant D^{c_1}$ with a sufficiently large $c_1 \ll 1$ and $D \geqslant D_0$(²). Using Siegel's theorem (see [11], p. 144) from (45) and (44) we deduce (2). (If the exceptional zero does not exist, then a better estimate follows from (43).)

It seems to be worth mentioning that (43) is included in (45) if we agree that whenever the exceptional zero does not exist, then $q_l = 1$ and ε_1 becomes the number (39).

Using the result of § 10 and an analogue of Siegel's theorem for Hecke L-functions (see [5]) by the same method we can prove an analogue of (45) for prime ideals $\mathfrak{p}$ of any class $\mathfrak{H}$ mod $\mathfrak{f}$ in any algebraic field.

13. Finally we are going to apply the theorem in the problem of the least interval $(x, x+x')$ in which there is a prime $p \equiv l(\mathrm{mod}\, D)$. By θ_1, θ' and θ we shall denote positive constants < 1.

We start from the identity (cf. [6], § 22)

$$(46)\quad h\sum_{n\equiv l(\mathrm{mod}\, D)} \Lambda(n)\exp\left(-\frac{1}{4y}\log^2\frac{n}{x}\right) = i\sqrt{\frac{y}{\pi}}\int_{2-i\infty}^{2+i\infty}\sum_{\chi}\bar{\chi}(l)\frac{L'}{L}(s, \chi)x^s e^{s^2 y}ds$$

(²) This is not a serious restriction, since for $D < D_0$ the exceptional zero does not exist.

with

$$x \geqslant D^{c_1}, \quad y = x^{-1+\theta_1}.$$

Denoting by S the left-hand side of (46) and moving the contour of integration to the line $\sigma = -1$ we prove that

$$(47)\quad S = 2\sqrt{\pi y}\Big\{xe^y - \sum_{\chi,\varrho_\chi}\bar{\chi}(l)x^\varrho e^{\varrho^2 y}\Big\} + O(x^{-\frac{1}{2}-\frac{1}{2}\theta_1}D\log Dx).$$

Let G denote the rectangle $(0 \leqslant \sigma \leqslant 1,\ |t| \leqslant T)$ with

$$(48)\quad T = x^{1/c_1}, \quad c_1 < 2 + \frac{2\theta_1}{1-\theta_1}.$$

The part of the sum in (47) over the ϱ's outside G being $< x^{-2}$, we have

$$(49)\quad S = 2\sqrt{\pi y}\Big\{xe^y - \sum_{\varrho\in G} x^\varrho e^{\varrho^2 y}\bar{\chi}_\varrho(l) + O(x^{-1/3})\Big\}.$$

First let us suppose that the exceptional zero does not exist. Then the expression in brackets is $> \frac{1}{2}x$ (cf. § 11), whence

$$(50)\quad S > \sqrt{\pi y}\,x > \tfrac{3}{2}x^{\frac{1}{2}+\frac{1}{2}\theta_1}.$$

Let

$$(51)\quad x' = x^{\frac{1}{2}+\frac{1}{2}\theta'}, \quad \text{where} \quad \theta' \in (\theta_1, 1).$$

Then

$$\frac{1}{4y}\log^2(x \pm x')/x > c_2 x^{1-\theta_1}\frac{x'^2}{x^2} = c_2 x^{\theta'-\theta_1},$$

whence it follows that in (46) the contribution of terms with $n \notin I\ (x-x', x+x')$ does not exceed x^{-2}. Hence, by (50),

$$\sum_{\substack{n\in I\\ n\equiv l(\mathrm{mod}\, D)}} \Lambda(n) > \tfrac{5}{4}x^{\frac{1}{2}+\frac{1}{2}\theta_1}$$

and thus

$$(52)\quad \sum_{\substack{p\in I\\ p\equiv l(\mathrm{mod}\, D)}} 1 > x^{\frac{1}{2}+\frac{1}{2}\theta_1}/\log x.$$

After substituting x for $x-x'$ the interval I takes the form $(x, x+x^\theta)$ with $\theta < 1$ (cf. (51)).

In what follows we suppose that the exceptional zero $\beta' = 1-\delta'$ does exist. Then the principal term of the expression in brackets in (49) is $\geqslant x(1-x^{-\delta'}/\beta')\geqslant xc_3\delta'\log D$, whereas the sum $\sum\limits_{\varrho}$ over all the other $\varrho\in G$ by § 11 satisfies

$$\leqslant x\min\{(\delta'\log T)^{c_4\frac{\log x}{\log T}},\ e^{-\frac{1}{2}c_0\frac{\log x}{\log T}}\}. \tag{53}$$

The constants c_0 and c_4 depend merely on the distribution of the zeros of the function $Z(s)$ but not on the constant c_1 in $x\geqslant D^{c_1}$, $T=x^{1/c_1}$. Now we take a sufficiently large

$$c_1\geqslant 4+2/c_4+4/c_0$$

and consider that for any $c_1\geqslant 4$ there is a number $\theta_1<1$ satisfying (48) and thus by (51) the interval I $(x-x',x+x')$ is of the form $(x,x+x^\theta)$ with $\theta<1$.

Our further arguments depend on whether the inequality $\delta'\log T\leqslant 1/c_1$ does or does not hold.

If $\delta'\log T\leqslant 1/c_1$, then the first term in the brackets in (53) satisfies

$$(\delta'\log T)^{c_4c_1}<(\delta'\log T)^2,$$

whereas

$$1-x^{-\delta'}/\beta'=1-e^{-c_1\delta'\log T}/\beta'\geqslant\tfrac{1}{3}c_1\delta'\log T$$

and thus

$$S>c_5x^{\frac{1}{2}+\frac{1}{2}\theta_1}(\delta'\log D),$$

$$\sum_{\substack{p\in I\\ p\equiv l(\mathrm{mod}\,D)}}1>c_6\frac{x^{\frac{1}{2}+\frac{1}{2}\theta_1}(\delta'\log D)}{\log x}. \tag{54}$$

If, on the contrary, $\delta'\log T>1/c_1$, then $\delta'\log x>1$ and $1-x^{-\delta'}/\beta'>1-e^{-1}-\delta'$, whereas the second term in the brackets in (53) is $e^{-\frac{1}{2}c_0c_1}<e^{-2}$. In this case (50) holds true, whence (52).

By (52) and (54) we have proved *the existence of a constant* $0<1$ *such that for* $D\geqslant D_0$ *and any* $x\geqslant D^{c_1}$ (with a sufficiently large $c_1\lessdot 1$) *in the interval* $(x,x+x^\theta)$ *there is a prime* $p\equiv l\,(\mathrm{mod}\,D)$ (3).

By the same method we can prove an analogous result for primes which are norms of prime ideals of a given class $\mathfrak{H}$ mod $\mathfrak{f}$ in any algebraic field. This improves the results of [4] and [5].

(3) This is of interest only for small x or $x\leqslant x_0=\exp(D^\varepsilon)$, where ε is any positive constant and $D>D_0(\varepsilon)$. If $x>x_0$, then the result has been proved for $\theta=\frac{2}{3}$ (cf. [11], p. 323).

An analogous theorem for L-functions of a semigroup (4)

14. The method of the present paper may be used as well for the L-functions $\zeta(s,\chi)$ of a semigroup considered in [6]. Now an analogue of the properties mentioned in § 2 holds by [6] (19), (22) and Lemma 11, whereas that of § 3 can be proved by applying the sieve method as in § 4. Then by the arguments of §§ 5, 6, 7 and 9 we can prove that the number of zeros of the function $\prod\limits_{\chi}\zeta(s,\chi)$ in the rectangle $(1-\lambda/\log T\leqslant\sigma\leqslant 1,$ $|t|\leqslant T)$ (with $T\geqslant D$ and $0\leqslant\lambda<\frac{1}{2}\vartheta\log T$) does not exceed $e^{c\lambda}$. Using this result and [6], Lemma 25, we can apply the method of §§ 12 and 13. This provides (i) a shorter way to the estimates [6], (10) and [6], § 25 for the number $\pi(x,H)$ of the generators and (ii) we can prove the existence of a constant $\theta<1$ such that, whenever $x\geqslant D^{c_3}$, in any class H there is a generator $b\in(x,x+x^\theta)$.

In [6] the estimate $\lessdot e^{c\lambda}$ for the number of zeros was proved only for the rectangles $(1-\lambda/\log D\leqslant\sigma\leqslant 1,$ $|t|\leqslant e^\lambda/\log D)$. Applying this weak result in a more complicated way we acquired merely the interval (x,xD^{c_4}) containing a generator $b\in H$.

15. The remaining part of the present paper will be devoted to improvements of the corresponding results (proved in [7]) about the two-dimensional distribution of the generators of a semigroup $\mathfrak{G}$. Now we suppose that the elements of $\mathfrak{G}$ are complex numbers $\mathfrak{a}=\sqrt{a}e^{2\pi i\alpha}$, where $a=|\mathfrak{a}|^2\geqslant 1$, $\alpha=(2\pi)^{-1}\arg\mathfrak{a}$, and that $a=1$ implies $\alpha=0$. Nex we suppose that the numbers $\mathfrak{a}\in\mathfrak{G}$ are distributed into classes H_j $(1\leqslant j\leqslant h;\ 1\leqslant h\leqslant D)$ forming a group Γ and satisfying

$$\begin{aligned}&\sum_{\substack{\mathfrak{a}\in H_j\\ a\leqslant x}}1=\varkappa x+O(D^{c_1}x^{1-\vartheta}),\qquad \varkappa=D^l\quad(l\gtreqless 0),\\ &\sum_{\substack{\mathfrak{a}\in H_j\\ a\leqslant x,0\leqslant\alpha<\varphi}}1=\varkappa\varphi x+O(D^{c_1}x^{1-\vartheta'})\quad(0<\vartheta'\leqslant\vartheta\leqslant 1)\end{aligned} \tag{55}$$

(uniformly in $0<\varphi\leqslant 1$), where the constants l, ϑ, ϑ', c_1 do not depend on j. In the case of $\vartheta\leqslant\frac{1}{2}$ and an even h we take it for granted that for any subgroup Γ' of Γ with the index 2

$$\lim_{x\to\infty}\Bigg(\sum_{\substack{\mathfrak{a}\in\Gamma'\\ a\leqslant x}}\frac{1}{a}-\sum_{\substack{\mathfrak{a}\notin\Gamma'\\ a\leqslant x}}\frac{1}{a}\Bigg)>D^{-c_2}.$$

(4) The subsequent paragraphes, 14-18, constitute in fact a continuation of my papers [6] and [7] on the abstract theory of primes. But, since they are closely associated with the arguments of §§ 1-13, it is more convenient to include this subject matter here than to write a separate paper.

All the constants used further on may depend on ϑ', ϑ, l, c_1, c_2. Let $\chi(H)$ be the characters of the classes H and let for any $\mathfrak{a} \in H$

$$\chi(\mathfrak{a}) = \chi(H), \qquad \xi(\mathfrak{a}) = e^{2\pi i a},$$

$$X(\mathfrak{a}) = \chi(\mathfrak{a})\xi(\mathfrak{a})^m = \chi(\mathfrak{a})e^{2\pi i m a} \qquad (|m| \leqslant M) \tag{56}$$

(m integer). For any

$$T \geqslant D(1+M) > 1$$

the functions

$$\zeta(s, X) = \sum_{\mathfrak{a}} X(\mathfrak{a})a^{-s} \qquad (\sigma > 1)$$

by [7] (§§ 4 and 5) possess properties analogous to those of § 2, which is the basis for the proof of the theorem.

Let $\Lambda(\mathfrak{a}) = \log b$ if $\mathfrak{a}$ is a power of a generator $\mathfrak{b} = \sqrt{b}e^{2\pi i a}$, and $= 0$ otherwise. The numbers $R(a) = R_k(a)$ being defined by (15) (with a instead of n), let us write

$$e_{\mathfrak{a}} = e_{\mathfrak{a}}(\tau, k) = \frac{\Lambda(\mathfrak{a})}{a^{1+i\tau}} R_k(a) \qquad (-T \leqslant \tau \leqslant T).$$

Suppose that there is a zero $\varrho = \varrho_X = \varrho_{\chi,m}$ of the function $\zeta(s, X)$ in the square $Q(1-\lambda/\log T \leqslant \sigma \leqslant 1,\ |t-\tau| \leqslant \lambda/2\log T)(c_0 \leqslant \lambda < c_3 \log T)$. Then for an appropriate natural integer $k < c_4\lambda$

$$\sum_{\mathfrak{a},\mathfrak{a}_1} \frac{\chi(\mathfrak{a})}{\chi(\mathfrak{a}_1)} e_{\mathfrak{a}}\bar{e}_{\mathfrak{a}_1} e^{2\pi i m(a-a_1)} > e^{-c_5\lambda}, \tag{57}$$

by [7], § 10.

For any fixed $m \in [-M, M]$ let

$$f_m(s) = \prod_{\chi} \zeta(s, \chi\xi^m).$$

Arguing as in [6], § 18 we can prove that the number of zeros $\in Q$ of $f_m(s)$ does not exceed $e^{c_6\lambda}$. By N_Q we denote the number of zeros $\in Q$ of the function

$$Z(s) = \prod_{-M \leqslant m \leqslant M} f_m(s) = \prod_{X} \zeta(s, X).$$

Let $f_{m_j}(s)$ $(1 \leqslant j \leqslant V)$ be all the functions $f_m(s)$ which have a zero $\in Q$; then

$$N_Q \leqslant Ve^{c_6\lambda}. \tag{58}$$

For any m_j we choose some $\chi = \chi_j$ such that the function $\zeta(s, \chi_j\xi^{m_j})$ has a zero $\in Q$. Then for at least $(1/c_4\lambda)V$ of these functions inequality (57) holds with the same $k = k_1$, whence

$$\begin{aligned} V/c_4\lambda e^{c_5\lambda} &< \sum_{1 \leqslant j \leqslant V} \sum_{\mathfrak{a},\mathfrak{a}_1} \frac{\chi_j(\mathfrak{a})}{\chi_j(\mathfrak{a}_1)} e_{\mathfrak{a}}\bar{e}_{\mathfrak{a}_1} e^{2\pi i m_j(a-a_1)} \\ &= \sum_{\mathfrak{a},\mathfrak{a}_1} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{a}_1)}{a^{1+i\tau}a_1^{1-i\tau}} R_{k_1}(a)\bar{R}_{k_1}(a_1) \sum_{1 \leqslant j \leqslant V} \frac{\chi_j(\mathfrak{a})}{\chi_j(\mathfrak{a}_1)} e^{2\pi i m_j(a-a_1)} \\ &< \frac{e^{c_7\lambda}}{\log^2 T} \sum_{\substack{\mathfrak{a},\mathfrak{a}_1 \\ T^B < a,a_1 < T^{3B}}} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{a}_1)}{aa_1} \Bigl| \sum_{1 \leqslant j \leqslant V} \frac{\chi_j(\mathfrak{a})}{\chi_j(\mathfrak{a}_1)} e^{2\pi i m_j(a-a_1)} \Bigr| \end{aligned}$$

and thus

$$\begin{aligned} V &< \frac{e^{c_8\lambda}}{\log^2 T} \sum_{\substack{\mathfrak{a},\mathfrak{a}_1 \\ T^B < a,a_1 < T^{3B}}} \frac{\Lambda(\mathfrak{a})\Lambda(\mathfrak{a}_1)}{aa_1} \Bigl| \sum_{1 \leqslant j \leqslant V} \frac{\chi_j(\mathfrak{a})}{\chi_j(\mathfrak{a}_1)} e^{2\pi i m_j(a-a_1)} \Bigr| \\ &\leqslant \frac{e^{c_8\lambda}}{\log^2 T} \sum_{\substack{\mathfrak{a}_1 \\ T^B < a_1 < T^{3B}}} \frac{\Lambda(\mathfrak{a}_1)}{a_1} \sum_{H} \sum_{\substack{\mathfrak{a} \in H \\ T^B < a < T^{3B}}} \frac{\Lambda(\mathfrak{a})}{a} \Bigl| \sum_{1 \leqslant j \leqslant V} e^{2\pi i m_j(a-a_1)} \Bigr|. \end{aligned} \tag{59}$$

Writing $a - a_1 = \varphi$,

$$g(\varphi) = \sum_{1 \leqslant j \leqslant V} e^{2\pi i m_j\varphi},$$

we deduce that

$$\int_0^1 |g(\varphi)|^2 d\varphi = \int_0^1 \sum_{j,j'} e^{2\pi i\varphi(m_j - m_{j'})} d\varphi = V.$$

For any fixed H and a_1 let Y be the measure of the set of points a such that

$$|g(a-a_1)| > V^{3/4}; \tag{60}$$

then $Y \leqslant V^{-1/2}$. The points a satisfying (60) will be called the *exceptional* ones.

Now we divide the interval $0 \leqslant a \leqslant 1$ into $[MV^{1/3}]$ equal parts $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_k, \ldots$ of the length $c'/MV^{1/3}$ $(1 \leqslant c' < 2)$. If there is in $\mathscr{A}_k$ a non-exceptional number $a = a'$, say, then for any $a = a_{\mathfrak{a}} \in \mathscr{A}_k$ with $\Lambda(\mathfrak{a}) \neq 0$ we have

$$\Bigl| \sum_{1 \leqslant j \leqslant V} e^{2\pi i m_j(a-a_1)} \Bigr| < 2V^{3/4}. \tag{61}$$

The pairs of numbers $\mathfrak{a}_1, \mathfrak{a}$ will be called *normal* ones or *exceptional* ones according as (61) does or does not hold.

By [7], (44) we have

$$\sum_{\substack{\mathfrak{a}\epsilon H, a\leqslant x\\ a\equiv a_0+\vartheta\varphi (\mathrm{mod}\, 1)}} \Lambda(\mathfrak{a}) < c_9 \varphi x/h \tag{62}$$

for any a_0 and any $\varphi \geqslant x^{-\vartheta_0}$ $(0<\vartheta_0<\vartheta')$, provided that $x\geqslant T^{c_{10}}$ $\big(c_{10}=c_{10}(\vartheta_0)\big)$.

Further on we use (62) with $x\,\epsilon\,(T^B, T^{3B})$ where $B=B(\vartheta_0)\ll 1$ is large enough. Then $c'/MV^{1/3}>x^{-\vartheta_0}$ (since $M<T$ and $V<e^{c\lambda}\log M$ by [7], (49), and we may suppose that $e^{c\lambda}<\log M$). This enables us to use (62) for the intervals $\mathscr{A}_k$:

$$\sum_{\substack{\mathfrak{a}\epsilon H, a\leqslant x\\ a\epsilon\mathscr{A}_k}} \Lambda(\mathfrak{a}) < c_{11}\frac{1}{MV^{1/3}}\cdot\frac{x}{h}.$$

Hence

$$\sum_{\substack{\mathfrak{a}\epsilon H, a\epsilon\mathscr{A}_k\\ T^B<a<T^{3B}}} \frac{\Lambda(\mathfrak{a})}{a} < \frac{c_{11}}{hMV^{1/3}}\Big(\int\limits_{T^B}^{T^{3B}}\frac{x\,dx}{x^2}+1\Big) \ll \frac{1}{hMV^{1/3}}\log T. \tag{63}$$

Summing over all the intervals $\mathscr{A}_k$ containing exclusively exceptional points (if there are such $\mathscr{A}_k$) we have for a fixed $\mathfrak{a}_1$

$$\sum_{\substack{\mathfrak{a}\epsilon H\\ T^B<a<T^{3B}\\ \mathfrak{a}_1,\mathfrak{a}\,\mathrm{exc.}}} \frac{\Lambda(\mathfrak{a})}{a} \ll Y\frac{\log T}{h} \ll \frac{\log T}{V^{1/2}h},$$

$$\sum_{\substack{\mathfrak{a}\epsilon H\\ T^B<a<T^{3B}\\ \mathfrak{a}_1,\mathfrak{a}\,\mathrm{exc.}}} \frac{\Lambda(\mathfrak{a})}{a}\Big|\sum_{1\leqslant j\leqslant V} e^{2\pi i m_j(a-a_1)}\Big| \ll h^{-1}V^{1/2}\log T,$$

$$\sum_{H}\sum_{\substack{\mathfrak{a}\epsilon H\\ T^B<a<T^{3B}\\ \mathfrak{a}_1,\mathfrak{a}\,\mathrm{exc.}}} \frac{\Lambda(\mathfrak{a})}{a}\Big|\sum_{1\leqslant j\leqslant V} e^{2\pi i m_j(a-a_1)}\Big| \ll V^{1/2}\log T.$$

The corresponding sum over the normal pairs $\mathfrak{a}_1, \mathfrak{a}$ being $\ll V^{3/4}\log T$, we have

$$\sum_{\substack{\mathfrak{a}_1\\ T^B<a_1<T^{3B}}} \frac{\Lambda(\mathfrak{a}_1)}{a_1}\sum_{H}\sum_{\substack{\mathfrak{a}\epsilon H\\ T^B<a<T^{3B}}}\frac{\Lambda(\mathfrak{a})}{a}\Big|\sum_j\Big| \ll \sum_{\substack{\mathfrak{a}_1\\ T^B<a_1<T^{3B}}}\frac{\Lambda(\mathfrak{a}_1)}{a_1}V^{3/4}\log T \ll V^{3/4}\log^2 T.$$

This combined with (59) proves that $V<e^{c_{12}\lambda}$, whence by (58)

$$N_Q < e^{c_{13}\lambda}. \tag{64}$$

N_Q being the number of zeros of the function $Z(s)=\prod\limits_X \zeta(s,X)$ (with M in (56) satisfying $M<T/D$) in the square Q $(1-\lambda/\log T\leqslant\sigma\leqslant 1$, $|t-t_0|\leqslant\lambda/2\log T)$ $(t_0\ll T)$, the estimate (64) improves [7], Fundamental Lemma 9 (⁵). Simultaneously it improves the theorem of [7], where D_1 can now be replaced by D (cf. [7], footnote (²)).

An estimate similar to that of (64) can be proved for the number of zeros of the function $Z(s)$ in rectangles of the height $\ll T$. This will be performed in the following paragraph.

16. By the arguments of §§ 5-8 we first prove that the number of zeros of any function $\zeta(s,\chi\xi^m)$ with $|m|\leqslant M$ in the rectangle $(1-\lambda/\log T\leqslant\sigma\leqslant 1$, $|t|\leqslant T^2)$ does not exceed $e^{c_3\lambda}$. Using this estimate by the method of § 9, adapted for the function $Z(s)=\prod\limits_X\zeta(s,X)$, we choose the numbers w_j $(1\leqslant j\leqslant V)$ in such a manner that if N_λ denotes the number of zeros of $Z(s)$ in the rectangle (13), then

$$N_\lambda < e^{c_4\lambda}V. \tag{65}$$

Next to this we may suppose that all w_j's are multiples of $c'/\log T$, where c' stands for a sufficiently small constant.

By analogy to (34)

$$V<\frac{e^{c_5\lambda}}{\log^2 T}\sum_{\substack{\mathfrak{a}_1,\mathfrak{a}\\ T^B<a_1\leqslant a<T^{3B}}}\frac{\Lambda(\mathfrak{a}_1)\Lambda(\mathfrak{a})}{a_1 a}\Big|\sum_{1\leqslant j\leqslant V}\frac{X_j(\mathfrak{a}_1)}{X_j(\mathfrak{a})}\Big(\frac{a}{a_1}\Big)^{iw_j}\Big|$$

$$\leqslant\frac{e^{c_5\lambda}}{\log^2 T}\sum_{\substack{\mathfrak{a}_1\\ T^B<a_1<T^{3B}}}\frac{\Lambda(\mathfrak{a}_1)}{a_1}\sum_H\sum_{\substack{\mathfrak{a}\epsilon H\\ a_1\leqslant a<T^{3B}}}\frac{\Lambda(\mathfrak{a})}{a}\Big|\sum_{1\leqslant j\leqslant V}e^{2\pi i m_j(a_1-a)}\Big(\frac{a}{a_1}\Big)^{iw_j}\Big|.$$

The interval $0\leqslant a\leqslant 1$ being divided into $L=[MV^{1/3}]$ equal parts $\mathscr{A}_k$ $(1\leqslant k\leqslant L)$ as in the previous paragraphs, we consider that for all a of the same $\mathscr{A}_k$ the values of $e^{2\pi i m_j(a_1-a)}$ differ at most by $\ll V^{-1/3}$. Hence

$$V\leqslant\frac{e^{c_5\lambda}}{\log^2 T}\sum_{\substack{\mathfrak{a}_1\\ T^B<a_1<T^{3B}}}\frac{\Lambda(\mathfrak{a}_1)}{a_1}\sum_H\sum_{1\leqslant k\leqslant L}\sum_{\substack{\mathfrak{a}\epsilon H\\ a_1\leqslant a<T^{3B}\\ a\epsilon\mathscr{A}_k}}\frac{\Lambda(\mathfrak{a})}{a}\Big|\sum_{1\leqslant j\leqslant V}\Big\{\Big(\frac{a}{a_1}\Big)^{iw_j}+O(V^{-1/3})\Big\}\Big|$$

$$<\frac{e^{c_6\lambda}}{\log^2 T}\sum_{\substack{\mathfrak{a}_1\\ T^B<a_1<T^{3B}}}\frac{\Lambda(\mathfrak{a}_1)}{a_1}\sum_H\sum_{1\leqslant k\leqslant L}\sum_{\substack{\mathfrak{a}\epsilon H\\ a_1\leqslant a<T^{3B}\\ a\epsilon\mathscr{A}_k}}\frac{\Lambda(\mathfrak{a})}{a}\Big(\Big|\sum_{1\leqslant j\leqslant V}\Big(\frac{a}{a_1}\Big)^{iw_j}\Big|+V^{2/3}\Big).$$

(⁵) I have found the method of the present paper after many attempts to improve this unsatisfactory lemma.

Using (66), Lemma 3 and arguing as in § 5 we can prove that

(67) $$V < e^{c_7\lambda}\log u + e^{c_7\lambda}V^{2/3} = e^{c_7\lambda}(V^{2/3}+\log u).$$

Hence in the case of $\log u \leqslant V^{2/3}$ we have $V < 2e^{c_7\lambda}V^{3/2}$, whence $V < e^{c_8\lambda}$. This combined with (65) gives the desired result.

It remains to consider the case of $\log u > V^{2/3}$. Then, by (67),

(68) $$V < e^{c_9\lambda}\log u.$$

Before going on we need the following

LEMMA 3. *Let $T \geqslant D$, $T^{-2} \leqslant \varepsilon \leqslant 1$, $\varphi \geqslant x^{-\vartheta_0}$ $(0 < \vartheta_0 < \vartheta')$ and $x \geqslant T^B$ where the constant $B = B(\vartheta_0)$ is large enough. If I and $\mathscr{A}$ denote respectively the intervals $[x, xe^{\varepsilon}] = [x, x+x']$ and $a \equiv a_0 + \theta\varphi \pmod 1$ $(0 \leqslant \theta < 1)$, then for a suitable c_{10} (which does not depend on H and a_0)*

$$\sum_{\substack{\mathfrak{a}\in H\\ a\in I, a\in\mathscr{A}}} \Lambda(\mathfrak{a}) < c_{10}\varepsilon\varphi x/h.$$

The proof is nearly the same as that of Lemma 2. In [7], Lemma 5 let $\mathfrak{a}_n$ $(n = 1, \ldots, N)$ be all the numbers $\mathfrak{a}\in H$ with $a\in I$ and $a\in\mathscr{A}$. Then, by (55), $N = \varkappa\varphi x' + O(D^{c_1}x^{1-\vartheta'})$ and

$$\sum_{\substack{\mathfrak{a}_n\\ \mathfrak{d}\mid\mathfrak{a}_n}} 1 = N/d + O\big(D^{c_1}(x/d)^{1-\vartheta'}\big),$$

whence (in the notation of [7], Lemma 5) $f(\mathfrak{d}) = d$, $R_{\mathfrak{d}} \ll D^{c_1}(x/d)^{1-\vartheta'}$.

Using $z \geqslant x^{c''}$ (with a small constant $c'' < \frac{1}{2}$) and arguing as in [7], § 7 we deduce that $S_z > c_{11}h\varkappa\log x$ and thus $N/S_z \ll \varphi x'/h\log x$. In proving the same estimate for the term

$$W = D^{c_1}x^{1-\vartheta'} \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2\\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(a_1, a_2)}{a_1 a_2}\right)^{1-\vartheta'},$$

consider that by [7], § 8 the last sum does not exceed $c_{12}D^{c_{13}-c_1}(h\varkappa)^2 z^{2\vartheta'}$, whence $W \ll D^{c_{13}}(h\varkappa)^2 x^{1-\vartheta'}z^{2\vartheta'}$. Putting

(69) $$z^{2\vartheta'} = \frac{\varphi x^{\vartheta'}(x'/x)}{D^{c_{13}}(h\varkappa)^2 h\log x},$$

we get the desired estimate. Under the conditions of the present lemma we have in (69) $z > x^{c''}$ and $z < x^{2/3}$. Hence for the number of the generators $\mathfrak{b} = \sqrt{b}e^{2\pi i a}$ we can prove the estimate

$$\sum_{\substack{\mathfrak{b}\in H\\ b\in I, a\in\mathscr{A}}} 1 < c_{14}\varphi x'/h\log x < c_{15}\varphi\varepsilon x/h\log x,$$

whence

(70) $$\sum_{\substack{\mathfrak{a}\in H\\ a\in I, a\in\mathscr{A}}} \Lambda(\mathfrak{a})/a \ll \varepsilon\varphi/h.$$

Let us now return to the function $Z(s)$ and the rectangle (1), our aim being the elimination of the factor $\log u$ in (68). To this end we use (66), (68), (70) and proceed as in § 7. Finally we get the following result:

Let the characters X be defined by (56) and let $T \geqslant D(1+M)$. Then the number of zeros of the function $Z(s) = \prod_X \zeta(s, X)$ in the rectangle

$$(1-\lambda/\log T \leqslant \sigma \leqslant 1,\ |t| \leqslant T) \quad (c_0 \leqslant \lambda \leqslant \tfrac{1}{3}\vartheta'\log T)$$

does not exceed $e^{c\lambda}$.

On the two-dimensional distribution of generators

17. In this paragraph let $\mathscr{A}$ denote any interval $a_1 \leqslant a \leqslant a_2$ of the length $\varphi\in[\Delta, 1-2\Delta]$, where $\Delta \geqslant D^{-c_3}$ with arbitrarily large $c_3 \ll 1$. We shall use the function

$$f(a) = \sum_{-\infty<m<\infty} d_m e^{2\pi i m a},$$

whose values are $\geqslant 0$, $\leqslant 1$ such that $f(a) = 1$ if $a\in\mathscr{A}$ and $f(a) = 0$ outside the interval $(a_1-\Delta, a_2+\Delta)$, and whose coefficients satisfy:

$$d_0 = a_2 - a_1 + \Delta,$$

(71) $$|d_m| \leqslant d_0, \quad d_m \ll \min(d_0, |m|^{-1}, \Delta^{-r}|m|^{-r-1})$$

for any integer $r \geqslant 1$ with the constant in the notation depending on r (cf. [7], § 11).

We start from an analogue of (36)

$$\sum_{\substack{\mathfrak{a}\\ a\leqslant x}} X(\mathfrak{a})\Lambda(\mathfrak{a}) + \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} \frac{x^s}{s}\cdot\frac{\zeta'}{\zeta}(s, X)\,ds \ll \frac{x^\eta}{T(\eta-1)} + \frac{x\log^2 x}{T} + \log x$$

$$(1 < \eta < 2;\ x > 1;\ T > 1),$$

with $x \geqslant D^{c_4}$, $\eta = 1+1/\log x$, $T < \sqrt{x}$; then the right-hand side is $\ll T^{-1}x\log^2 x$. Hence, by (56),

(72) $$\sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x}} \Lambda(\mathfrak{a})\xi(\mathfrak{a})^m + h^{-1}\sum_{\chi}\bar{\chi}(H)\frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT}\frac{x^s}{s}\cdot\frac{\zeta'}{\zeta}(s, \chi\xi^m)\,ds \ll T^{-1}x\log^2 x.$$

By the definition of $f(a)$ we have, say,

(73)
$$\sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x}}\Lambda(\mathfrak{a})f(a) = \sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x}}\Lambda(\mathfrak{a})\sum_{-\infty<m<\infty} d_m\,\xi(\mathfrak{a})^m = \sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x}}\sum_{|m|\leqslant M} + \sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x}}\sum_{|m|>M} = U_1+U_2.$$

Hence, by (73) and (72)

$$U_1 = -\frac{h^{-1}}{2\pi i}\sum_{\chi}\bar{\chi}(H)\sum_{|m|\leqslant M} d_m\int_{\eta-iT}^{\eta+iT}\frac{x^s}{s}\cdot\frac{\zeta'}{\zeta}(s,\chi\xi^m)\,ds+O(T^{-1}x\log^2 x\log M).$$

Choose
$$T = D^{c_5}e^{c_6\sqrt{\log x}},$$

where the constants c_5 and c_6 are large enough. If the exceptional zero (see [7], § 5) does not exist, then we use

$$\varepsilon = \exp\left(-\frac{1}{2}c_0\frac{\log x}{\log T}\right),\qquad \Delta = d_0\varepsilon,\qquad M = (d_0\varepsilon)^{-2},\qquad r=1$$

and arguing as in §§ 11 and 12 we prove that

$$U_1 = \frac{x}{h}d_0\big(1+O(\varepsilon)\big)+O(T^{-1}x\log^3 x\log M),$$

$$U_2 \ll \frac{x}{h}\Delta^{-1}M^{-1} = \frac{x}{h}d_0\varepsilon.$$

Since for appropriate c_5, c_6

$$\frac{x\log^3 x\log M}{T} < \frac{x}{h}d_0\varepsilon,$$

the last term in U_1 is of no importance. Denoting by Φ the left-hand side of (73), we have

(74)
$$\Phi = \frac{x}{h}d_0\big(1+O(\varepsilon)\big).$$

From the definition of $f(a)$ it follows that

$$\Phi = \sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x,\,\alpha\in\mathscr{A}}}\Lambda(\mathfrak{a})+\Phi_1+\Phi_2,$$

where Φ_1 and Φ_2 do not exceed analogous sums over intervals $\mathscr{A}_1$, $\mathscr{A}_2$ (say) of the length Δ. They may be estimated in the same manner as (74), except that now $\alpha_2 = \alpha_1+\Delta$. We have

$$\Phi_1+\Phi_2 \ll \frac{x}{h}\Delta = \frac{x}{h}d_0\varepsilon,$$

whence, by (74),

(75)
$$\sum_{\substack{\mathfrak{a}\in H\\ a\leqslant x,\,\alpha\in\mathscr{A}}}\Lambda(\mathfrak{a}) = \frac{x}{h}d_0\big(1+O(\varepsilon)\big).$$

Denoting by $\pi(x; H, \mathscr{A})$ the number of generators $\mathfrak{b} = \sqrt{b}e^{2\pi i\alpha}\in H$ such that $b\leqslant x$ and α is in the interval $\mathscr{A}$ of the length φ, from (75) we can deduce that

$$\pi(x; H, \mathscr{A}) = \varphi\frac{x}{h\log x}\left\{1+\theta c_7\frac{\log(h/\varepsilon\varphi)}{\log x}+O(\varepsilon)\right\}\qquad (0<\theta<1)$$

(cf. § 12). If the exceptional zero $\beta' = 1-\delta'$ does exist, then

(76)
$$\pi(x; H, \mathscr{A}) = \varphi\frac{x}{h\log x}\left\{q_H+\theta c_7\frac{\log(h/\varepsilon_1\varphi)}{\log x}+O(\varepsilon_1)\right\}\qquad (0<\theta<1),$$

where

$$q_H = 1-\chi'(H)\frac{x^{-\delta'}}{1-\delta'},\qquad \varepsilon_1 = \{(\delta'\log T)^{-c_8}+e^{c_0/2}\}^{-\log x/\log T}$$

(χ' being the exceptional character). For $\varphi\geqslant D^{-c_3}$ and $x\geqslant D^{c_4}$ (with a sufficiently large $c_4 = c_4(c_3)$) the right-hand side of (76) is evidently positive; for large x it is asymptotically $\varphi x/h\log x$.

18. In this final paragraph we shall investigate the existence of a generator $\mathfrak{b} = \sqrt{b}e^{2\pi i\alpha}\in H$ in a region ($x<b<x+x^\theta$; $\alpha\in\mathscr{A}$) with a constant $\theta<1$ and a small interval $\mathscr{A}$ of the length $\varphi = x^{-c}$ ($c>0$).

Denoting by C_1 a sufficiently large constant we suppose that
$$x\geqslant D^{2C_1}.$$

Let
$$g = \tfrac{1}{24}\vartheta',\qquad \sigma_1 = 1-g$$

and let $f(a)$ be the function defined in the previous paragraph with $\alpha_2 = \alpha_1$ and $\Delta = \frac{1}{2}\varphi$. Then for any $y>0$ we have, say,

(77)
$$h\sum_{\substack{\mathfrak{a}\in H\\ \alpha\in\mathscr{A}}}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}}\exp\left(-\frac{1}{4y}\log^2\frac{a}{x}\right) \geqslant h\sum_{\mathfrak{a}\in H}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}}\exp\left(-\frac{1}{4y}\log^2\frac{a}{x}\right)f(a)$$
$$= h\sum_{\mathfrak{a}\in H}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}}\exp\left(-\frac{1}{4y}\log^2\frac{a}{x}\right)\sum_{-\infty<m<\infty}d_m\,\xi(\mathfrak{a})^m$$
$$= h\sum_{\mathfrak{a}\in H}\sum_{|m|\leqslant M}+h\sum_{\mathfrak{a}\in H}\sum_{|m|>M} = U_1+U_2.$$

By [7], § 11

$$U_1 = \sum_{\chi} \overline{\chi}(H) \sum_{|m|\leqslant M} d_m i \sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, \chi\xi^m) x^{s-\sigma_1} e^{(s-\sigma_1)^2 y} ds$$

$$= 2\sqrt{\pi y}\, x^g e^{g^2 y}(d_0 S - S') + O(D\log^4 D)$$

where

$$S' = \sum_{\chi} \overline{\chi}(H) \sum_{|m|\leqslant M} d_m \sum_{\varrho_{\chi,m}(\neq\beta')} x^{-\delta} \exp\{(-\delta(2g-\delta)-\gamma^2+2i\gamma(g-\delta))y + i\gamma\log x\},$$

$$S = 1 - E\chi'(H)x^{-\delta'}\exp\{-\delta'(2g-\delta')y\},$$

$\varrho_{\chi,m} = 1-\delta+i\gamma$ denotes the zeros of $\zeta(s, \chi\xi^m)$ in general, β' the exceptional zero, χ' the exceptional character, and $E = 1$ or 0 according as χ' exists or not.

Further on we use

$$T = x^{1/C_1}, \qquad y = x^{-1+\theta_1}$$

with a positive constant $\theta_1 < 1$ such that

$$C_1 < 2 + \frac{2\theta_1}{1-\theta_1}.$$

Let G be the rectangle $(1-2g \leqslant \sigma \leqslant 1, |t| \leqslant T)$. In the case where β' does not exist, we have (cf. § 13)

(78) $$U_1 > 2\sqrt{\pi y}\, x^g e^{g^2 y} d_0 \Big(1 - \sum_{\varrho \in G} x^{-\delta}\Big) - c_3 D\log^4 D > \tfrac{1}{2}\sqrt{\pi y}\, x^g \varphi - c_3 D \log^4 D.$$

We suppose that the constant c in $\varphi \geqslant x^{-c}$ satisfies

$$c < \min\{g, (3C_1)^{-1}\};$$

then

(79) $$U_1 > \tfrac{1}{2}\sqrt{y}\, x^g \varphi.$$

By [7], § 12 and (71)

$$U_2 \ll \sqrt{y}\, x^g \sum_{|m|>M} |d_m| \ll \sqrt{y}\, x^g (\Delta M)^{-r}.$$

Choose $M = x^{1/2C_1} - 1$ and $r = 3$. Then $U_2 < \frac{1}{2}U_1$, whence by (79) and (77)

$$h \sum_{\substack{\mathfrak{a}\in H \\ a\in\mathscr{A}}} \frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}} \exp\Big(-\frac{1}{4y}\log^2\frac{a}{x}\Big) > \frac{1}{4}\sqrt{y}\, x^g \varphi.$$

Putting $x' = x^{1/2+\theta'/2}$ (where $\theta_1 < \theta' < 1$) and arguing as in § 13 we can prove that in the last sum the contribution of terms with $a \notin I\ (x-x', x+x')$ does not exceed x^{-2}, whence

(80) $$h \sum_{\substack{\mathfrak{a}\in H \\ a\in I, a\in\mathscr{A}}} \frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}} > \frac{1}{8}\varphi x^{-\frac{1}{2}+\frac{1}{2}\theta_1+g}.$$

Since $\sigma_1 > \frac{3}{4}$, the contribution of terms with $\mathfrak{a} = \mathfrak{b}^2, \mathfrak{b}^3, \ldots$ does not exceed $O(D^l x^{-1/2}\log^2 x)$, which is much smaller than the right-hand side of (80), whence

(81) $$h \sum_{\substack{\mathfrak{b}\in H \\ b\in I, a\in\mathscr{A}}} \frac{\log b}{b^{\sigma_1}} > \frac{\varphi}{16} x^{-\frac{1}{2}+\frac{1}{2}\theta_1+g}.$$

If the exceptional zero $\beta' = 1-\delta'$ exists, then the expression in the brackets in (78) is $\geqslant c_4\delta'\log D$ (cf. § 13), whence $U_1 > c_5\varphi\sqrt{y}\, x^g\delta'\log D$, $U_2 < \frac{1}{2}U_1$, etc. and finally

$$h \sum_{\substack{\mathfrak{b}\in H \\ b\in I, a\in\mathscr{A}}} \frac{\log b}{b^{\sigma_1}} > c_6(\delta'\log D)\varphi x^{-\frac{1}{2}+\frac{1}{2}\theta_1+g}.$$

From this and (81) we deduce that *for appropriate constants* $\theta = 1-c'$, c, C (which may depend on the constants $c_1, c_2, \vartheta, \vartheta', l$ of § 15), *for any* α_1 *and any class* H *in the region*

$$\{x < b < x+x^\theta;\ \alpha_1 < a < \alpha_1 + x^{-c}\ (\mathrm{mod}\,1)\} \quad \textit{with} \quad x \geqslant D^C$$

there is a generator $\mathfrak{b} = \sqrt{b}\, e^{2\pi i a} \in H$.

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Reçu par la Rédaction le 4. 6. 1964

ACTA ARITHMETICA
XI (1965)

О нулях аналитических функций с заданной арифметикой коэффициентов и представлении чисел

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§ 1. Нули аналитических функций с целыми коэффициентами. Если целочисленность аналитической функции на каком — либо [1] кольце сразу вызывает ограничения на ее рост в том или ином смысле, то целочисленность коэффициентов a_n, $n = 0, 1, 2, \ldots$, функций

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_2(z) = \sum_{n=0} \frac{a_n}{n!} z^n$$

практически не накладывает никаких условий на арифметическую природу их нулей.

Докажем, в подтверждение этого, ряд теорем.

Теорема I. *Если* $a_1, a_2, \ldots; |a_k| < 1$, *последовательность действительных чисел и* $m \geqslant 1$ *целое, то можно найти последовательность целых чисел* $t_0, t_1, t_2, \ldots$, $t_0 \neq 0$, $|t_k| \leqslant m$, $k = 0, 1, 2, \ldots$ *такую, что*

$$f(a_k) = 0, \quad k = 1, 2, \ldots, n, \ldots; \quad f(z) = \sum_{0}^{\infty} t_k z^k, \tag{1.1}$$

при условии

$$1 + m > \prod_{1}^{n} |a_k|^{-1}, \quad n \geqslant 1. \tag{1.2}$$

Это условие не меняется и в случае кратных корней, $f^{(n)}(a_k) = 0$, $n = 0, \ldots, p-1$. *Тогда в нем надо брать* $|a_k|^p$ *вместо* $|a_k|$.

Замечание к теореме. Если предполагать числа a_k комплексными, то условие (1.2) заменится условиями

$$1 + m > \prod_{1}^{n} |a_k|^{-2}, \quad 1 + m > \prod_{1}^{n} |a_k|^{-1}, \quad k = 1, 2, 3, \ldots, \tag{1.3}$$

если, соответственно, $|t_k| \leqslant m$ целые или $|t_k'| < m$, $|t_k''| < m$, $t_k = t_k' + it_k''$, где t_k' и t_k'' целые рациональные.