Connectivity functions and retracts

by

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1. Introduction. In studying the fixed point property and other aspects of topological spaces and mappings it has often been found useful to consider a category of functions more extensive than the class of continuous functions (see, for example, [2]-[8], especially [7]). In view of this, it seems desirable to unify some of the known results and attempt to clarify the nature of and relationships between such functions in order to obtain further useful results. This paper represents such an attempt with regard to connected and connectivity functions in particular.

1.1. Definition. The *graph function* $f_\gamma\colon S \to S\times T$ of a function $f\colon S\to T$ is defined by $f_\gamma(s) = \big(s, f(s)\big)$ for $s \in S$.

Since a function with values in a product space is continuous iff its composition with each projection mapping is continuous, one easily obtains the following well-known result:

1.2. Lemma. *f is continuous iff f_γ is continuous (and both are equivalent to f_γ being a homeomorphism).*

It is also well known that if a function is continuous, the image of each connected set is connected but that the converse is false even for real-valued functions of a real variable (e.g. every derivative function has the "Darboux Property" or "Intermediate Value Property", hence preserves connected sets, but of course need not be continuous). This observation yields one category of functions referred to in the opening paragraph, and from it a rather natural modification of the statement of 1.2 yields yet another.

1.3. Definition. A function $f\colon S\to T$ is a *connected function* iff $f(C)$ is connected for each connected subset C of S. Also, f is a *connectivity function* iff f_γ is a connected function. Furthermore, in analogy to the C^n classification of functions (C^0 being the continuous functions) we let C^{-1} denote the class of connectivity functions and C^{-2} the class of connected functions.

The class a function belongs to depends, of course, on the topology of its domain and of its range and these will be specified unless it is clear from the context or is immaterial, as in 2.1 below which is true for any (fixed) domain and range space.

2. Characterizations and comparisons. From 1.2 it is clear that continuity implies connectivity but the converse is easily disproved, e.g. set

$$F(0) = 0 \quad \text{and} \quad F(x) = \sin\frac{1}{x}$$

for $x \neq 0$. Given $f\colon S \to T$, if π denotes the projection mapping of $S \times T$ onto T, then $f = \pi f_\gamma$ so that $f \in C^{-1}$ implies $f \in C^{-2}$ since the composition of connected functions is clearly connected. On the other hand, one can readily obtain a function of $[0, 1]$ onto itself which takes on every value in every interval yet has no fixed point (see [5]). Such a function is in C^{-2} but not in C^{-1} since it preserves all connected sets but its graph function preserves no non-trivial connected sets. The next theorem merely summarizes the foregoing observations.

2.1. Theorem. *$C^{-2} \supset C^{-1} \supset C^{0}$ and each inclusion is proper.*

In contrast to the example preceding 2.1, Kuratowski and Sierpiński [4] have shown that a real-valued function of a real variable of Baire class one is a connected function iff it is a connectivity function.

Note. If σ and τ are collections of sets and f is a function, $f^{-1}(\tau)$ will denote the collection of sets of the form $f^{-1}(V)$ where $V \in \tau$, and $\sigma \times \tau$ the collection of sets of the form $U \times V$ where $U \in \sigma$ and $V \in \tau$. Also, if σ is a subbasis for a topology on a set S, the resulting space will be denoted by (S, σ) or simply by S if no confusion is likely. If P is a topological property, "A is σ-P" will mean the object A has property P in the space (S, σ). Thus U is σ-open iff U is the union of sets which are intersections of finite subcollections of σ.

2.2. Definition. The *connectivity structure* of (S, σ), denoted by $C(\sigma)$, is the class of all σ-connected subsets of S.

The next theorem gives a strong motivation for the name given to the functions of class C^{-1} and relates this class more closely to the class C^{0}. It implies that connectivity functions are those whose inverses preserve, in a certain sense, the connectivity structure of their domain and that one can alter the topology of the domain without affecting its connectivity structure so as to make a connectivity function continuous. We first prove a lemma for functions in general.

2.3. Lemma. *Given any function $f\colon (S, \sigma) \to (T, \tau)$, if $\sigma' = \sigma \cup f^{-1}(\tau)$ and $K = A \cup B$ is a σ'-separation of a set $K \subset S$, then $f_\gamma(K) = f_\gamma(A) \cup f_\gamma(B)$ is a $(\sigma \times \tau)$-separation. That is, $K \notin C(\sigma')$ implies $f_\gamma(K) \notin C(\sigma \times \tau)$.*

Proof. If $K = A \cup B$ is a σ'-separation, there are σ-open sets V_α, V_β and τ-open sets W_α, W_β for $\alpha \in A$, $\beta \in B$ such that

$$A \subset U = \bigcup_{\alpha \in A} \left(f^{-1}(W_\alpha) \cap V_\alpha\right), \qquad B \subset U' = \bigcup_{\beta \in B} \left(f^{-1}(W_\beta) \cap V_\beta\right)$$

where $K \cap U \cap U' = \emptyset$. Thus the $(\sigma \times \tau)$-open sets

$$\bigcup_{\alpha \in A} V_\alpha \times W_\alpha \quad \text{and} \quad \bigcup_{\beta \in B} V_\beta \times W_\beta$$

contain $f_\gamma(A)$ and $f_\gamma(B)$, respectively, and separate $f_\gamma(K)$. This proves the lemma.

2.4. Theorem. *A function $f\colon (S, \sigma) \to (T, \tau)$ is a connectivity function iff $C(\sigma) = C(\sigma')$ where $\sigma' = \sigma \cup f^{-1}(\tau)$.*

Proof. Suppose $C(\sigma) = C(\sigma')$. Then $K \in C(\sigma)$ implies $K \in C(\sigma')$ and since $f\colon (S, \sigma') \to (T, \tau)$ is continuous,

$$f_\gamma(K) \in C(\sigma' \times \tau) \subset C(\sigma \times \tau)$$

and $f \in C^{-1}$. Conversely, suppose $f \in C^{-1}$. Since σ' yields a finer topology than does σ, it suffices to show $C(\sigma') \supset C(\sigma)$. Thus suppose $K \notin C(\sigma')$. By 2.3, $f_\gamma(K) \notin C(\sigma \times \tau)$ so $f \in C^{-1}$ implies $K \notin C(\sigma)$ and the proof is complete.

The following corollaries are easy consequences.

2.5. Corollary. *Let $i\colon (S, \sigma) \to (S, \sigma')$ be the identity and $\sigma' \supset \sigma$. Then $i \in C^{-2}$, $i \in C^{-1}$ and $C(\sigma) = C(\sigma')$ are three equivalent statements.*

Note. Actually one can replace $\sigma' \supset \sigma$ by the requirement that σ' generates a finer topology, i.e. that each $U \in \sigma$ be σ'-open.

2.6. Corollary. *$f\colon (S, \sigma) \to (T, \tau)$ is a connectivity function iff $i\colon (S, \sigma) \to (S, \sigma')$ is a connected function where $\sigma' = \sigma \cup f^{-1}(\tau)$. That is, $f \in C^{-1}$ iff $i \in C^{-2}$.*

Although the classes C^n for $n \neq -1$ are closed under composition (i.e., the composition of connected functions is connected, of continuous functions is continuous, etc.), the same is not true for $n = -1$. Examples are easily constructed (see [7]) of connectivity functions f and g for which $gf \notin C^{-1}$ even if $f \in C^{0}$. However, if $g \in C^{0}$, then $gf \in C^{-1}$ and in fact the next theorem shows this to be characteristic.

2.7. Theorem. *Given $f\colon (R, \varrho) \to S$ and $g\colon S \to (T, \tau)$, gf is a connectivity function iff $f\colon (R, \varrho) \to \left(S, g^{-1}(\tau)\right)$ is a connectivity function. Thus $gf \in C^{-1}$ iff S has a topology for which $f \in C^{-1}$ and $g \in C^{0}$.*

Proof. By 2.4, $gf\colon (R, \varrho) \to (T, \tau)$ is a connectivity function iff $C(\varrho) = C(\varrho')$ where $\varrho' = \varrho \cup (gf)^{-1}(\tau)$ and $f\colon (R, \varrho) \to \left(S, g^{-1}(\tau)\right)$ is a connectivity function iff $C(\varrho) = C(\varrho'')$ where $\varrho'' = \varrho \cup f^{-1}\left(g^{-1}(\tau)\right)$, but clearly $\varrho' = \varrho''$.

2.8. Corollary. *If $f\colon (R, \varrho) \to (S, \sigma)$ is a connectivity function and $g\colon (S, \sigma) \to (T, \sigma)$ is continuous, then $gf\colon (R, \varrho) \to (T, \tau)$ is a connectivity function.*

Proof. Since $g^{-1}(\tau)$ generates a topology for S coarser than does σ, the corollary follows immediately from 2.7.

A class of functions weaker than C^{-1} but not comparable to C^{-2} can be defined in a fashion analogous to the definition of C^{-1} and enjoys similar properties. It can be used to obtain alternate proofs of some of the above results for connectivity functions but will not be used here so we will only give the definition and cite some results. Only one of the proofs will be given since they are all simple or similar to previous ones.

2.9. DEFINITION. $f\colon S\to T$ is a *quasi-connectivity function* iff f_γ preserves components, that is, iff $f_\gamma(C)$ is a component of $f_\gamma(S)$ whenever C is a component of S. Let $K(\sigma)$ denote the class of all σ-components of S.

2.10. THEOREM. *A connectivity function is a quasi-connectivity function.*

2.11. THEOREM. *A function $f\colon (S,\sigma)\to(T,\tau)$ is a quasi-connectivity function iff $K(\sigma)=K(\sigma')$ where $\sigma'=\sigma\cup f^{-1}(\tau)$.*

2.12. THEOREM. *Given $f\colon (R,\varrho)\to S$ and $g\colon S\to(T,\tau)$, gf is a quasi-connectivity function iff $f\colon (R,\varrho)\to\big(S,g^{-1}(\tau)\big)$ is a quasi-connectivity function.*

Proof of 2.11. If $K(\sigma)=K(\sigma')$ and $A\in K(\sigma)$, then by 1.2, since $f\colon (S,\sigma')\to(T,\tau)$ is continuous, $f_\gamma(A)$ is a $(\sigma'\times\tau)$-component of $f_\gamma(S)$ and is then $(\sigma\times\tau)$-connected because $\sigma\subset\sigma'$. Then since $A\in K(\sigma)$, it follows that $f_\gamma(A)$ is in fact a $(\sigma\times\tau)$-component of $f_\gamma(S)$. Conversely if $A\in K(\sigma)$ implies $f_\gamma(A)$ is a $(\sigma\times\tau)$-component of $f_\gamma(S)$, then to show $K(\sigma)=K(\sigma')$ it suffices to show $K(\sigma)\subset C(\sigma')$. But by 2.3, if $A\notin C(\sigma')$, then $f_\gamma(A)\notin C(\sigma\times\tau)$, whence $f_\gamma(A)$ certainly is not a $(\sigma\times\tau)$-component of $f_\gamma(S)$ and $A\notin K(\sigma)$. This completes the proof.

2.13. Remark. One significant difference between connectivity and quasi-connectivity functions is the easily verified fact that any restriction of the former is again a connectivity function but a restriction of a quasi-connectivity function may not be quasi-connectivity (although it will be in linearly ordered spaces since the two classes coincide on such spaces).

3. Connectivity retraction. John Stallings has shown in [7] that connectivity functions can be very useful in proving fixed point theorems for continuous functions. It is the purpose of this section to make a study of the retraction of spaces by connectivity functions both for its intrinsic interest and in the hope of providing the means for discovering additional spaces having the fixed point property, particularly among non-separating plane continua. The hope for this is provided by results of this paper and of Stalling's which imply that any connectivity retract of an n-cell has the fixed point property.

Locally connected non-separating plane continua are known to have the fixed point property and it has long been known [1] that a (continuous) retract of a locally connected space is locally connected. This might dash some of the hopes expressed above except that the latter is not true for connectivity retracts (Example 3.15). After preliminary definitions, results paralleling the standard ones for retracts will be given.

3.1. DEFINITION. A connectivity function f of a space S onto $A\subset S$ which is the identity on A (i.e., $f(x)=x$ for $x\in A$) is a *connectivity retract function* (CRF) and is said to retract, or be a retraction of, S onto A. A is a *connectivity retract* (CR) of S iff such a CRF exists.

The following theorem and corollary are direct consequences of 2.7 and the above definition:

3.2. THEOREM. *If $S\supset A\supset B$ and $f\colon S\to A$, $g\colon A\to B$ extend the identity on A and B respectively, then $gf\colon S\to B$ is a CRF iff there is a topology for A with respect to which f is a connectivity function and g is continuous.*

3.3. COROLLARY. *A (continuous) retract of a connectivity retract is a connectivity rectract.*

A commonly used fact of (continuous) retracts is that a retract of a Hausdorff space is closed. That this is false for connectivity retracts of metric spaces even is easily shown. For example, it is clear from 1.1 or 2.4 that any function defined on a space (such as the subspace R of rationals in E^1) whose connected subsets are degenerate is a connectivity function. Thus any subset of R is a connectivity retract of R whether closed or not. This can be generalized as follows:

3.4. THEOREM. *If A is a non-null subset of S containing at most one point of any component of S not contained in A, then A is a connectivity retract of S.*

Proof. If $p\in A$, the function f which is the identity on A and maps each component of S onto its intersection with A (or onto p if the intersection is null) is either constant or the identity on each connected subset of S, hence the restriction thereto has a connected graph and, by Definition 1.1, f is a connectivity function, therefore a CRF.

On the other hand, the following shows connectivity retracts of Hausdorff spaces are closed if they are connected:

3.5. THEOREM. *A connectivity retract of a Hausdorff space has closed components.*

Proof. Let f be a CRF of a Hausdorff space S onto A. If C is a component of A, it is of course closed in A so suppose $p\in S-A$. Then p and $f(p)$ are distinct points and so have disjoint neighborhoods U and V. $U\times V$ contains $\big(p,f(p)\big)=f_\gamma(p)$ but no point of the form (q,q), hence no point of $f_\gamma(C)$. It follows that $f_\gamma(C\cup\{p\})$ is disconnected and (since $f\in C^{-1}$) that $C\cup\{p\}$ is disconnected. Thus p is not a limit point of C and C is closed in S.

3.6. COROLLARY. *A connectivity retract of a Hausdorff space is closed if it is connected or if its components form a discrete collection (or if the space itself has either of these properties).*

Proof. Since a collection of disjoint sets is discrete iff a limit point of its union is a limit point of exactly one of its members, 3.6 follows directly from 3.5 with the aid of the following observation:

3.7. Remark. If f is a CRF of S onto A and S' is a union of components of S, the fact that f is a connected function implies $f|S'$ is a CRF of S' onto $S'\cap A$ provided each component of S' meets A (moreover, the components of A are precisely the intersections of A with such components of S). Conversely, if f' is a CRF of S' onto A', f' extends to a CRF of S onto A' by simply mapping S-S' onto any given point of A' (cf. proof of 3.4).

An appropriate combination of the statements in 3.7 yields a result with no apparent counterpart for continuous retracts:

3.8. THEOREM. *If A is a connectivity retract of S, so is any union of components of A.*

3.9. COROLLARY. *If A is a connectivity retract of a Hausdorff space S and C is a closed subset of A, then $A-C$ is a connectivity retract of S iff C is a union of components of A.*

Proof. The "if" part follows from 3.8. If $A-C$ is a CR of S and K is a component of A, then K is closed by 3.5. Thus $K\cap C$ and (in view of 3.7) $K-C$ are both closed, hence one is empty and C is a union of components of A.

3.10. COROLLARY. *If A is a connectivity retract of a Hausdorff space S, then $A-\{p\}$ is a connectivity retract of S iff p is a limit point of no component of A.*

Proof. If $p\notin A$, this is a trivial consequence of 3.5, so assume $p\in A$. Since $\{p\}$ is closed, it is a component of A iff it is a limit point of no component of A, so 3.10 follows from 3.9.

Extending a common practice, if every continuous (respectively, connectivity) function of S into itself maps a point onto itself, we say that S has the fixed point property (connectivity fixed point property) and abbreviate it fpp (cfpp). If S has the cfpp it has the fpp, of course, but example 3.14 below shows the converse is false and also that, unlike retracts, a connectivity retract of a space with the fpp need not have the fpp. Some partial results along this line can be obtained, however.

3.11. LEMMA. *A is a connectivity retract of S iff every continuous function $f\colon A\to T$ (T an arbitrary space) extends to a connectivity function $F\colon S\to T$.*

Proof. Using 2.8, the proof is exactly analogous to the corresponding result for continuous retracts. If we set $T=A$ and observe that no point of S-A is fixed under F, we immediately arrive at the next result.

3.12. THEOREM. *If S has the connectivity fixed point property and A is a connectivity retract of S, then A has the fixed point property.*

The next theorem is a slight generalization of an observation in [7] attributed to Borsuk.

3.13. THEOREM. *Suppose that S has the fixed point property. If S is a (finite) polyhedron or has an order topology, then every connectivity retract of S has the fixed point property.*

Proof. The polyhedral case follows directly from 3.12 in view of Theorem 7 of [7] which implies that such a polyhedron has the cfpp. If S has an order topology and the fpp, then it satisfies the Dedekind Cut Property. Thus by 3.6 (and 2.1) every proper CR of S is a closed segment (initial, terminal or intermediate) of S and hence is clearly also a retract of S and has the fpp.

In our abstract, 603-110 of the American Mathematical Society Notices (p. 464 of vol. 10 (1964)), 3.13 was erroneously stated without restriction on the space, but the following example shows the necessity for limitation.

3.14. EXAMPLE. In the xy-plane let S' consist of the four sides and horizontal diagonal of the square with vertices $(0,\pm4)$, $(\pm4,0)$. Let S be the subspace of the plane obtained from S' by replacing the portion of S' between $(0,0)$ and $(1,0)$ by the corresponding portion of the graph of $y=\sin\pi/x$ and the portion of S' for which $|y|>3$ by the points $(0,\pm3)$ together with the set of points for which

$$|y-\sin\pi/x|=3\,,\qquad 0<|x|<1\,.$$

The arc-component, L, of S containing the origin, θ, is a topological triod minus two of its endpoints. Another arc-component of S is the subset, R, of those points of S for which $x>0$. R is topologically a triod minus all its endpoints. Let $f\colon S\to S$ be a continuous function. If $f(\theta)$ is either of the points $(0,\pm3)$ then $f(L)=f(\theta)$ and this point maps into itself. If T denotes either L or R and $f(\theta)\in T-\{\theta\}$, then $f(T)\subset T$. Denote the triod point of T by p and (if $p\neq f(p)$) the branch of T containing $f(p)$ by A. The set consisting of all points of A which f maps toward p (along A) or into $T-A$ and the set of points in A which f maps into A but away from p are both open and, unless one of the points $(0,\pm3)$ is a fixed point, both are non-empty. A, being connected, must then contain a fixed point of f. It follows that S has the fpp. Actually, the same type argument shows that the plane continuum $\bar{S}$ also has the fpp. S can be retracted onto $S-D$, where D is the "diagonal" of S (those points of S not in the closure of the unbounded component of $E^2-\bar{S}$), by vertical projection of D onto either the upper or lower half of $S-D$. This function is clearly continuous except at the origin and using 2.4 one can easily see that it is a connectivity function. $S-D$ is symmetric about the origin so does not have the fpp. Thus a CR of a space with fpp need not have fpp (even for plane continua since the same is true of $\bar{S}$). As a consequence of 3.12, we note that S (and $\bar{S}$) is an example of a space with the fpp but not having the cfpp.

QUESTIONS. A number of thoughts are suggested by the preceeding results: (1) If S is locally connected and has the fpp, does every CR of S have the fpp? (2) Does every CR of a space with the cfpp have the cfpp? (3) Is every CR of a disk locally connected? (4) Is every non-separating plane continuum a CR of a disk?

An affirmative answer to (4) would, as a result of 3.13, settle the long-standing question as to whether such continua have the fpp (this is pointed out in [7] in slightly different form). Even if the answer is negative, it would be interesting to know what continua are connectivity retracts of a disk, particularly if (3) has a negative answer (since the fpp is known to hold for locally connected non-separating plane continua). The next example suggests that the answer to (3) may indeed be negative.

3.15. EXAMPLE. Let K be the closure of the graph of

$$y = \sin \pi/x\,, \quad 0 < x \leqslant 1\,,$$

and S the union of K and all line segments

$$y = m/2^n\,, \quad 0 \leqslant x \leqslant 1/n$$

($n = 1, 2, \ldots$ and m an integer with $|m| \leqslant 2^n$). The locally connected continuum S can be retracted onto the non-locally connected continuum K by projecting each point $(x, y) \in S$ $(x > 0)$ vertically onto $(x, \sin \pi/x)$. Using 2.4 it is not difficult to show this to be a CRF. Thus, unlike retraction, connectivity retraction does not preserve local connectedness, even for plane continua.

3.16. THEOREM. *Every connectivity retract of an n-cell in E^n $(n \geqslant 2)$ is a non-separating subcontinuum of E^n.*

Proof. Let f be a CRF of an n-cell $D \subset E^n$ onto K. By 3.6, K is a continuum so it suffices to prove K is nonseparating. If K separates E^n there is an essential mapping g of K onto the unit sphere, $S \subset E^n$. Then by 2.8, gf is a connectivity function of D onto S and by Corollary 1 of [7], gf is almost continuous (i.e., every neighborhood of its graph contains the graph of a continuous function of D into S). Let U be the neighborhood of the graph of gf which is the union of $(D-K)\times S$ and the set of all points (p, q) where $p \in K$ and the distance from q to $gf(p) = g(p)$ is less than 2 (the diameter of S). Then U contains the graph of a mapping h of D into S. But then if $p \in K$, $g(p)$ and $h(p)$ are not antipodal points of S so $h|K$ is homotopic to g. This implies that $h|K$, and hence also h, is essential. No mapping of an n-cell is essential, so this contradiction completes the proof.

The proof of 3.16 raises another question: (5) Is every connectivity function of a non-separating subcontinuum of E^n into an $(n-1)$-sphere almost continuous? If so, then the above proof modifies to prove 3.16 with the n-cell replaced by an arbitrary non-separating continuum. Of particular interest is the case $n = 2$.

Added in proof. In a paper soon to appear in this journal, J. L. Cornette obtains an affirmative answer to question (3) and negative answers to (4) and (5)

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