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On a cyclic sum of Mordell

by

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1. Introduction. In [1] and [2] Mordell and Diananda respectively have considered for what values of λ the inequality

$$(1.1) \qquad \left(\sum_{i=1}^{n} x_i \right)^2 - \lambda \sum_{i=1}^{n} x_i(x_{i+1} + \ldots + x_{i+m}) \geqslant 0,$$

where $x_{n+i} = x_i \geqslant 0$ for all i, holds. The results obtained so far are summarised by two theorems in [2], which we state here for the sake of completeness.

THEOREM A. *Let $x_{n+i} = x_i$ for all i. Then for given $m, n > 0$ there is a constant $\lambda(m, n)$, $0 \leqslant \lambda(m, n) \leqslant \dfrac{n}{m}$, such that (1.1) is true for all $x_1, \ldots, x_n \geqslant 0$ if $\lambda \leqslant \lambda(m, n)$ and false for some $x_1, \ldots, x_n \geqslant 0$ if $\lambda > \lambda(m, n)$.*

THEOREM B. *The constants $\lambda(m, n)$ are such that*

(i) $\lambda(m, n+1) \geqslant \lambda(m, n) \geqslant \lambda(m+1, n)$,

(ii) $\lambda(m, n) = \dfrac{n}{m}$ *if $n | m+2$ or $2m$ or $2m+1$ or $2m+2$,*

    or if $n | m+3$ and $n = 8$ or 9 or 12,
    or if $n | m+4$ and $n = 12$,

    and $\lambda(m, n) < \dfrac{n}{m}$ otherwise,

(iii) $\lambda(m, n) = \dfrac{2m+2}{m}$ *if $n > 2m+2$,*

(iv) $\lambda(m, n) = \dfrac{12n}{n+12m-6}$ *if $n | 2m-1$ and $n > 6$,*

(v) $\lambda(m, n) = \dfrac{\lambda(m-kn, n)}{1+k\lambda(m-kn, n)}$ *if $kn < m$ ($k = 1, 2, \ldots$).*

From these theorems we see that the only upper bound known for $\lambda(m, n)$ when $m+2 < n < 2m-1$ is $\min\left\{\dfrac{n}{m}, \dfrac{12}{7}\right\}$. In this paper we prove that:

(1) For a fixed positive integer $t \neq 3$ and $m \geqslant \max\{2t, \tfrac{3}{2}t+2\}$,

$$(1.2) \qquad \lambda(m, 2m-t) = \frac{4(t+2)}{3t+4}.$$

(2) When $t = 3$,

$$(1.3) \qquad \lambda(7, 11) \leqslant \lambda(m, 2m-t) \leqslant \frac{4(t+2)}{3t+4} \qquad \text{for} \qquad m \geqslant 7.$$

(3) For $m+2 < n \leqslant 2m-1$,

$$(1.4) \qquad \lambda(m, n) \leqslant \frac{2(r+1)\{(r+1)(m+1)-rn\}}{(2r+1)\{(r+1)m-rn\}+2r(r+1)},$$

where r is the integer such that $\dfrac{r+2}{r+1} \leqslant \dfrac{n}{m} < \dfrac{r+1}{r}$.

$\dfrac{n}{m}$ is a better bound for $\lambda(m, n)$ only for $\lambda(6, 9)$ in the range considered. Although this case shows that strict inequality holds in (1.4) in at least one case, (1.2) shows that equality does hold for $r = 1$, $t \neq 3$. It therefore seems likely that equality holds in (1.4) except possibly for a few particular cases.

2. In this section we prove that, for a fixed positive integer t and $m \geqslant \tfrac{3}{2}t+1$, $\lambda(m, 2m-t) \leqslant 4(t+2)/(3t+4)$, and also that, if strict inequality holds, then any sequence which requires $\lambda = \lambda(m, 2m-t)$ in (1.1) can contain at most $2t+3$ positive terms.

Consider (1.1); the case $\sum\limits_{r=1}^{n} x_r = 0$ being trivial since $x_r \geqslant 0$, we may suppose by homogeneity that $\sum\limits_{r=1}^{n} x_r = 1$. Hence (1.1) becomes

$$(2.1) \qquad 1 \geqslant \lambda \sum_{r=1}^{n} x_r(x_{r+1}+\ldots+x_{r+m}) = \lambda f_m(x_1, \ldots, x_n),$$

where $f_m(x_1, \ldots, x_n) = \sum\limits_{r=1}^{n} x_r(x_{r+1}+\ldots+x_{r+m})$. Clearly (2.1) holds for all sufficiently small λ, so $\lambda(m, n) > 0$ and we have

$$(2.2) \qquad f_m(x_1, \ldots, x_n) \leqslant \frac{1}{\lambda(m, n)}.$$

DEFINITION. A *maximal sequence* is a sequence of non-negative numbers $x_1, \ldots, x_n$ where $\sum\limits_{r=1}^{n} x_r = 1$ such that $f_m(x_1, \ldots, x_n) = \dfrac{1}{\lambda(m, n)}$.

Let $y_1, \ldots, y_n$ be a maximal sequence of $f_m(x_1, \ldots, x_n)$ then since $y_r \geqslant 0$ for all r we may temporarily write $y_i = z_i^2$. Since $\sum\limits_{r=1}^{n} z_r^2 = 1$, by Lagrange's method the condition for $f_m(y_1, \ldots, y_n)$ to have a maximum is that, if $u = f_m(z_1^2, \ldots, z_n^2) - k\{\sum\limits_{i=1}^{n} z_i^2 - 1\}$ where k is a constant, then $\dfrac{\partial u}{\partial z_i} = 0$ for $i = 1, \ldots, n$, i.e.

$$2z_i \sum_{0<|j-i|\leqslant m} z_j^2 = 2z_i k \qquad (i = 1, \ldots, n).$$

Hence reverting to the y_i we must have at least one of

$$(2.3) \quad y_i = 0 \quad \text{and} \quad \sum_{0<|j-i|\leqslant m} y_j = k \quad \text{for each } i \quad (i = 1, \ldots, n).$$

Let P be the subset of $1, 2, \ldots, n$ for which only the second equality of (2.3) holds and $n = 2m-t$ where $1 \leqslant t < m-2$, then, since $\sum\limits_{r=1}^{n} y_r = 1$, (2.3) gives

$$(2.4) \qquad 1 - y_r + y_{r+m-t} + y_{r+m-t+1} + \ldots + y_{r+m} = k \qquad (r \,\epsilon\, P).$$

If P contains p members then on adding the p equations of (2.4) we obtain

$$p - \sum_{r\epsilon P} y_r + \sum_{r\epsilon P}(y_{r+m-t}+\ldots+y_{r+m}) = pk.$$

Since $\sum\limits_{r\epsilon P} y_r = 1$ and, for fixed s $(s = 0, 1, \ldots, t)$ $\sum\limits_{r\epsilon P} y_{r+m-s} \leqslant 1$ we therefore have $p - 1 + (t+1) \geqslant pk$, i.e.

$$(2.5) \qquad k \leqslant 1 + t/p.$$

Now, using (2.3) we obtain

$$2f_m(y_1, \ldots, y_n) = \sum_{r\epsilon P} y_r\left\{\sum_{0<|j-r|\leqslant m} y_j\right\} = \sum_{r\epsilon P} y_r k = k.$$

Hence, since $y_1, \ldots, y_{2m-t}$ is a maximal sequence, we have from (2.5)

$$(2.6) \qquad \lambda(m, 2m-t) = \frac{2}{k} \geqslant \frac{2p}{p+t}.$$

In the above we have only assumed that $y_1, \ldots, y_n$ is a maximal sequence of $f_m(x_1, \ldots, x_n)$, so, if $f_m(x_1, \ldots, x_n)$ has more than one maximal sequence, we may choose for $y_1, \ldots, y_n$ a maximal sequence which contains

at least as many positive terms as any of the remaining maximal sequence, and then (2.6) will be true for the value of p of this maximal sequences.

Consider $n = 2m - t$ when t is even, say $t = 2s$, then for the sequence defined by $z_1 = z_2 = \ldots = z_{s+1} = 1/2(s+1) = z_{m-s+1} = z_{m-s+2} = \ldots = z_{m+1}$ and $z_r = 0$ otherwise, we have, for $m \geqslant \frac{3}{2}t + 1$,

$$(2.7) \quad f_m(z_1, \ldots, z_n)$$

$$= \frac{2}{4(s+1)^2} \{(2s+2-1) + (2s+2-2) + \ldots + (2s+2-s-1)\}$$

$$= \frac{2}{4(s+1)^2} \{2(s+1)^2 - \tfrac{1}{2}(s+1)(s+2)\} = \frac{3s+2}{4(s+1)}$$

$$= \frac{3t+4}{4(t+2)} \quad \text{since } t = 2s.$$

Now consider $n = 2m - t$ when t is odd, say $t = 2s - 1$, then for the sequence defined by $z_1 = z_2 = \ldots = z_s = 1/(2s+1) = z_{m-s+2} = z_{m-s+3} = \ldots = z_{m+1}$, $z_{s+1} = 1/2(2s+1) = z_{m-s+1}$ and $z_r = 0$ otherwise, we have, for $m \geqslant \frac{3}{2}(t+1)$,

$$(2.8) \quad f_m(z_1, \ldots, z_n) = \frac{3t+4}{4(t+2)}.$$

Hence for $n = 2m - t$, where $m \geqslant \frac{3}{2}t + 1$, we see from (2.7) and (2.8) that there is a sequence $z_1, z_2, \ldots, z_n$ with $\sum_{r=1}^{n} z_r = 1$ and $z_r \geqslant 0$ such that $f_m(z_1, \ldots, z_n) = (3t+4)/4(t+2)$. Thus from (2.2) we have:

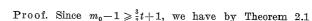THEOREM 2.1. *If t is a positive integer and $m \geqslant \frac{3}{2}t + 1$, then*

$$\lambda(m, 2m - t) \leqslant \frac{4(t+2)}{3t+4}.$$

Further if $\lambda(m, 2m - t) < \dfrac{4(t+2)}{3t+4}$, from (2.6) we have $\dfrac{2p}{p+t} < \dfrac{4(t+2)}{3t+4}$, i.e. $p < 2(t+2)$. Hence we have:

LEMMA 2.1. *If $\lambda(m, 2m - t) < 4(t+2)/(3t+4)$, then a maximal sequence $y_1, \ldots, y_{2m-t}$ can contain at most $2t + 3$ positive terms when $m \geqslant \frac{3}{2}t + 1$.*

3. In this section we show that $\lambda(m, 2m - t)$ is a non-decreasing function of m for $m \geqslant \max\{2t, \frac{3}{2}t + 2\}$ and so deduce that (1.2) holds for $t = 1$ 2, 4 and that (1.3) holds when $t = 3$.

LEMMA 3.1. *If, for a fixed positive integer t, $\lambda(m, 2m - t) = 4(t+2)/(3t+4)$ for $m = m_0 \geqslant \max\{2t+1, 4\}$ then $\lambda(m_0 - 1, 2m_0 - t - 2) \leqslant \lambda(m_0, 2m_0 - t)$.*

Proof. Since $m_0 - 1 \geqslant \frac{3}{2}t + 1$, we have by Theorem 2.1

$$\lambda[m_0 - 1, 2(m_0 - 1) - t] \leqslant \frac{4(t+2)}{3t+4} = \lambda(m_0, 2m_0 - t).$$

Let $z_1, \ldots, z_{2m-t}$ where $\sum_{r=1}^{2m-t} z_r = 1, z_r \geqslant 0$ and $m \geqslant \frac{3}{2}t + 1$, be a sequence and suppose it contains two zeros which are separated by at least $(m - t - 2)$ terms. Take two such zeros; we may suppose without loss of generality that one of them is $z_1 = 0$ and the other $z_b = 0$ so that $m - t \leqslant b \leqslant m + 2$. Consider the sequence $y_1, \ldots, y_{2m-t-2}$ consisting of $z_2, z_3, \ldots, z_{b-1}, z_{b+1}, z_{b+2}, \ldots, z_{2m-t}$, then $f_{m-1}(y_1, \ldots, y_{2m-t-2}) \geqslant f_m(z_1, \ldots, z_{2m-t})$ because, for $r = 2, 3, \ldots, b-1$, $z_r(z_{r+1} + \ldots + z_{r+m}) \leqslant y_{r-1}(y_r + \ldots + y_{r+m-2})$ since $z_{r+1} + \ldots + z_{r+m}$ must include z_b since $b \leqslant m+2$, and for $r = b+1, b+2, \ldots, 2m-t$, $z_r(z_{r+1} + \ldots + z_{r+m}) \leqslant y_{r-2}(y_{r-1} + \ldots + y_{r+m-3})$ since $z_{r+1} + \ldots + z_{r+m}$ must include z_1 since $b \geqslant m - t$.

In particular, if $z_1, \ldots, z_{2m-t}$ is a maximal sequence which contains two zeros which are separated by at least $(m - t - 2)$ terms, then there is a sequence $y_1, y_2, \ldots, y_{2m-t-2}$ such that

$$f_{m-1}(y_1, \ldots, y_{2m-t-2}) \geqslant f_m(z_1, \ldots, z_{2m-t}) = \frac{1}{\lambda(m, 2m-t)}$$

so that $\lambda(m-1, 2m-t-2) \leqslant \lambda(m, 2m-t)$.

Now suppose $\lambda(m, 2m-t) < 4(t+2)/(3t+4)$ and $m \geqslant 2t+3$ then, by Lemma 2.1, a maximal sequence $z_1, \ldots, z_{2m-t}$ can contain at most $(2t+3)$ positive terms so, since $m \geqslant 2t+3$, there are at least $(m-t)$ zeros and so two zeros must be separated by at least $(m-t-2)$ terms. Hence, in virtue of Theorem 2.1 and Lemma 3.1 we have from the above:

THEOREM 3.1. *For a fixed positive integer t, $\lambda(m, 2m-t)$ is a non-decreasing function of m for $m \geqslant 2t+2$.*

From the special result $\lambda(4, 7) = \frac{12}{7}$ proved in [1], we now see from Theorems 2.1 and 3.1 that (1.2) holds when $t = 1$, a result proved by Diananda in [2].

Now supposing $t \geqslant 2$ consider $m = 2t + s$ for $s = 1, 2$; by Theorem 2.1 either $\lambda(m, 2m-t) = 4(t+2)/(3t+4)$ in which case $\lambda(m-1, 2m-t-2) \leqslant \lambda(m, 2m-t)$ by Lemma 3.1 or $\lambda(m, 2m-t) < 4(t+2)/(3t+4)$. If the latter holds and $z_1, z_2, \ldots, z_{2m-t}$ is a maximal sequence we have from the above that $\lambda(m-1, 2m-t-2) \leqslant \lambda(m, 2m-t)$ if $z_1, \ldots, z_{2m-t}$ contains two zeros which are separated by at least $(m-t-2)$ terms. Hence $\lambda(m-1, 2m-t-2) > \lambda(m, 2m-t)$ can only possibly hold if each maximal sequence has $(t+s-1)$ consecutive terms which contain all the zeros. Thus, using the notation of Section 2, consider, if possible, a maximal sequence $z_1, \ldots, z_{2m-t}$ such that $1, 2, \ldots, 2t+s+1$ all belong to P.

(i) If $s = 2$, by Lemma 2.1 there can be at most $(2t+3)$ positive terms and so $z_r = 0$ for $r = 2t+4, \ldots, 3t+4$. Thus, from (2.4) with $r = t+2$, we have $k = 1-z_{t+2} < 1$ which is impossible since, from (2.6),

$$k = \frac{2}{\lambda(m, 2m-t)} > \frac{2(3t+4)}{4(t+2)} > 1.$$

(ii) If $s = 1$, (2.4) holds for $r = 1$, $t+2$ and $2t+2$ and on adding these three equations we obtain

$$3k = 3 + \sum_{r=1}^{3t+2} z_r - z_{t+2} - z_{2t+2} < 4.$$

Thus if $t \geqslant 4$ we have a contradiction since

$$k > \frac{2(3t+4)}{4(t+2)} \geqslant \frac{4}{3} \quad \text{for} \quad t \geqslant 4.$$

Hence from the above we may strengthen Theorem 3.1 to:

THEOREM 3.2. *For a fixed positive integer t, $\lambda(m, 2m-t)$ is a non-decreasing function of m for $m \geqslant \max\{2t, \frac{3}{2}t+2\}$.*

From Theorem B(ii) $\lambda(5, 8) = \frac{8}{5}$ and $\lambda(8, 12) = \frac{3}{2}$ so from Theorems 2.1 and 3.2 we see that (1.2) holds when $t = 2$ and 4 and (1.3) when $t = 3$.

4. From Theorems 2.1 and 3.2 it follows that $\lambda(2t, 3t) \leqslant \lambda(m, 2m-t) \leqslant 4(t+2)/(3t+4)$ for $m \geqslant 2t$ and $t > 4$. Hence, to show that (1.2) holds for $t > 4$, we need only prove that $\lambda(2t, 3t) = 4(t+2)/(3t+4)$. To do this we firstly obtain a number of lemmas which give us information concerning the terms of a maximal sequence of $\lambda(2t, 3t)$.

Notation. Throughout this section we assume $t > 4$.

DEFINITION. The *dual* of the sequence $x_1, x_2, \ldots, x_n$ is the sequence $x_n, x_{n-1}, \ldots, x_1$.

Clearly the dual of a maximal sequence is a maximal sequence.

LEMMA 4.1. *If $y_1, \ldots, y_{3t}$ is a maximal sequence of $\lambda(2t, 3t)$ then the following situations cannot arise:*

   (i) $y_{j+st} \neq 0$ *for $s = 0, 1$ and 2.*

   (ii) $y_{j+st} = 0$ *for $s = 0, 1$ and 2.*

   (iii) $y_j \neq 0$, $y_{j+2t} = 0 = y_{j-1}$.

   (iv) $y_j \neq 0, y_{j+t} = 0 = y_{j+1}$.

   (v) $y_j \neq 0, y_{j-1} = y_{j+1} = 0$.

   (vi) $y_{j+s} \neq 0, y_{j+s+t} \neq 0$ *for $s = 0, 1$ and 2.*

Proof. (I) Suppose (i) occurs; then (2.4) holds for $r = j, j+t$ and $j+2t$ and on adding these three equations we have $3k = 3 + \sum_{j=1}^{3t} y_j = 4$ so, using (2.6) and Theorem 2.1,

$$\frac{4}{3} = k = \frac{2}{\lambda(2t, 3t)} \geqslant \frac{2(3t+4)}{4(t+2)} > \frac{4}{3}$$

since $t > 4$ and we have a contradiction.

(II) Suppose (ii) occurs then, since $y_j \neq 0$ for some j, by (i) we may assume $y_{t+1} = 0 = y_{2+st}$ $(s = 0, 1, 2)$, $y_1 \neq 0$. However, then $f_{2t}(\frac{1}{2}y_1, \frac{1}{2}y_1, y_3, y_4, \ldots, y_{3t}) > f_{2t}(y_1, y_2, \ldots, y_{3t})$ which is impossible since $y_1, \ldots, y_{3t}$ is a maximal sequence.

(III) Suppose (iii) occurs then we may suppose $y_1 = 0 = y_{2t+2}$, $y_2 \neq 0$. Further we may assume $y_{t+1} = 0$, for otherwise $f_{2t}(y_2, y_1, y_3, y_4, \cdots \ldots, y_{3t}) > f_{2t}(y_1, \ldots, y_{3t})$. However, if $z_1, \ldots, z_{3t}$ is the sequence $y_1, \ldots, y_{3t}$ with y_{2t+1} and y_{2t+2} interchanged, clearly $f_{2t}(z_1, \ldots, z_{3t}) \geqslant f_{2t}(y_1, \ldots, y_{3t})$ so $z_1, \ldots, z_{3t}$ is a maximal sequence. Since $z_{1+st} = 0$ for $s = 0, 1, 2$ this contradicts (ii).

(IV) Using (iii) on the dual sequence (iv) clearly cannot occur.

(V) Suppose (v) occurs then by (iii) and (iv) $y_{j+2t} \neq 0$ and $y_{j+t} \neq 0$. Since $y_j \neq 0$ this contradicts (i).

(VI) Suppose (vi) occurs then by (i) $y_{j+s+2t} = 0$ $(s = 0, 1, 2)$. Subtracting the equations obtained by putting $r = j+1$ and $j+2$ in (2.4) we have $y_{j+t+1} - y_{j+1} = -y_{j+2}$, i.e. $y_{j+1} > y_{j+t+1}$. However, subtracting the equations obtained by putting $r = j+t$ and $j+t+1$ in (2.4) we have $-y_{j+t} = -y_{j+t+1} + y_{j+1}$, i.e. $y_{j+1} < y_{j+t+1}$ which contradicts the above.

LEMMA 4.2. *Let $y_1, \ldots, y_{3t}$ be a maximal sequence of $\lambda(2t, 3t)$:*

   (i)   *if $y_j \neq 0$, $y_{j+1} \neq 0$, $y_{t+j} = 0 = y_{2t+j+1}$, then $y_j = y_{j+1}$,*

   (ii)   *if $y_j \neq 0$, $y_{j+t} \neq 0$, then $y_{j+2t} = 0$, $y_{j+t+1} \neq 0$, $y_{j-1} \neq 0$,*

   (iii)   *if $y_{j+st} \neq 0$, $y_{j+st+1} \neq 0$ for $s = 0$ and 1, then $y_j = y_{j+t+1} = y_{j+1} + y_{j+t}$.*

Proof.

(I) Suppose the conditions of (i) are satisfied then subtracting the equations obtained by putting $r = j$ and $j+1$ in (2.4) we have $y_j = y_{j+1}$ since $y_{t+j} = 0 = y_{2t+j+1}$.

(II) Suppose $y_s \neq 0$, $s = j, j+t$, then by Lemma 4.1 (i) $y_{j+2t} = 0$. From Lemma 4.1 (iv) $y_{j+t+1} \neq 0$ and from Lemma 4.1 (iii) $y_{j-1} \neq 0$.

(III) Suppose the conditions of (iii) are satisfied, then by Lemma 4.1 (i) $y_{j+s} = 0$ $(s = 2t, 2t+1)$. Subtracting the equations obtained by putting $r = j$ and $j+1$ in (2.4) we have $y_{j+1} = y_j - y_{j+t}$. Subtracting the equa-

tions obtained by putting $r = j+t$ and $j+t+1$ in (2.4), $y_{j+t+1} = y_{j+t} + y_{j+1}$ and the proof is complete.

LEMMA 4.3. *If* $y_1, \ldots, y_{3t}$ *is a maximal sequence of* $\lambda(2t, 3t)$ *then, for each* r, *there is at most one value of* ω *in* $r \leqslant \omega \leqslant r+t-1$ *such that* $y_\omega \neq 0$, $y_{\omega+1} = 0$.

Proof. Suppose that for some r there are at least two such values of ω. We may clearly assume that $\omega = r = 1$ is one such value and that another is $\omega = a$ so $3 \leqslant a \leqslant t$ and $y_2 = 0 = y_{a+1}$. By Lemma 4.1 (iv) $y_{t+1} \neq 0$, $y_{t+a} \neq 0$ so by Lemma 4.2 (ii) $y_{2t+1} = 0 = y_{2t+a}$, $y_{t+2} \neq 0$, $y_{t+a+1} \neq 0$, $y_{a-1} \neq 0$. Since $y_2 = 0$ there is a β with $2 < \beta \leqslant a-1$ such that $y_\beta \neq 0$, $y_{\beta-1} = 0$. By Lemma 4.1 (iii) $y_{\beta+2t} \neq 0$ so by Lemma 4.1 (i) $y_{\beta+t} = 0$. Hence there is a γ with $\beta+2t \leqslant \gamma < a+2t$ such that $y_\gamma \neq 0$, $y_{\gamma+1} = 0$ and further a λ with $t+2 \leqslant \lambda < \beta+t$ such that $y_\lambda \neq 0$, $y_{\lambda+1} = 0$. Also since $y_{a+t+1} \neq 0$ and $y_{2t+1} = 0$ there is a μ with $t+a+1 \leqslant \mu \leqslant 2t$ such that $y_\mu \neq 0$, $y_{\mu+1} = 0$. Clearly $1 < \beta < a < \lambda < \mu < \gamma < 3t$.

Let Ω be the set of numbers $1, a, \gamma, \lambda, \mu$, then for $j \epsilon \Omega$ $y_j \neq 0$, $y_{j+1} = 0$ so by Lemma 4.1 (iv) $y_{j+t} \neq 0$. Hence (2.4) holds for $r = j$ and $j+t$ so on addition, $2k = 2 + \sum\limits_{i=j+t+1}^{j+3t-1} y_i$.

Thus

$$10k = 10 + \sum_{j\epsilon\Omega} \sum_{i=j+t+1}^{j+3t-1} y_i = 15 - \sum_{j\epsilon\Omega} \sum_{i=j+1}^{j+t} y_i .$$

Since $1 < a < t+1 < \lambda < a+t < \mu \leqslant 2t < \lambda+t$ and $2t+2 < \gamma < a+2t < \mu+t < 3t+1 < \gamma+t$, we have

$$\sum_{j\epsilon\Omega} \sum_{i=j}^{j+t} y_i \geqslant \sum_{i=1}^{3t} y_i + \sum_{j\epsilon\Omega} (y_j + y_{j+t}) = 1 + \sum_{j\epsilon\Omega} (y_j + y_{j+t}).$$

Hence

(4.1) $$10k \leqslant 14 - \sum_{j\epsilon\Omega} (y_j + y_{j+t}).$$

Hence, using (2.6) and Theorem 2.1,

(4.2) $$0 < \sum_{j\epsilon\Omega} (y_j + y_{j+t}) \leqslant 14 - 10k \leqslant 14 - \left\{ 10 + \frac{10t}{2(t+2)} \right\} = \frac{8-t}{t+2}$$

so unless $t \leqslant 7$ we have a contradiction.

By the construction if $j_1 \epsilon \Omega$, $j_2 \epsilon \Omega$ and $j_1 \neq j_2$ then $j_1 - j_2 \not\equiv 0 \pmod{t}$ so when $t = 5$, $\sum\limits_{j\epsilon\Omega} (y_j + y_{j+t}) = 1$ since $y_{j+2t} = 0$ by Lemma 4.1(i). Thus (4.1) gives

$$\frac{13}{10} \geqslant k \geqslant \frac{3t+4}{2(t+2)}$$

which is impossible.

Hence we may suppose $t = 6$ or 7. Let i be such that none of i, $i+t$ and $i+2t$ belong to Ω. From Lemma 4.1 (i) and (ii) only two effective cases arise:

(I) Suppose $y_i \neq 0$ and $y_{i+t} \neq 0$, then on putting $r = i$ and $i+t$ in (2.4) and adding we obtain

$$2 + \sum_{j=i+t+1}^{i+3t-1} y_j = 2k \geqslant 2 + \frac{t}{t+2} .$$

Thus

$$1 \geqslant \sum_{j=i+t}^{i+3t} y_j \geqslant \frac{t}{t+2} + y_i + y_{i+t} .$$

i.e.

(4.3) $$y_i + y_{i+t} \leqslant \frac{2}{t+2} < \frac{t+4}{4(t+2)}$$

since $t = 6$ or 7.

(II) Suppose $y_i \neq 0$, $y_{i+t} = 0 = y_{i+2t}$, then by (2.4)

$$1 - y_i + y_{i+t} + \ldots + y_{i+2t} = k \geqslant 1 + \frac{t}{2(t+2)} .$$

Thus

$$1 \geqslant y_i + \sum_{j=i+t}^{i+2t} y_j \geqslant \frac{t}{2(t+2)} + 2y_i ,$$

i.e.

(4.4) $$y_i \leqslant \frac{t+4}{4(t+2)} .$$

Since $t = 6$ or 7 from the above there can be at most two such i and so from (4.3) and (4.4)

(4.5) $$\sum_{i(\bmod t)\notin\Omega} y_i \leqslant \frac{t+4}{2(t+2)} .$$

However, using (4.2)

$$\sum_{i(\bmod t)\notin\Omega} y_i = 1 - \sum_{j\epsilon\Omega} (y_j + y_{j+t}) \geqslant 1 - \frac{8-t}{t+2} = \frac{4t-12}{2(t+2)} > \frac{t+4}{2(t+2)} .$$

This contradicts (4.5) and the lemma is proved.

THEOREM 4.1. *For a fixed integer* $t > 4$ *and* $m \geqslant 2t$, $\lambda(m, 2m-t) = 4(t+2)/(3t+4)$.

Proof. Let $y_1, \ldots, y_{3t}$ be a maximal sequence of $\lambda(2t, 3t)$, then without loss of generality we may assume $y_1 \neq 0$, $y_2 = 0$ so by Lemma 4.1 (iv)

$y_{t+1} \neq 0$ and by Lemma 4.2 (ii) $y_{2t+1} = 0$, $y_{t+2} \neq 0$, $y_{3t} \neq 0$. Thus on using Lemma 4.3 there is exactly one λ with $t+2 \leqslant \lambda \leqslant 2t$ such that $y_\lambda \neq 0$, $y_{\lambda+1} = 0$ so $y_r \neq 0$ for $r = t+2, \ldots, \lambda$ and $y_r = 0$ for $r = \lambda+1, \ldots$ $\ldots, 2t+1$. Further by Lemma 4.1 (iv) $y_{\lambda+t} \neq 0$ so by Lemma 4.3 $y_r \neq 0$ for $r = \lambda+t, \ldots, 3t+1$. Thus $y_\omega = 0$ for $\omega = t-1$ and $\lambda+t-2$ in either case if $y_\omega \neq 0$ then by Lemma 4.3 $y_{\omega+1} \neq 0$ and then we have a contradiction to Lemma 4.1 (vi). Hence $y_r = 0$ for $r = 2, \ldots, t-1$ and $r = 2t+1, \ldots, \lambda+t-2$. Hence by Lemma 4.2 (i) $y_r = y_{t+1} = a$, say, for $r = t+1, \ldots, \lambda-1$ and $y_s = y_{3t} = b$, say, for $s = \lambda+t, \ldots, 3t$.

Two cases now arise:

(I) Suppose $y_t \neq 0$ then by Lemma 4.2 (iii) $b = y_{3t} = y_1 + y_t = y_{t+1} = a$. Further $y_t + y_{\lambda+t-1} = a$ by Lemma 4.2 (iii) if $y_{\lambda+t-1} \neq 0$ and by Lemma 4.2 (i) if $y_{\lambda+t-1} = 0$. Thus

$$1 = \sum_{i=1}^{3t} y_i = (t+2)a \quad \text{so} \quad a = \frac{1}{t+2}.$$

Hence on putting $r = 3t$ and $\lambda-1$ in (2.4) we have

$$1-a+y_t+(\lambda-t-1)a+y_\lambda = k,$$
$$1-a+y_{\lambda+t-1}+(2t-\lambda+1)a+y_1 = k.$$

Thus $2k = 2 + ta = (3t+4)/(t+2)$ so by (2.6) $\lambda(2t, 3t) = 4(t+2)/(3t+4)$.

(II) By cyclic symmetry the case $y_{\lambda+t-1} \neq 0$ is covered by (I) so we may now assume $y_t = 0 = y_{\lambda+t-1}$. By Lemma 4.2 (i) $y_\lambda = a$, $y_1 = b$ so letting $v = \lambda-t$ and $\omega = 2t-\lambda+2$ we have $v+\omega = t+2$ and, since $\sum_{i=1}^{3t} y_i = 1$, $va + \omega b = 1$. By calculation

$$f_{2t}(y_1, \ldots, y_{3t}) = \frac{1}{2} v(v-1)a^2 + 2abv\omega + \frac{1}{2} \omega(\omega-1)b^2$$

$$= \frac{1}{2}(va+\omega b)^2 - \frac{1}{2} va^2 - \frac{1}{2} \omega b^2 + abv\omega$$

$$= \frac{1}{2} - \frac{1}{2(v+\omega)} \{v^2 a^2 + \omega v a^2 + \omega^2 b^2 + \omega v b^2\} + abv\omega$$

$$= \frac{1}{2} - \frac{1}{2(t+2)} \{(va+\omega b)^2 + v\omega(a-b)^2\} + \frac{1}{4}\{(va+\omega b)^2 - (va-\omega b)^2\}$$

$$\leqslant \frac{1}{2} - \frac{1}{2(t+2)} + \frac{1}{4} = \frac{3t+4}{4(t+2)}$$

with equality if $a = b$ and $v = \omega$. Thus $\lambda(2t, 3t) = 4(t+2)/(3t+4)$.

Hence in both cases $\lambda(2t, 3t) = 4(t+2)/(3t+4)$ and so the theorem now follows by Theorems 2.1 and 3.2.

5. By constructing sequences we now obtain the upper bound for $\lambda(m, n)$ given by (1.4) and show that this is a better bound than n/m in the range considered except for $n = 9$, $m = 6$.

For given n and m with $m+2 < n < 2m$ let

(i) r be the integer such that $\dfrac{r+2}{r+1} \leqslant \dfrac{n}{m} < \dfrac{r+1}{r}$,

(ii) s and θ integers such that $0 \leqslant s < r$ and $m = r\theta+s$,

(iii) $t = (r+1)\theta - n$.

(A) $\dfrac{(r+1)\theta-t}{r\theta+s} < \dfrac{r+1}{r}$ if and only if $-s < r(s+t)$, i.e. since

(5.1) $$0 \leqslant s < r, \quad s+t \geqslant 0.$$

(B) $\dfrac{(r+1)\theta-t}{r\theta+s} \geqslant \dfrac{r+2}{r+1}$ if and only if

(5.2) $$\theta \geqslant (r+1)(s+t)+s.$$

(I) Suppose $t \neq 0$ and such that $(r+1)$ and t have no common factor. Let N be the integer such that $0 \leqslant T \leqslant r$ where

(5.3) $$T = -t+N(r+1).$$

Further let

(5.4) $$a = s+t-N+1$$

and

(5.5) $$\beta = \theta-s-t-2.$$

Now $a \geqslant 1$ from (5.1) if $N \leqslant 0$ and from (ii) and $t-N = Nr-T \geqslant 0$ for $N \geqslant 1$. Further $\beta \geqslant 1$ since $n \geqslant m+3$.

For $\omega = 0, 1, \ldots, r$ let a_ω be such that $0 \leqslant a_\omega \leqslant r$ and $\omega T = a_\omega [\bmod (r+1)]$ then the a_ω form a complete set of residues $\bmod (r+1)$. Let

$$c = \begin{cases} 1 & \text{if} \quad a_\omega > T, \\ 2 & \text{if} \quad a_\omega \leqslant T \end{cases}$$

and S_ω be the sequence $s_{\omega 1}, \ldots, s_{\omega a+c}$ where $s_{\omega 1} = a_\omega s_{\omega a+c} = a_{r-\omega+2}$ and $s_{\omega j} = r+1$, $j = 2, \ldots, a+c-1$. Hence $\sum_{j=1}^{a+c} s_{\omega j} = a(r+1)+T$. Finally let $W = (r+1)\{a(r+1)+T\}$ and V_j denote the sequence comprising of j zeros. Since the sequence defined by $S_r, V_\beta, S_{r-1}, V_\beta, S_{r-2}, \ldots, V_\beta, S_1, V_{\beta-1}, S_0, V_\beta$ has n terms by (5.4) and (5.5), denote this sequence with each term divided by W by $y_1, \ldots, y_n$. Hence

$$\sum_{i=1}^{n} y_i = 1$$

and

$$W^2 f_m(y_1, \ldots, y_n)$$

$$\geqslant \sum_{\omega=0}^{r} \left\{ a_\omega [W - (r+1)] + \sum_{j=1}^{a+c-2} (r+1) [W - a_\omega - j(r+1)] + a_{\omega+r-2} \left[W - \frac{W}{r+1} \right] \right\}$$

$$= (W - r - 1) \frac{r(r+1)}{2} + (r+1) \sum_{j=r+1}^{a(r+1)+T} (W - j) + \frac{r}{r+1} W \frac{r(r+1)}{2}$$

$$= \frac{W}{2(r+1)} \{ (2r+1)W - (r+1)^2 \}.$$

From (5.3) and (5.4) $W = \{(r+1)(s+1) + rt\}(r+1)$ so

$$f_m(y_1, \ldots, y_n) \geqslant \frac{(2r+1)[rt + (r+1)s] + 2r(r+1)}{2(r+1)[rt + (r+1)(s+1)]}.$$

(II) Suppose $t \neq 0$ and such that $(r+1)$ and t have a common factor $p \geqslant 2$ not divisible by $(r+1)$, say $r+1 = p(u+1)$ and $t = p\omega$ where $(u+1)$ and ω have no common factor; clearly $u \neq 0$. Now $s + \omega(p-1) \geqslant 0$ is trivial if $\omega \geqslant 0$ and follows from (5.1) since $\omega p = t$ if $\omega < 0$. Thus let P and Q be the integers such that

$$(5.6) \qquad s + \omega(p-1) = uQ + P$$

where $Q \geqslant 0$ and $0 \leqslant P < u$, $\varphi = \theta + Q$, $h = (u+1)\varphi - (u+1)Q - \omega$, $g = u\varphi + P$. Now $P + (u+1)Q + \omega = s + t + Q \geqslant 0$ from (5.1) so from (A) $h/g < (u+1)/u$. Using (5.6)

$$W = (u+1)[P + (u+1)Q + \omega] + P = (u+1)(s+t+Q) + P$$

$$= (u+1)(s+t) + s + t - \omega + Q.$$

Thus from (5.2):

if $\omega \geqslant 0$, $W \leqslant (u+2)(s+t) + Q \leqslant (r+1)(s+t) + Q \leqslant \theta + Q = \varphi$,

if $\omega < 0$, $s + t - \omega < s$ so $W < (u+1)(s+t) + s + Q \leqslant (r+1)(s+t) + s + Q \leqslant \theta + Q = \varphi$.

Hence from (B) $h/g \geqslant (u+2)/(u+1)$.

Further $h - g = \varphi - (u+1)Q - P - \omega = \theta + Q - Q - s - t + \omega - \omega = \theta - s - t > 2$ since $n > m + 2$.

Thus by (I) there is a sequence $y_1, \ldots, y_h$ such that $\sum_{i=1}^{h} y_i = 1$ and

$$f_g(y_1, \ldots, y_h) \geqslant \frac{(2u+1)\{u[(u+1)Q+\omega]+(u+1)P\}+2u(u+1)}{2(u+1)\{u[(u+1)Q+\omega]+(u+1)(P+1)\}}$$

$$= 1 - \frac{rt+(r+1)(s+2)}{2(r+1)\{rt+(r+1)(s+1)\}} p.$$

Hence for the sequence $z_1, \ldots, z_n$ where $z_{i+sh} = \frac{1}{p} y_i$ for $s = 0, 1, \ldots, p-1$ $(i = 1, \ldots, h)$ we have

$$f_m(z_1, \ldots, z_n) = \frac{p-1}{p} + \frac{1}{p} f_g(y_1, \ldots, y_h) \geqslant \frac{(2r+1)\{rt+(r+1)s\}+2r(r+1)}{2(r+1)\{rt+(r+1)(s+1)\}}.$$

(III) Suppose t is a multiple of $(r+1)$, say $t = W(r+1)$ where possibly $W = 0$; consider the sequence $y_1, \ldots, y_n$ defined by $y_{i+\lambda(\theta-W)}$ $= \frac{1}{(r+1)(s+1+rW)}$ $(\lambda = 0, 1, \ldots, r, i = 1, 2, \ldots, s+1+rW)$, $y_i = 0$ otherwise; this is possible since $s+1+rW < \theta - W - 1$ since $n > m+2$.

Hence

$$f_m(y_1, \ldots, y_n) \geqslant \frac{r}{r+1} + \left\{ \frac{1}{(r+1)(s+1+rW)} \right\}^2 \frac{(s+1+rW)(s+rW)(r+1)}{2}$$

$$= \frac{(2r+1)\{rt+(r+1)s\}+2r(r+1)}{2(r+1)\{rt+(r+1)(s+1)\}}$$

on substituting for W and simplifying.

Hence from (2.2) we have in each case,

$$\lambda(m, n) \leqslant \frac{2(r+1)\{rt+(r+1)(s+1)\}}{(2r+1)\{rt+(r+1)s\}+2r(r+1)}$$

$$= \frac{2(r+1)\{(r+1)(m+1)-rn\}}{(2r+1)\{(r+1)m-rn\}+2r(r+1)}$$

since $(r+1)m - rn = (r+1)s + rt$.

This is a better bound than $\frac{n}{m}$ for $\lambda(m, n)$ unless

$$\frac{2(r+1)\{rt+(r+1)(s+1)\}}{(2r+1)\{rt+(r+1)s\}+2r(r+1)} > \frac{(r+1)\theta - t}{r\theta + s},$$

i. e. unless

$$(5.7) \qquad \theta < 2(s+t) + 2 - \frac{t}{r+1}.$$

Now putting $s + t = k \geqslant 0$ from (5.1), from (5.2) $\theta \geqslant (r+1)k + s$ and since $n \geqslant m+3$, $\theta \geqslant k+3$. If (5.7) holds we must therefore have

$$(5.8) \qquad 2k+2 - \frac{k-s}{r+1} > k+3, \quad \text{i.e.} \quad rk > r+1-s$$

and

$$(5.9) \qquad 2k+2 - \frac{k-s}{r+1} > (r+1)k+s, \quad \text{i.e.} \quad 2(r+1) > r^2 k + rs.$$

Since $0 \leqslant s < r$, from (5.8) $k \geqslant 1$ and $k = 1$ only if $s \geqslant 2$; in this case however $r \geqslant 3$ and then (5.9) is not satisfied. Hence $k \geqslant 2$ and so from (5.9) $2r^2 < 2(r+1)$, i.e. $r \leqslant 1$. Thus $r = 1$ and $s = 0$. From (5.8) $k \geqslant 3$ and from (5.9) $k < 4$ so $k = 3$. Now when $r = 1$, $s = 0$, $t = 3$ we have $(r+1)(s+t)+s+1 > 2(s+t)+2-\dfrac{t}{r+1}$ so the only case for which $\dfrac{n}{m}$ is a better bound in the range $m+2 < n < 2m$ is $r = 1$, $s = 0$, $t = 3$, $\theta = 6$, i.e. $n = 9$, $m = 6$.

References

[1] L. J. Mordell, *On the inequality* $\displaystyle\sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}} \geqslant \frac{n}{2}$ *and some others*, Abh. Math. Sem. Univ. Hamburg. 22 (1958), pp. 229-241.

[2] P. H. Diananda, *On a conjecture of L. J. Mordell regarding an inequality involving quadratic forms*, Journ. London Math. Soc. 36 (1961), pp. 185-192.

Further developments in the comparative prime-number theory IV

(Accumulation theorems for residue-classes representing quadratic non-residues $\bmod k$)

by

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1. In the second and third papers of this series we introduced a new approach instead of that of Chebyshev, in order to find a sense in which there are more primes $\equiv l_1 \bmod k$ than $\equiv l_2 \bmod k$ if and only if l_1 is a quadratic non-residue, l_2 quadratic residue $\bmod k$. We succeeded in obtaining results in this direction when the Haselgrove-condition is satisfied for k, i.e. when there is an $E = E(k) > 0$ such that no $L(s, \chi)$ belonging to the modulus k vanishes for[1]

$$(1.1) \qquad \sigma \geqslant \tfrac{1}{2}, \qquad |t| \leqslant E(k) \qquad (s = \sigma + it).$$

For the sake of brevity we shall call such k-values "good" k-values. We made a comparison in the second paper for the residue-classes

$$\equiv 1 \bmod k \qquad \text{and} \qquad \equiv l \bmod k$$

(l quadratic non-residue $\bmod k$) in the third one for the residue-classes

$$\equiv 1 \bmod k \qquad \text{and} \qquad \equiv l \bmod k$$

(l quadratic residue $\bmod k$).

In this paper we shall pass to the more general case, when we compare the residue-classes

$$(1.2) \qquad \equiv l_1 \bmod k \qquad \text{and} \qquad \equiv l_2 \bmod k$$

(l_1, l_2 both quadratic non-residues).

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[1] Though no k-value is known for which this would be false, it is desirable to prove its truth at least for an infinity of k-values.