Cartesian products and dyadic spaces

by

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The present paper is devoted to the investigation of some properties related to separability in Cartesian products, their subspaces and continuous images. In particular the minimal power of sets dense in products and in some of their subsets, and the maximal power of some families of pairwise disjoint non-empty subsets of products are investigated.

Some theorems are obtained which generalize the well-known Hewitt-Marczewski-Pondiczery Theorem (which asserts that the Cartesian product of $2^{\mathfrak{m}}$ topological spaces each of which contains a dense subset of power $\mathfrak{m}$, contains a dense subset of power $\mathfrak{m}$) and Marczewski's classical theorem (which claims that in the product of spaces with countable bases every family of pairwise disjoint non-empty open subsets is countable). The first of the two general theorems concerning continuous images of Cartesian products of compact spaces gives an evaluation of the weight (see footnote ([1]), p. 288) of such an image by means of the weight at the points of its dense subset and of the weight of the factors of the Cartesian product. The second one asserts that if the weight at a point of the image is equal to $\mathfrak{m} \geqslant \aleph_0$, then (under the assumption that the weight of the factors of the Cartesian product does not exceed $\mathfrak{n} < \mathfrak{m}$) the image contains the one-point compactification of the discrete space of power $\mathfrak{m}$. It is therefore a theorem on the structure of continuous images of Cartesian products of compact spaces.

The last paragraph of the paper concerns dyadic spaces, i.e. continuous images of a Cartesian product of a certain number of copies of the two-point discrete space. A few theorems about this class of spaces follow from the theorems on Cartesian products obtained earlier. It seems that the most important is the theorem which claims that if the weight at the point x of the dyadic space X is equal to $\mathfrak{m} \geqslant \aleph_0$, then X contains the discrete space of power $\mathfrak{m}$ which can be compactified by adjoining x_0. From this theorem a number of known theorems on dyadic spaces follow. At the end of the paper an example of a dyadic space is given which solves a question, raised, by P. R. Halmos, concerning a class of Boolean Algebras.

The Cartesian product of the family $\{X_s\}_{s\in S}$ of sets is denoted by $\underset{s\in S}{P} X_s$. By the *cube* in the product $\underset{s\in S}{P} X_s$ we mean its subset of the form $K = \underset{s\in S}{P} K_s$, where $K_s \subset X_s$ for $s \in S$. The set K_s will be called the *s-th face* of the cube K and the set $D(K) = \{s \in S\colon K_s \neq X_s\}$ will be called the set of its *distinguished indices*. If $\overline{\overline{D(K)}} \leqslant \mathfrak{m}$, then we shall say that K is an $\mathfrak{m}$-*cube*. If $\{X_s\}_{s\in S}$ is the family of topological spaces, then the symbol $\underset{s\in S}{P} X_s$ denotes the Cartesian product of this family with the Tychonoff topology. In this case the projection $p_{S_0}\colon \underset{s\in S}{P} X_s \to \underset{s\in S_0}{P} X_s$ is a continuous mapping for every $S_0 \subset S$.

A subset of a topological space X which is the intersection of $\mathfrak{m} \geqslant \aleph_0$ open sets is called a $G_\delta^{\mathfrak{m}}$-*set*, and the union of an arbitrary number of $G_\delta^{\mathfrak{m}}$-sets is called a $G_{\delta\Sigma}^{\mathfrak{m}}$-*set*. Instead of $G_\delta^{\aleph_0}$ and $G_{\delta\Sigma}^{\aleph_0}$ we shall write G_δ and $G_{\delta\Sigma}$. Every $G_\delta^{\mathfrak{m}}$-set in the Cartesian product $\underset{s\in S}{P} X_s$ of topological spaces is the union of $\mathfrak{m}$-cubes, and even of cubes of a special kind, namely of cubes whose faces are $G_\delta^{\mathfrak{m}}$-sets in the corresponding spaces (see Lemma 3 of [8]). Under the additional assumption that the points of the space X_s are $G_\delta^{\mathfrak{m}}$-sets for every $s \in S$ (for example, if X_s's are T_1-spaces with bases of power $\leqslant \mathfrak{m}$) the class of $G_{\delta\Sigma}^{\mathfrak{m}}$-sets coincides with the class of unions of $\mathfrak{m}$-cubes. This follows from the fact that an arbitrary $\mathfrak{m}$-cube is a union of $\mathfrak{m}$-cubes whose distinguished faces are one-point sets.

By I we denote the closed interval $[0,1]$, and by D the two-point discrete space. The Cartesian product of $\mathfrak{m}$ copies of I, i.e. the Tychonoff cube of weight $\mathfrak{m}$, is denoted by $I^{\mathfrak{m}}$. The symbol $D^{\mathfrak{m}}$ denotes the Cantor cube of weight $\mathfrak{m}$, i.e. the Cartesian product of $\mathfrak{m}$ copies of D. The Cantor cube $D^{\aleph_0}$ is the well-known Cantor perfect set in the real line.

1. Density character of Cartesian products and of some of their subsets. The well-known Hewitt-Marczewski-Pondiczery Theorem (see [14], [17] and [18]) asserts that the density character (1) of a Cartesian product $\underset{s\in S}{P} X_s$, where $d(X_s) \leqslant \mathfrak{m} \geqslant \aleph_0$ for $s \in S$ and $\overline{\overline{S}} \leqslant 2^{\mathfrak{m}}$, does not exceed $\mathfrak{m}$. It is easy to see that the density character of subsets of $\underset{s\in S}{P} X_s$ can be greater than $\mathfrak{m}$, even under the assumption that $w(X_s) \leqslant \mathfrak{m}$ for $s \in S$. For example, the Tychonoff cube $I^{2^{\aleph_0}}$ contains a discrete space of power $2^{\aleph_0}$. Theorem 1 below shows that under this stronger assumption the evaluation of density character remains true for a fairly large class of subspaces of $\underset{s\in S}{P} X_s$, in fact for all unions of $\mathfrak{m}$-cubes and, in particular, for all $G_{\delta\Sigma}^{\mathfrak{m}}$-sets in $\underset{s\in S}{P} X_s$. Let us note that this theorem is not true under the assumption that $d(X_s) \leqslant \mathfrak{m}$ for $s \in S$. To notice this it suffices to remark that the Tychonoff cube $I^{2^{\aleph_0}}$, which satisfies the condition $d(I^{2^{\aleph_0}}) \leqslant \aleph_0$, contains a subspace (which is obviously a $\aleph_0$-cube in the Cartesian product of the only factor $I^{2^{\aleph_0}}$) whose density character is equal to $2^{\aleph_0}$.

(1) By the *density character* of a topological space X we mean the least cardinal number which is the power of some dense subset of X. By the *weight* of the topological space X we mean the least cardinal number which is the power of some base of X. The density character of the space X is denoted by $d(X)$ and its weight by $w(X)$. If $d(X) = \aleph_0$, then we say that X is a *separable space*.

In the proof of the theorem we shall use a lemma whose content we shall now explain in a few words. Let us suppose that to every s in a set S a topological space X_s is assigned which has a base $\mathfrak{B}_s$ of power less than or equal to $\mathfrak{m} \geqslant \aleph_0$. For an arbitrary set X of power $\mathfrak{m}$ and for every $s \in S$ there exists a map f_s from X onto $\mathfrak{B}_s$. To every finite sequence $x_1, x_2, \ldots, x_n$ of elements of X and every sequence $S_1, S_2, \ldots, S_n$ of pairwise disjoint and non-empty subsets of S corresponds a family of open sets of $\underset{s\in S}{P} X_s$, namely the family $\mathfrak{R}$ of all sets of the form $\bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i))$, where $s_i \in S_i$ for $i = 1, 2, \ldots, n$. It turns out that every set $W \subset \underset{s\in S}{P} X_s$ which is the union of $\mathfrak{m}$-cubes contains a subset Q, of power not greater than $\mathfrak{m}$, which meets every member of $\mathfrak{R}$ which has a non-empty intersection with W. This in not quite obvious, since the power of the family $\mathfrak{R}$ is equal to $\overline{\overline{S}}_1 \cdot \overline{\overline{S}}_2 \cdot \ldots \cdot \overline{\overline{S}}_n$, and there are no assumptions regarding the power of the sets S_i. The following lemma contains a set-theoretical formulation of this fact.

LEMMA. *Let $\{X_s\}_{s\in S}$ be a family of sets and $\{f_s\}_{s\in S}$ a family of functions, where f_s maps a fixed set X onto some family of non-empty subsets of X_s, and let W be a subset of $\underset{s\in S}{P} X_s$ which is the union of $\mathfrak{m}$-cubes. For every sequence $(x_1, x_2, \ldots, x_n;\ S_1, S_2, \ldots, S_n)$, where $x_i \in X$, $0 \neq S_i \subset S$ for $i = 1, 2, \ldots, n$ and $S_i \cap S_j = 0$ for $i \neq j$, there exists a subset Q of W such that $\overline{\overline{Q}} \leqslant \mathfrak{m}$ and for every sequence $s_1, s_2, \ldots, s_n$, where $s_i \in S_i$ for $i = 1, 2, \ldots, n$,*

$$\text{if } W \cap \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) \neq 0\,, \text{ then } Q \cap \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) \neq 0\,.$$

Proof. We shall apply induction with respect to the number of members of the sequence $S_1, S_2, \ldots, S_n$ which contain more than one point; if this number is equal to zero, then the lemma is obvious.

Suppose that the lemma is true if the number of members of the sequence $S_1, S_2, \ldots, S_n$ which contain more than one point is less than k and consider the sequence $(x_1, x_2, \ldots, x_n;\ S_1, S_2, \ldots, S_n)$ whose k members contain at least two points. We can assume that the sets S_i for $i > k$ contain only one point, i.e. that $S_i = \{s_i\}$ for $i = k+1, k+2, \ldots, n$. It is enough to consider the case where $W \cap \bigcap_{i=k+1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) \neq 0$ [2], i.e. the case where there exists a point $q = \{q_s\} \in W$ such that $p_{s_i}(q) = q_{s_i} \in f_{s_i}(x_i) \subset X_{s_i}$ for $i = k+1, k+2, \ldots, n$. The point q is contained in an $\mathfrak{m}$-cube lying in W. Thus there exists a set $S' \subset S$ of power less than or equal to $\mathfrak{m}$ such that every point $\{x_s\} \in \mathop{P}_{s\in S} X_s$ with $x_s = q_s$ for $s \in S'$ is contained in W. In particular the point $q' = \{q'_s\}$, where $q'_s = q_s$ for $s \in S \setminus [\bigcup_{i=1}^{k} (S_i \setminus S')]$ and q'_s is an arbitrary point of the (non-empty) set $f_s(x_i)$ for $s \in S_i \setminus S'$, $i = 1, 2, \ldots, k$, is contained in W. For every sequence $s_1, s_2, \ldots, s_k$, where $s_i \in S_i \setminus S'$ for $i = 1, 2, \ldots, k$ we have

(1) $$q' \in \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) .$$

Let Σ denote the set of all sequences $(S'_1, S'_2, \ldots, S'_k)$ such that for some $j \leqslant k$ we have $S'_i = S_i$ for $i \neq j$ and S'_j is a one-point subset of $S_j \cap S'$. The power of Σ is not greater than $\mathfrak{m}$. For every $\sigma = (S'_1, S'_2, \ldots, S'_k) \in \Sigma$ the sequence $S'_1, S'_2, \ldots, S'_k, S_{k+1}, \ldots, S_n$ has only $k-1$ members which contain more than one point. Thus, by the inductive assumption, there exists a subset Q_σ of W such that $\overline{\overline{Q}}_\sigma \leqslant \mathfrak{m}$ and for every sequence $s_1, s_2, \ldots, s_n$, where $s_i \in S'_i$ for $i = 1, 2, \ldots, k$

(2) $$\text{if } W \cap \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) \neq 0 , \text{ then } Q_\sigma \cap \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}(x_i)) \neq 0 .$$

It is easy to verify, on the grounds of (1) and (2), that the set $Q = \{q'\} \cup \bigcup_{\sigma\in\Sigma} Q_\sigma$ satisfies the conditions of the lemma.

THEOREM 1. *Every subset W of the Cartesian product $\mathop{P}_{s\in S} X_s$, where $w(X_s) \leqslant \mathfrak{m} \geqslant \aleph_0$ for $s \in S$ and $\overline{\overline{S}} \leqslant 2^{\mathfrak{m}}$, which is the union of $\mathfrak{m}$-cubes contains a dense subset of power less than or equal to $\mathfrak{m}$.*

Proof. For every $s \in S$, let $\mathfrak{B}_s$ be a base of the space X_s such that $\overline{\overline{\mathfrak{B}}}_s \leqslant \mathfrak{m}$ and $0 \notin \mathfrak{B}_s$. Choose an arbitrary set X of power $\mathfrak{m}$ and a function f_s which transforms X onto $\mathfrak{B}_s$ for every $s \in S$. Denote by $\mathfrak{B}$ an arbitrary base of the Tychonoff cube $I^{\mathfrak{m}}$ which satisfies $\overline{\overline{\mathfrak{B}}} = \mathfrak{m}$. Since $\overline{\overline{I^{\mathfrak{m}}}} = 2^{\mathfrak{m}}$, we can suppose that $S = I^{\mathfrak{m}}$. Let Σ be a set of all sequences $(x_1, x_2, \ldots, x_n; S_1, S_2, \ldots, S_n)$, where $x_i \in X$, $0 \neq S_i \in \mathfrak{B}$ for $i = 1, 2, \ldots, n$ and $S_i \cap S_j = 0$ for $i \neq j$. For every $\sigma = (x_1, x_2, \ldots, x_n; S_1, S_2, \ldots, S_n) \in \Sigma$, choose a set Q_σ which satisfies the conditions of the Lemma and put $Q = \bigcup_{\sigma\in\Sigma} Q_\sigma$. Since $\overline{\overline{\Sigma}} = \mathfrak{m}$ and $\overline{\overline{Q}}_\sigma \leqslant \mathfrak{m}$ for $\sigma \in \Sigma$, we have $\overline{\overline{Q}} \leqslant \mathfrak{m}$. We shall show that Q is dense in W.

[2] The intersection of an empty family of subsets of a set is equal to the set itself; thus if $i = n$ this condition yields $W \neq 0$.

The sets $\bigcap_{i=1}^{n} p_{s_i}^{-1}(V_i)$, where $V_i \in \mathfrak{B}_{s_i}$ and $s_i \in S = I^{\mathfrak{m}}$ for $i = 1, 2, \ldots, n$, form a base of $\mathop{P}_{s\in S} X_s$. Let $U = \bigcap_{i=1}^{n} p_{s_i}^{-1}(V_i)$ be an arbitrary element of this base which meets W. Since $I^{\mathfrak{m}}$ is a Hausdorff space, there exist sets $S_1, S_2, \ldots, S_n \in \mathfrak{B}$ such that $s_i \in S_i$ for $i = 1, 2, \ldots, n$ and $S_i \cap S_j = 0$ for $i \neq j$. For $\sigma = (x_1, x_2, \ldots, x_n; S_1, S_2, \ldots, S_n) \in \Sigma$, where $x_i \in f_{s_i}^{-1}(V_i)$ for $i = 1, 2, \ldots, n$, we have, by the definition of Q_σ and by the inequality $W \cap U \neq 0$,

$$0 \neq Q_\sigma \cap U \subset Q \cap U .$$

Hence the set Q is dense in W.

The following theorem is a special case of Theorem 1 (and is equivalent to it if X_s's are T_1-spaces).

THEOREM 2. *Every $G^{\mathfrak{m}}_{\delta\Sigma}$-set in the Cartesian product $\mathop{P}_{s\in S} X_s$, where $w(X_s) \leqslant \mathfrak{m}$ for $s \in S$ and $\overline{\overline{S}} \leqslant 2^{\mathfrak{m}}$, contains a dense subset of power less than or equal to $\mathfrak{m}$.*

COROLLARY. *The Cartesian product $\mathop{P}_{s\in S} X_s$, where $d(X_s) \leqslant \mathfrak{m} \geqslant \aleph_0$ for $s \in S$ and $\overline{\overline{S}} \leqslant 2^{\mathfrak{m}}$, contains a dense subset of power less than or equal to $\mathfrak{m}$.*

Proof. The product $\mathop{P}_{s\in S} X_s$ contains a dense subset which is a continuous image of the product of $2^{\mathfrak{m}}$ copies of the discrete space of power $\mathfrak{m}$, in fact a cube whose faces have power not greater than $\mathfrak{m}$ and are dense in the respective spaces. Since, on the ground of Theorem 1, the Cartesian product of $2^{\mathfrak{m}}$ copies of the discrete space of power $\mathfrak{m}$ contains a dense subset of power $\mathfrak{m}$, the theorem follows from the fact that the density character of a continuous image of a space is not greater than the density character of the space.

The last corollary is the Hewitt-Marczewski-Pondiczery Theorem mentioned at the beginning of this paragraph.

We have noted that Theorem 2 is not valid under the weaker assumption that $d(X_s) \leqslant \mathfrak{m}$ for $s \in S$. It appears that under this assumption the theorem is not valid even for $G^{\mathfrak{m}}_{\delta}$-sets. Indeed, it is well known (see e.g. [11], problem 6Q.2, p. 97) that in the space $\beta N \setminus N$, where βN denotes the Čech-Stone compactification of the set N of positive integers, there exists a family $\mathfrak{R}$ of pairwise disjoint non-empty open sets which has the power $2^{\aleph_0}$. The set $\beta N \setminus N$ is therefore a G_δ-set in the Cartesian product of separable spaces (which has only one factor βN) and does not contain any countable dense subset.

2. Families of pairwise disjoint subsets of Cartesian products. Marczewski's classical theorem proved in [16] claims that in the Cartesian product of spaces with countable bases every family of pairwise disjoint non-empty open sets is countable. In a later paper [17], E. Marczewski has proved a stronger theorem, which implies that the same is true in the Cartesian product of separable spaces. We shall now prove another theorem of this type.

THEOREM 3. *Every family $\mathfrak{R}$ of pairwise disjoint $\mathfrak{m}$-cubes in the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}\geqslant\aleph_0$ for $s\in S$, non-empty and open in the union of the family $\mathfrak{R}$, has power less than or equal to $\mathfrak{m}$.*

Proof. Suppose, on the contrary, that in $\underset{s\in S}{P} X_s$ there exists a family $\{K_t\}_{t\in T}$ of non-empty $\mathfrak{m}$-cubes such that $K_t\cap K_{t'}=0$ for $t\neq t'$, any set K_t is open in $\bigcup_{t\in T} K_t$ and $\overline{\overline{T}}>\mathfrak{m}$. We have therefore $K_t=\underset{s\in S}{P} K_s^{(t)}$, where $K_s^{(t)}\subset X_s$, and $\overline{\overline{D(K_t)}}\leqslant\mathfrak{m}$. Denote by $\mathfrak{n}$ the least cardinal number greater than $\mathfrak{m}$ and choose a subset $T_0\subset T$ of power $\mathfrak{n}$. Since $\overline{\overline{T}}_0\leqslant 2^{\mathfrak{m}}$, we conclude that the power of the set $S_0=\bigcup_{t\in T_0} D(K_t)$ does not exceed $2^{\mathfrak{m}}$. The family $\{K_t'\}_{t\in T_0}$, where $K_t'=\underset{s\in S_0}{P} K_s^{(t)}$, consists of pairwise disjoint (cf. Lemma 1 in [8]) non-empty $\mathfrak{m}$-cubes in $\underset{s\in S_0}{P} X_s$.

By the assumption of the theorem, for every $t_0\in T_0$ there exists an open set $G_{t_0}\subset\underset{s\in S}{P} X_s$ such that $G_{t_0}\cap\bigcup_{t\in T} K_t=K_{t_0}$. It is easy to see that $p_{S_0}(G_{t_0})\cap\bigcup_{t\in T_0} K_t'=K_{t_0}'$. Hence the family $\{K_t'\}_{t\in T_0}$ consists of $\mathfrak{n}>\mathfrak{m}$ pairwise disjoint, non-empty $\mathfrak{m}$-cubes every one of which is closed-and-open in the union $\bigcup_{t\in T_0} K_t'$. Since this contradicts Theorem 1, we conclude that our theorem is true.

COROLLARY. *If a subset W of the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}\geqslant\aleph_0$ for $s\in S$, is the union of $\mathfrak{m}$-cubes, then every family of pairwise disjoint, non-empty open sets in W has power less than or equal to $\mathfrak{m}$.*

The following theorem easily follows from Theorem 3.

THEOREM 4. *Every family $\mathfrak{R}$ of pairwise disjoint $G_\delta^{\mathfrak{m}}$-sets in the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}\geqslant\aleph_0$ for $s\in S$, non-empty and open in the union of the family $\mathfrak{R}$, has power less than or equal to $\mathfrak{m}$.*

COROLLARY 1. *Every family of pairwise disjoint, non-empty open sets in a $G_{\delta\Sigma}^{\mathfrak{m}}$-set in the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}$ for $s\in S$, has power less than or equal to $\mathfrak{m}$.*

COROLLARY 2. *Every family of pairwise disjoint, non-empty open sets in the Cartesian product of spaces with countable bases is countable.*

The last corollary is the theorem of Marczewski mentioned at the beginning of this paragraph.

THEOREM 5. *For every subset W of the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}\geqslant\aleph_0$ for $s\in S$, which is the union of $\mathfrak{m}$-cubes, there exists a set $S_0\subset S$ of power not greater than $\mathfrak{m}$, such that $\overline{W}=\overline{p_{S_0}(W)}\times\underset{s\in S\setminus S_0}{P} X_s$.*

Proof. We shall consider the class of families of pairs of sets $\{(K_t, G_t)\}_{t\in T}$ where $K_t\subset W\cap G_t$, K_t is a non-empty $\mathfrak{m}$-cube and G_t an open set in $\underset{s\in S}{P} X_s$ such that

(3) $$K_t\cap G_{t'}=0 \quad \text{for} \quad t\neq t'.$$

Since property (3) is of finite character, there exists a family $\{(K_t, G_t)\}_{t\in T}$ maximal in the class in question; we shall show that for this family

(4) $$\overline{\bigcup_{t\in T} K_t}=\overline{W}.$$

The inclusion $\overline{\bigcup_{t\in T} K_t}\subset\overline{W}$ is obvious; suppose that there exists a point $x\in\overline{W}\setminus\overline{\bigcup_{t\in T} K_t}$. Hence there exists an open set $G\subset\underset{s\in S}{P} X_s$ such that $G\cap W\neq 0$ and $G\cap\bigcup_{t\in T} K_t=0$. The intersection $G\cap W$, as a non-empty intersection of sets which are unions of $\mathfrak{m}$-cubes, contains a non-empty $\mathfrak{m}$-cube K. Putting

$$K_{t_0}=K \quad \text{and} \quad G_{t_0}=G$$

for some $t_0\notin T$ and adjoining the pair (K_{t_0}, G_{t_0}) to the family $\{(K_t, G_t)\}_{t\in T}$, we obtain a family in our class. This contradicts the maximality of the family $\{(K_t, G_t)\}_{t\in T}$. Thus (4) has been proved.

We shall prove that the theorem is satisfied by $S_0=\bigcup_{t\in T} D(K_t)$. Since, by Theorem 3, we have $\overline{\overline{T}}\leqslant\mathfrak{m}$, $\overline{\overline{S}}_0\leqslant\mathfrak{m}$. The inclusion

$$p_{S_0}(W)\times\underset{s\in S\setminus S_0}{P} X_s\subset\overline{W}$$

follows from (4) and from the fact that $D(K_t)\subset S_0$ for every $t\in T$. From this inclusion we conclude that $\overline{p_{S_0}(W)}\times\underset{s\in S\setminus S_0}{P} X_s\subset\overline{W}$, which with the obvious converse inclusion completes the proof.

The following theorem is a special case of Theorem 5 (and is equivalent to it if X_s's are T_1-spaces).

THEOREM 6. *For every $G_{\delta\Sigma}^{\mathfrak{m}}$-set W in the Cartesian product $\underset{s\in S}{P} X_s$, where $w(X_s)\leqslant\mathfrak{m}\geqslant\aleph_0$ for $s\in S$, there exists a set $S_0\subset S$ of power not greater than $\mathfrak{m}$, such that $\overline{W}=\overline{p_{S_0}(W)}\times\underset{s\in S\setminus S_0}{P} X_s$.*

From the well-known fact that closed subsets of metric spaces are G_δ-sets and from Theorem 6 we obtain the following corollary.

COROLLARY 1. *The closure of any $G_{\delta\Sigma}$-set in the Cartesian product of metric separable spaces is a G_δ-set.*

From Theorem 6 one can obtain also Theorem 5 of [8], proved there in a completely different manner, which is an improvement of a result of Bockstein [3]:

COROLLARY 2. *For every open set U and a $G_{\delta\Sigma}^{\mathfrak{m}}$-set W disjoint with U in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$, where $w(X_s)\leqslant \mathfrak{m}$ for $s\in S$, there exists a set $S_0\subset S$ of power not greater than $\mathfrak{m}$ such that $p_{S_0}(U)\cap p_{S_0}(W)=0$.*

The proofs of Theorems 4 and 6 given here are modifications of some reasonings of [19] and are based on Theorem 1. It is worth noticing that these theorems can both be deduced from Theorem 5 of [8], formulated above as Corollary 2. We show this for Theorem 4; for Theorem 6 the argument is straightforward.

Indeed, let the family $\mathfrak{R}=\{W_t\}_{t\in T}$, where $W_t\cap W_{t'}=0$ for $t\neq t'$, satisfy the assumptions of Theorem 4 and let $\{U_t\}_{t\in T}$ be a family of open subset of $\underset{s\in S}{\boldsymbol{P}} X_s$ such that $U_{t_0}\cap\bigcup_{t\in T} W_t=W_{t_0}$ for $t_0\in T$. Without loss of generality we may assume that $S\cap T=0$; for every $t\in T$, let X_t be the two-point discrete space $D=\{0,1\}$.

The subsets U_t' and W_t' of the product $\underset{s\in S}{\boldsymbol{P}} X_s\times\underset{t\in T}{\boldsymbol{P}} X_t$, where

$$U_t'=p_t^{-1}(0)\cap(U_t\times\underset{t\in T}{\boldsymbol{P}} X_t)\quad\text{and}\quad W_t'=p_t^{-1}(1)\cap(W_t\times\underset{t\in T}{\boldsymbol{P}} X_t)\,,$$

are an open set and $G_\delta^{\mathfrak{m}}$-set, respectively. Since $U_t'\cap W_{t'}'=0$ for $t,t'\in T$, the sets $U=\bigcup_{t\in T} U_t'$ and $W=\bigcup_{t\in T} W_t'$ are disjoint and it follows from Theorem 5 of [8] that there exists a set $S_0\subset S\cup T$ such that $p_{S_0}(U)\cap p_{S_0}(W)=0$ and $\overline{\overline{S}}_0\leqslant\mathfrak{m}$. For every set $S'\subset S\cup T$ which does not contain $t_0\in T$ we have

$$p_{S'}(U_{t_0}')\cap p_{S'}(W_{t_0}')\supset p_{S'}(W_{t_0}')\neq 0;$$

thus $T\subset S_0$ and $\overline{\overline{T}}\leqslant\mathfrak{m}$.

The example considered at the end of the first paragraph shows that Theorems 3 and 4 are not valid, even for $\mathfrak{m}=\aleph_0$, under the weaker assumption that $d(X_s)\leqslant\mathfrak{m}$ for $s\in S$. Starting with the same example and reasoning as in the deduction of Theorem 4 from Theorem 5 of [8], one can construct an example which shows that Corollary 2 of Theorem 6 and hence also Theorems 5 and 6 are not true under this weaker assumption. In this case the set W is even a G_δ-set.

In the last part of this paragraph we shall prove, with the help of some methods of the present paper, an important special case of Theorem 6 of [8], which will be used in the following paragraph.

LEMMA. *The Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$, where $w(X_s)\leqslant 2^{\mathfrak{m}}$ for $s\in S$ and $\overline{\overline{S}}\leqslant 2^{2^{\mathfrak{m}}}$, contains a subset Q of power not greater than $2^{\mathfrak{m}}$ which meets every non-empty $G_\delta^{\mathfrak{m}}$-set.*

Proof. Let $\mathfrak{B}_s$ be a base of X_s such that $\overline{\overline{\mathfrak{B}}}_s\leqslant 2^{\mathfrak{m}}$ and let Q_s be the set obtained by choosing a point in every non-empty intersection of not more than $\mathfrak{m}$ elements of $\mathfrak{B}_s$. It is easy to see that $\overline{\overline{Q}}_s\leqslant 2^{\mathfrak{m}}$ and Q_s meets every non-empty $G_\delta^{\mathfrak{m}}$-set in X_s. The reasoning used in the deduction of Corollary from Theorem 2 shows that it is enough to establish the special case of our lemma, where $\overline{\overline{S}}=2^{2^{\mathfrak{m}}}$ and $X_s=X$ is for every $s\in S$ a discrete space of power $2^{\mathfrak{m}}$.

Let $S=I^{2^{\mathfrak{m}}}$ be the Tychonoff cube of weight $2^{\mathfrak{m}}$, let $\mathfrak{B}$ be a base of $I^{2^{\mathfrak{m}}}$ such that $\overline{\overline{\mathfrak{B}}}=2^{\mathfrak{m}}$, and let $\mathfrak{B}_{\mathfrak{m}}$ denote the family of all non-empty subsets of $I^{2^{\mathfrak{m}}}$ which are intersections of not more than $\mathfrak{m}$ elements of the base $\mathfrak{B}$. Let T be an arbitrary set of power $\mathfrak{m}$ and let Σ denote the set of all pairs $(\{B_t\}_{t\in T},\{x_t\}_{t\in T})$, where $B_t\in\mathfrak{B}_{\mathfrak{m}}$, $x_t\in X$ for $t\in T$, and $B_t\cap B_{t'}=0$ for $t\neq t'$. One can easily verify that $\overline{\overline{\Sigma}}=2^{\mathfrak{m}}$. Assigning to any $\sigma=(\{B_t\}_{t\in T},\{x_t\}_{t\in T})\in\Sigma$ the point $\varphi(\sigma)=\{x_s\}\in\underset{s\in I^{2^{\mathfrak{m}}}}{\boldsymbol{P}} X_s$, where

$$x_s=\begin{cases} x_t & \text{if}\quad s\in B_t \text{ for some } t\in T\,,\\ \bar{x} & \text{if}\quad s\in I^{2^{\mathfrak{m}}}\setminus\bigcup_{t\in T} B_t\end{cases}$$

and $\bar{x}$ is a fixed point of X, we obtain the set $Q=\varphi(\Sigma)$, which satisfies the lemma.

THEOREM 7. *Every family of pairwise disjoint, non-empty $G_\delta^{\mathfrak{m}}$-sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$, where $w(X_s)\leqslant 2^{\mathfrak{m}}$ for $s\in S$, has power less than or equal to $2^{\mathfrak{m}}$.*

Proof. We proceed as in the proof of Theorem 3. We suppose on the contrary that there exists in $\underset{s\in S}{\boldsymbol{P}} X_s$ a family $\{K_t\}_{t\in T}$ of pairwise disjoint non-empty $\mathfrak{m}$-cubes which are $G_\delta^{\mathfrak{m}}$-sets such that $\overline{\overline{T}}>2^{\mathfrak{m}}$. Next, we construct in the Cartesian product $\underset{s\in S_0}{\boldsymbol{P}} X_s$, where $S_0\subset S$ and $\overline{\overline{S}}_0\leqslant\mathfrak{n}$ the least cardinal number greater than $2^{\mathfrak{m}}$, a family of power $\mathfrak{n}$ consisting of pairwise disjoint and non-empty $G_\delta^{\mathfrak{m}}$-sets. Since $\mathfrak{n}\leqslant 2^{2^{\mathfrak{m}}}$, this is impossible by the Lemma.

COROLLARY. *Every family of pairwise disjoint, non-empty $G_\delta^{\mathfrak{m}}$-sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$, where X_s is regular and $d(X_s)\leqslant\mathfrak{m}$ for $s\in S$, has power less than or equal to $2^{\mathfrak{m}}$.*

Proof. The corollary follows from the well-known fact that any regular space X such that $d(X)\leqslant\mathfrak{m}$ has a base of power not greater than $2^{\mathfrak{m}}$. Indeed, it is easy to see that if X_0 is dense in X and has power not greater than $\mathfrak{m}$, then for an arbitrary base $\{U_s\}_{s\in S}$ of X, the family $\{V_s\}_{s\in S}$, where $V_s = \operatorname{Int}\overline{U_s \cap X_0}$, is the base of X and has power at most $2^{\mathfrak{m}}$.

Finally, let us remark that Theorem 7 is not true for families of $\mathfrak{m}$-cubes.

3. Continuous images of Cartesian products of compact spaces. We shall now deduce from the theorems 1 and 7 a theorem which characterizes the weight of a continuous image of the Cartesian products of compact spaces. This theorem is a generalization of the results of A. Esenin-Volpin and B. Efimov (see [10] and [5], [6]).

THEOREM 8. *Let $\{X_s\}_{s\in S}$ be a family of compact spaces such that $w(X_s) \leqslant \mathfrak{m} \geqslant \aleph_0$ for $s \in S$, and let $f\colon \underset{s\in S}{P} X_s \to X$ be a continuous mapping onto a Hausdorff space X. If X contains a dense subset X_0 such that the weight at every point* ([3]) *of X_0 does not exceed $\mathfrak{m}$, then the weight of the space X is less than or equal to $\mathfrak{m}$.*

Proof. For every $x \in X_0$, the counter-image $f^{-1}(x)$ is a non-empty $G_\delta^{\mathfrak{m}}$-set. It follows from Theorem 7 that the power of the family $\{f^{-1}(x)\}_{x\in X_0}$ does not exceed $2^{\mathfrak{m}}$, whence $\overline{\overline{X}}_0 \leqslant 2^{\mathfrak{m}}$. For every $x \in X_0$, let us choose a point $x' \in f^{-1}(x)$ and an $\mathfrak{m}$-cube $K(x)$, such that

$$x' \in K(x) \subset f^{-1}(x)\,. \tag{5}$$

The power of the set $S_0 = \bigcup_{x\in X_0} D\big(K(x)\big)$ does not exceed $2^{\mathfrak{m}}$. Since the theorem is obvious if $X_s = 0$ for some $s \in S$, we can assume that for every $s \in S$ there exists a point $a_s \in X_s$. Let us consider the product $\underset{s\in S}{P} X'_s \subset \underset{s\in S}{P} X_s$, where $X'_s = X_s$ for $s \in S_0$ and $X'_s = \{a_s\}$ for $s \in S \setminus S_0$, and the function $f' = f \mid \underset{s\in S}{P} X'_s\colon \underset{s\in S}{P} X'_s \to X$. Since the product $\underset{s\in S}{P} X'_s$ is compact and $f'(\underset{s\in S}{P} X'_s)$ contains the set X_0 dense in X, the mapping f' maps $\underset{s\in S}{P} X'_s$ onto the space X. The function f' determines a function f_0 which maps the product $\underset{s\in S_0}{P} X_s$ onto X. Hence we can confine our attention, without loss of generality, to the special case of the theorem, namely we can assume that $\overline{\overline{S}} \leqslant 2^{\mathfrak{m}}$.

Let us consider again the family $\{K(x)\}_{x\in X_0}$ of $\mathfrak{m}$-cubes satisfying (5). By Theorem 1, there exists a set Q dense in $\bigcup_{x\in X_0} K(x)$, such that $\overline{\overline{Q}} \leqslant \mathfrak{m}$.

([3]) By the *weight at a point x of a topological space X* we mean the least cardinal number which is the power of a base for the neighbourhood system of x. It is easy to see that the weight at a point x of a compact space X is not greater than $\mathfrak{m} \geqslant \aleph_0$ if and only if $\{x\}$ is a $G_\delta^{\mathfrak{m}}$-set. If X is a regular space and X_0 is dense in it, then for $x \in X_0$ the weight of x in X and of x in X_0 are, as can be easily verified, equal.

Hence there exists a set $X_1 \subset X_0$ of power not greater than $\mathfrak{m}$ such that $Q \subset \bigcup_{x\in X_1} K(x)$. The power of the set $S_1 = \bigcup_{x\in X_1} D\big(K(x)\big)$ is at most $\mathfrak{m}$. Let us consider the product $\underset{s\in S}{P} X''_s$, where $X''_s = X_s$ for $s \in S_1$ and $X''_s = \{a_s\}$ for $s \in S\setminus S_1$, and the function $f'' = f \mid \underset{s\in S}{P} X''_s\colon \underset{s\in S}{P} X''_s \to X$. It can easily be seen that the function f'' maps the product $\underset{s\in S}{P} X''_s$ onto the space X. The theorem follows now from the fact (see the Appendix in [2]) that the weight of a continuous image of a compact space is not greater than the weight of the space itself, because $w(\underset{s\in S}{P} X''_s) \leqslant \mathfrak{m}$.

COROLLARY 1. *Let $\{X_s\}_{s\in S}$ be a family of compact metric spaces and let $f\colon \underset{s\in S}{P} X_s \to X$ be a continuous mapping onto a Hausdorff space X. If X contains a dense subset X_0 which satisfies the first axiom of countability, then X has a countable base, i.e. X is a compact metrizable space.*

We obtain also the following Theorem, proved by B. Efimov in [6], as a corollary to Theorem 8.

COROLLARY 2. *Let $\{X_s\}_{s\in S}$ be a family of compact spaces such that $w(X_s) \leqslant \mathfrak{m} \geqslant \aleph_0$ for $s \in S$, and let $f\colon \underset{s\in S}{P} X_s \to X$ be a continuous mapping onto a Hausdorff space X. If the weight at every point of X does not exceed $\mathfrak{m}$, then the weight of the space X is less than or equal to $\mathfrak{m}$.*

The following theorem gives some information on the structure of continuous images of Cartesian products of compact spaces. We begin with two preliminary lemmas.

LEMMA 1. *Let Y be a compact space containing a discrete space R of power at least $\mathfrak{m} \geqslant \aleph_0$ as a dense subset, and let $f\colon Y \to Z$ be a continuous mapping onto a Hausdorff space Z, such that $f(Y\setminus R) \cap f(R) = 0$. If the image $f(Y\setminus R)$ is a one-point set, then Z is the one-point compactification of the discrete space $f(R)$ of power greater than or equal to $\mathfrak{m}$.*

Proof. For every $z \neq z_0 \in f(Y\setminus R)$ the counter-image $f^{-1}(z)$ is a compact subset of R, whence it is finite. It follows that $\overline{\overline{f(R)}} = \overline{\overline{R}}$ and that the set $f\big(Y\setminus f^{-1}(z)\big) = Z\setminus\{z\}$ is compact, i.e. that the point $z \neq z_0$ is isolated in Z.

Let f be a mapping of the Cartesian product $\underset{s\in S}{P} X_s$ onto the space X, and let x_0 be a point of X and s_0 an element of S. If there exist two points $\{r_s\}, \{t_s\} \in \underset{s\in S}{P} X_s$ such that

$$f(\{r_s\}) \neq x_0 = f(\{t_s\}) \quad \text{and} \quad r_s = t_s \quad \text{for} \quad s \neq s_0\,.$$

then we say that the value x_0 *depends on* s_0.

LEMMA 2. *Let $\{X_s\}_{s\in S}$ be a family of compact spaces such that $w(X_s) \leqslant \mathfrak{n} < \mathfrak{m} \geqslant \aleph_0$ for $s \in S$, and let $f\colon \underset{s\in S}{P} X_s \to X$ be a continuous mapping onto a Hausdorff space X. If the power of the set S_0 of all those $s \in S$ on which a value x_0 depends is less than $\mathfrak{m}$, then the weight at the point $x_0 \in X$ is less than $\mathfrak{m}$.*

Proof. It is enough to show that for the set $A = f^{-1}(x_0)$ we have

$$A = p_{S_0}(A) \times \underset{s\in S\setminus S_0}{P} X_s . \tag{6}$$

Indeed, if (6) holds then the (compact) set A has in the space $\underset{s\in S}{P} X_s$ a base for a neighbourhood system of power less than $\mathfrak{m}$, and accordingly the point x_0 has in the space X a base for a neighbourhood system of power less than $\mathfrak{m}$. To prove (6) it suffices to remark that the set of all points which are in $p_{S_0}(A) \times \underset{s\in S\setminus S_0}{P} X_s$ and differ from a point of A only in a finite number of coordinates which all correspond to the indices in $S\setminus S_0$ is dense in $p_{S_0}(A) \times \underset{s\in S\setminus S_0}{P} X_s$ and contained in A.

THEOREM 9. *Let $\{X_s\}_{s\in S}$ be a family of compact spaces such that $w(X_s) \leqslant \mathfrak{n} < \mathfrak{m} \geqslant \aleph_0$ for $s \in S$, and let $f\colon \underset{s\in S}{P} X_s \to X$ be a continuous mapping onto a Hausdorff space X. If the weight at the point $x_0 \in X$ is equal to $\mathfrak{m}$, then the space X contains a discrete space M of power $\mathfrak{m}$, such that $M \cup \{x_0\}$ is the one-point compactification of M.*

Proof. Let S_0 be the set of all $s \in S$ on which depends the value x_0. Choose, for every $s' \in S_0$, points $r(s'), t(s') \in \underset{s\in S}{P} X_s$ such that

$$r(s') \notin A\,,\ t(s') \in A \quad \text{and} \quad p_s[r(s')] = p_s[t(s')] \quad \text{for} \quad s \neq s'\,, \tag{7}$$

where $A = f^{-1}(x_0)$, and put

$$R = \{r(s')\colon s' \in S_0\}\,, \qquad T = \{t(s')\colon s' \in S_0\}\,.$$

From Lemma 2 it follows that $\overline{\overline{S}}_0 \geqslant \mathfrak{m}$. We shall show that $\overline{\overline{R}} = \overline{\overline{S}}_0$; since $\overline{\overline{S}}_0 \geqslant \aleph_0$ it suffices to verify that counter-images of the function $r\colon S_0 \to R$ are finite. Suppose, on the contrary that there exists an $S' \subset S_0$ such that $\overline{\overline{S'}} \geqslant \aleph_0$ and

$$r(s') = r_0 \quad \text{for every} \quad s' \in S'$$

and put $T' = \{t(s')\colon s' \in S'\} \subset T \subset A$. For an arbitrary neighbourhood $V = \bigcap_{i=1}^{n} p_{s_i}^{-1}(V_i)$ of r_0 there exists $s' \in S'\setminus\{s_1, s_2, \dots, s_n\}$. Since $r(s') = r_0 \in V$, we infer from (7) that also $t(s') \in V$. It follows that $r_0 \in \overline{T'} \subset A$, which is impossible by the first condition of (7). Thus $\overline{\overline{R}} = \overline{\overline{S}}_0 \geqslant \mathfrak{m}$.

We shall now prove that all accumulation points of R are in A. Since A is closed, every point $\{x_s\} \notin A$ has a neighbourhood $V = \bigcap_{i=1}^{n} p_{s_i}^{-1}(V_i)$ disjoint with A. From (7) it follows that only the points $r(s')$ with $s' \in S_0 \cap \{s_1, s_2, \dots, s_n\}$ can be contained in V. Hence the neighbourhood V of the point $\{x_s\}$ contains only a finite number of points of the set R and $\{x_s\}$ is not an accumulation point of R.

The spaces $Y = \overline{R}$ and $Z = f(R) \cup \{x_0\}$ satisfy the assumptions of Lemma 1. Since the weight at the point x_0 in X, and hence the weight at the point x_0 in Z, are equal to $\mathfrak{m}$, we have $\overline{\overline{f(R)}} = \mathfrak{m}$ and the Theorem holds with $M = f(R)$.

4. Dyadic spaces. By a *dyadic space* we mean a Hausdorff space which is a continuous image of a Cartesian product of a certain number of copies of the two-point discrete space, i.e. of a Cantor cube. Dyadic spaces are compact. This class of spaces was defined by P. S. Aleksandrov in [1] and was subsequently investigated by a number of mathematicians. The most important results are to be found in the papers of E. Marczewski [16], A. Esenin-Volpin [10], N. Šanin [20] and B. Efimov [4], [5]. In particular, in [16], E. Marczewski, answering a question raised by P. S. Aleksandrov, points out the first example of a compact non-dyadic space. Namely, he remarks that from the fact that every family of pairwise disjoint open sets in a Cantor cube is countable (see Corollary 2 to Theorem 4) it follows that the same is true in any dyadic space, and hence the one-point compactification of the discrete space of power $2^{\aleph_0}$ is not dyadic. Simple proofs of all important theorems on dyadic spaces are given in [7] and [9].

In this paragraph we shall formulate some theorems on dyadic spaces which follow from the theorems of the preceding paragraphs. We shall also describe an example of a dyadic space which solves a problem from P. R. Halmos's paper [12].

Theorem 38 of [20] (simple proof in [9]). which asserts that any dyadic space of weight $\mathfrak{m}$ is a continuous image of the Cantor cube $D^{\mathfrak{m}}$, and Theorem 2 imply

THEOREM 10. *Every $G_{\delta\Sigma}^{\mathfrak{m}}$-set in a dyadic space of weight not greater than $2^{\mathfrak{m}}$ contains a dense subset of power less than or equal to $\mathfrak{m}$.*

Let us note that from the above Theorem it follows that the Čech-Stone compactification βN of the set N of positive integers is not dyadic (see the remark at the end of the first paragraph), which was proved in a different manner in [9].

From Theorems 4 and 7 we obtain two theorems on families of pairwise disjoint non-empty subsets of dyadic spaces.

THEOREM 11. *Every family $\mathfrak{R}$ of pairwise disjoint, $G_\delta^{\mathfrak{m}}$-sets in a dyadic space, non-empty and open in the union of the family $\mathfrak{R}$, has power less than or equal to $\mathfrak{m}$.*

COROLLARY 1. *Every family of pairwise disjoint non-empty open sets in a $G_{\delta\Sigma}^{\mathfrak{m}}$-set in a dyadic space has power less than or equal to $\mathfrak{m}$.*

Theorem 11 implies also Marczewski's theorem proved in [16]:

COROLLARY 2. *Every family of pairwise disjoint, non-empty open sets in a dyadic space is countable.*

THEOREM 12. *Every family of pairwise disjoint, non-empty $G_\delta^{\mathfrak{m}}$-sets in a dyadic space has power less than or equal to $2^{\mathfrak{m}}$.*

Since every closed subset of the Cantor cube $D^{\aleph_0}$ is a retract (4) of it (see [15], p. 169 or [13] p. 183), Theorem 6 implies:

THEOREM 13. *Every subspace of a dyadic space, which is the closure of a $G_{\delta\Sigma}$-set, is a dyadic space.*

From the above theorem we obtain the following result of [4].

COROLLARY. *Every subspace of a dyadic space which is the closure of an open set or a closed G_δ-set is a dyadic space.*

From Theorem 8 we deduce the following theorem, proved in a different manner by B. Efimov in [5], which is an improvement of a theorem due to A. Esenin-Volpin [10] and formulated below as Corollary 2.

THEOREM 14. *If the weight at every point of a dense subset X_0 of a dyadic space X does not exceed $\mathfrak{m} \geqslant \aleph_0$, then the weight of the space X is less than or equal to $\mathfrak{m}$.*

COROLLARY 1. *If the weight at every point of a dense subset X_0 of a dyadic space X does not exceed $\aleph_0$ then X is a compact metrizable space.*

COROLLARY 2. *If the weight at every point of a dyadic space X does not exceed $\mathfrak{m} \geqslant \aleph_0$, then the weight of the space X is less than or equal to $\mathfrak{m}$.*

From Theorem 9 we obtain the following theorem, whose special case for $\mathfrak{m} = \aleph_1$ is proved in [4]; it seems that Theorem 15 cannot be proved by the method employed by B. Efimov in [4].

THEOREM 15. *If the weight at the point x_0 of a dyadic space X is equal to $\mathfrak{m} \geqslant \aleph_0$, then X contains a discrete space M of power $\mathfrak{m}$, such that $M \cup \{x_0\}$ is the one-point compactification of M.*

From the above theorem, the Corollary to Theorem 13, and Corollary 2 to Theorem 14 we obtain the following result of [4]:

COROLLARY 1. *A dyadic space X is hereditarily dyadic with respect to closed sets if and only if it is metrizable.*

(4) A subset A of the topological space X is called a *retract* of X if there exists a continuous mapping $r: X \to A$ such that $r(a) = a$ for $a \in A$; a mapping r with the above property is called a *retraction of X onto A*.

This theorem implies also the following theorem, proved independently by M. Katětov and B. Efimov (see footnote (2) in [9] and [5]).

COROLLARY 2. *Every non-isolated point of a dyadic space is the limit of a sequence of distinct points.*

As M. Katětov remarked, from Corollary 2 one can obtain the following result of [7].

COROLLARY 3. *Every dyadic subspace of a basically disconnected space (5) is finite.*

Proof. It is enough to remark that no basically disconnected space contains a convergent sequence of distinct points. Suppose, on the contrary, that in a basically disconnected space X there exists a convergent sequence $x_1, x_2, \ldots$ Hence, there exist in X two sequences $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ of closed-and-open sets such that $x_{2i-1} \in U_i$, $x_{2i} \in V_i$ for $i = 1, 2, \ldots$ and $U_i \cap U_j = 0 = V_i \cap V_j$ if $i \neq j$, $U_i \cap V_j = 0$ for $i, j = 1, 2, \ldots$ It follows that the closures of disjoint cozero-sets $U = \bigcup_{i=1}^{\infty} U_i$ and $V = \bigcup_{i=1}^{\infty} V_i$ have a non-empty intersection, which is impossible, since X is supposed to be basically disconnected.

From Corollary 3 we obtain the following result of the theory of Boolean algebras (see [13], in particular § 31, where all notions occurring in Corollary 4 are defined) noted also in [7].

COROLLARY 4. *Every projective Boolean algebra which is a homomorphic image of a σ-complete algebra is finite.*

Finally, Theorem 15 implies the following theorem of N. Šanin ([20], Theorem 51):

COROLLARY 5. *If the topology of the dyadic space X is induced by a linear order $<$ in the set X, then X is a metrizable space.*

Proof. By Corollary 1 to Theorem 14, it suffices to show that the weight at every point $x \in X$ does not exceed $\aleph_0$. Suppose, on the contrary, that the weight at point $x_0 \in X$ is equal to $\mathfrak{m} > \aleph_0$. From Theorem 15 it follows that a discrete space M of power $\mathfrak{m}$, such that $M \cup \{x_0\}$ is compact, is contained in X. Without loss of generality we can assume that $x < x_0$ for every $x \in M$. It is easy to see that there exists an accumulation point of M smaller (with respect to the order $<$) than x_0, which is impossible, since M has only one accumulation point.

(5) A Tychonoff space X is called *basically disconnected* if the closures of every pair of disjoint cozero-sets in X are disjoint, where by a *cozero-set* in X we mean the counter-image $f^{-1}(R \setminus \{0\})$ under a continuous real-valued function $f: X \to R$. It can easily be seen that the family of cozero-sets is closed under countable unions. Every basically disconnected space has a base composed of closed-and-open sets, which are obviously cozero-sets. Every extremally disconnected space is basically disconnected.

It is well known that every closed subset of the Cantor cube $D^{\aleph_0}$ is a retract of it. This is not true for $\mathfrak{m} > \aleph_0$, because by the classical Vedenissoff theorem of [21] every zero-dimensional space (i.e. a Hausdorff space which has a base composed of closed-and-open sets) of weight $\mathfrak{m} \geqslant \aleph_0$ can be topologically embedded in the Cantor cube $D^{\mathfrak{m}}$, and there exist non-dyadic compact zero-dimensional spaces. In this situation the problem arises whether every *dyadic* subspace of $D^{\mathfrak{m}}$ is a retract of it. This problem, formulated in the language of Boolean algebras, was raised by P. R. Halmos (Problem 1 in [12]). We shall show that the answer is negative, i.e. we shall construct a dyadic zero-dimensional space which is not homeomorphic to any retract of $D^{\mathfrak{m}}$.

THEOREM 16. *If a subset X of the Cantor cube $D^{\mathfrak{m}}$ is a retract of it, then for every pair U, V of disjoint open subsets of X there exist open F_σ-sets $U_0, V_0 \subset X$ such that*

$$U \subset U_0,\ V \subset V_0 \quad \text{and} \quad U_0 \cap V_0 = 0 . \tag{8}$$

Proof. Let $r\colon D^{\mathfrak{m}} = \underset{s\in S}{\mathbf{P}} D_s \to X$ be a retraction of $D^{\mathfrak{m}}$ onto X. The sets $r^{-1}(U)$ and $r^{-1}(V)$ are disjoint and open in $D^{\mathfrak{m}}$, whence by Corollary 2 to Theorem 6 there exists a countable set $S_0 \subset S$ such that

$$p_{S_0}r^{-1}(U) \cap p_{S_0}r^{-1}(V) = 0 .$$

The sets $p_{S_0}r^{-1}(U)$ and $p_{S_0}r^{-1}(V)$ are open in $\underset{s\in S_0}{\mathbf{P}} D_s$, whence they are F_σ-sets. The sets

$$U_1 = p_{S_0}r^{-1}(U) \times \underset{s\in S\setminus S_0}{\mathbf{P}} D_s \quad \text{and} \quad V_1 = p_{S_0}r^{-1}(V) \times \underset{s\in S\setminus S_0}{\mathbf{P}} D_s$$

are disjoint and open F_σ-sets in $D^{\mathfrak{m}}$; thus the sets

$$U_0 = X \cap U_1 \quad \text{and} \quad V_0 = X \cap V_1$$

are disjoint and open F_σ-sets in X. Since for $B \subset X$ we have

$$B \subset r^{-1}(B) \subset p_{S_0}^{-1}p_{S_0}r^{-1}(B) = p_{S_0}r^{-1}(B) \times \underset{s\in S\setminus S_0}{\mathbf{P}} D_s ,$$

both inclusions in (8) are also true.

Let us now consider the Cantor cube $D^{\mathfrak{m}}$ where $\mathfrak{m} > \aleph_0$ and let Z be the set of all points of $D^{\mathfrak{m}}$ which have at most one coordinate equal to zero. It is easy to see that Z is the one-point compactification of the discrete space of power $\mathfrak{m}$. Since Z is not dyadic, we conclude by the Corollary to Theorem 13 that Z is not a G_δ-set in $D^{\mathfrak{m}}$. It follows that $D^{\mathfrak{m}}\setminus Z$ is not an F_σ-set, i.e. it cannot be represented as a countable union of compact sets.

Denote by Y the space obtained from $D^{\mathfrak{m}}$ by the identification of Z to a point; let z be the point of Y which is the image of Z. The space Y is obviously dyadic. Let Y_1, Y_2 be two disjoint copies of Y and z_1, z_2— their distinguished points. The required space X is obtained by the identification of z_1 and z_2 in the discrete union $Y_1 \cup Y_2$ of spaces Y_1 and Y_2; let $\varphi\colon Y_1 \cup Y_2 \to X$ denote the quotient map.

The space X is dyadic, as a continuous image of the dyadic space $Y_1 \cup Y_2$. It is also zero-dimensional, i.e. every point of X has a base for the neighbourhood system composed of closed-and-open sets. This is obvious for all points distinct from $x_0 = \varphi(z_1) = \varphi(z_2)$, and for x_0 it follows from the fact that for every open set $W \subset D^{\mathfrak{m}}$ which contains Z there exists closed-and-open set $C \subset D^{\mathfrak{m}}$ such that $Z \subset C \subset W$.

The sets $U = X\setminus\varphi(Y_2)$ and $V = X\setminus\varphi(Y_1)$ are disjoint and open in X but there exists no pair of open F_σ-sets $U_0, V_0 \subset X$ satisfying (8). Suppose, on the contrary, that (8) holds for a pair U_0, V_0 of open F_σ-sets in X. The point x_0 can be contained in at most one set of this pair, whence the set $X\setminus\varphi(Y_i)$ is equal to one of the sets U_0, V_0 for $i = 1$ or 2 and is a F_σ-set. Thus, $X\setminus\varphi(Y_i)$ can be represented as a countable union of compact sets and the set $D^{\mathfrak{m}}\setminus Z$, homeomorphic to $X\setminus\varphi(Y_i)$, would have the same property, which is impossible. We now conclude from Theorem 16 that the space X is not homeomorphic to any retract of the Cantor cube.

The above result can be formulated in the language of Boolean algebras as the following theorem, which gives a solution of Problem 1 from [12].

THEOREM 17. *A free Boolean algebra with $\mathfrak{m} > \aleph_0$ generators contains a subalgebra which is not projective.*

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Ordnungsfähigkeit zusammenhängender Räume

von

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Ziel der Arbeit ist eine topologische Kennzeichnung der zusammenhängenden (Satz 1) und der lokal-zusammenhängenden (Satz 2, 2a) ordnungsfähigen Räume.

DEFINITIONEN.

1) Ein topologischer Raum T heißt *ordnungsfähig*, wenn er einem geordneten Raum homöomorph ist, d.h. wenn es eine lineare Ordnung R auf T so gibt, daß die offenen Intervalle eine Basis der Topologie von T bilden.

2) Ein Punkt x einer zusammenhängenden Menge M heißt *Randpunkt von* M, wenn $M-\{x\}$ zusammenhängend ist, sonst Schnittpunkt von M.

3) Ein topologischer Raum heißt *randendlich*, wenn jede seiner zusammenhängenden Teilmengen höchstens zwei Randpunkte enthält.

HILFSSÄTZE.

1) *Sind x, y zwei verschiedene Elemente des zusammenhängenden, lokal-zusammenhängenden T_1-Raumes T, so gibt es eine Komponente K von $T'=T-\{x,y\}$, die x und y als Häufungspunkte besitzt. $K_1=K\cup\{x,y\}$ ist als Unterraum lokal-zusammenhängend.*

Beweis: a) $\mathfrak{M}$ sei die Menge aller Komponenten von T'. Jedes K aus $\mathfrak{M}$ ist offen-abgeschlossen in T'. Gäbe es ein K in $\mathfrak{M}$, das weder x noch y als Häufungspunkt besäße, so wäre dieses K offen-abgeschlossen in T, im Widerspruch zum Zusammenhang von T. Jedes K aus $\mathfrak{M}$ besitzt also x oder y als Häufungspunkt. Ist U eine zu y disjunkte, zusammenhängende Umgebung von x und enthält ein K aus $\mathfrak{M}$ Elemente von U, so ist x Häufungspunkt von K; denn sonst wäre $K\cap U$ offen-abgeschlossen in U. Also umfaßt $X=\bigcup\{K\mid K\in\mathfrak{M},\,x\in K^-\}\cup\{x\}$ die Menge U, ist somit Umgebung von x, also offen in T. Analog ist $Y=\bigcup\{K\mid K\in\mathfrak{M},\,y\in K^-\}\cup\{y\}$ offen in T. Hätte keine Komponente x und y als Häufungspunkte, so wären X, Y disjunkt, im Widerspruch zum Zusammenhang von T.