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Linear-compact congruence topologies in *-lattices *

by

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1. Introduction. The notion of "linear-compactness" was first introduced by Lefschetz in topological linear spaces. This concept has been further extended to topological groups and modules by Leptin. This paper gives a formulation for linear-compactness in a class of topological lattices the C-lattices. Here the general properties of linear-compact C*-lattices are analysed and it is shown that the study of any Hausdorff linear-compact C*-lattice can, in some sense, be reduced to the study of certain discrete linear-compact lattices. We then proceed to establish that the centre of a discrete linear-compact C*-lattice is finite which enables us to prove that the centre of a linear-compact Hausdorff C*-lattice is compact. Next we investigate the structure of the compact complemented modular C*-lattices from which we deduce that any linear-compact Hausdorff C-Boolean algebra is the direct product of (two element) simple Boolean algebras. Hence the question naturally arises as to whether every linear-compact C*-lattice admits such a direct product decomposition into simple lattices. In this paper we shall answer this question in the affirmative for a certain class of C*-lattices viz., the generalized continuous geometries. We also define the concept of a PC*-lattice and show that a Hausdorff PC*-generalized continuous geometry is linear-compact if and only if its centre is compact. The paper ends with a brief discussion on some unsolved problems concerning the PC*-lattices.

2. Preliminaries and basic results. In our notations and terminology in lattice theory and topology we shall generally follow [2] and [5], respectively.

It is seen that in a lattice L, given any set $C = [\theta_i]$ $(i \in I)$ of congruences directed below in the lattice of congruences, the subsets $V_i = [(x, y)/x\theta_i y]$ $(i \in I)$ define a uniformity V on L. Further the lattice sum and product in L are uniformly continuous with respect to V. A complete study of these uniformities, termed "congruence uniformites"

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(the induced topologies are called congruence topologies, and C is said to be a *base* of nuclear congruences for (L, V)), has been made in [9]. By a C-*lattice* (L, T) we shall mean a lattice L together with a congruence topology T.

In [9] it has been shown that

(2.1) *The direct product (with the product topology) of* C-*lattices is a* C-*lattice.*

(2.2) *Any sublattice of a* C-*lattice is a* C-*lattice in its relative topology.*

(2.3) *If (L, T) is a* C-*lattice and θ is a congruence on L which permutes with every congruence in a base $[\theta_i]$ $(i \in I)$ of nuclear congruences of (L, T), then the quotient topology of L/θ is a congruence topology.*

(2.4) *If (L, T) is a* C-*lattice and θ is a congruence on L as in* (2.3) *then L/θ is Hausdorff in its quotient topology if and only if each congruence class $\theta(x)$, $x \in L$, is closed in (L, T).*

(2.5) *Let (L, V) be a* C-*lattice with a Hausdorff congruence uniformity V. Then (L, V) can be uniformly imbedded as a dense sublattice of the projective limit of the quotient lattices L/θ_i $(i \in I)$ each with the discrete topology, $[\theta_i]$ $(i \in I)$ being a base of nuclear congruences for (L, V).*

A lattice L with 0 is said to be a $*$-*lattice* if every congruence θ on L is of the form: $x\theta y \leftrightarrow x \vee t = y \vee t$ for some $t \in \theta(0)$. Any relatively complemented lattice with zero is a $*$-lattice. In any $*$-lattice L, (1) any two congruences permute and (2) there is a 1-1 correspondence between congruences and congruence ideals (i.e. zero classes under congruences). If L is a $*$-lattice so is the quotient lattice L/A $(= L/\theta_A)$ for any congruence ideal A of L and consequently the homomorphic image of a $*$-lattice is a $*$-lattice.

We shall call a $*$-lattice L with a congruence topology T as the C$*$-*lattice* (L, T). We have

(2.6) *Let (L, T) be a* C$*$-*lattice. If A is any congruence ideal of L, then*

(1) *A is open in (L, T) if and only if some $\theta_i(0) \subseteq A$ (where θ_i is a congruence in the nuclear base $[\theta_i]$ $(i \in I)$ of congruences for (L, T)).*

(2) *A is open $\Rightarrow$ each residue class $\theta_A(x)$ is open and closed (where θ_A is the congruence determined by A).*

(3) *$\bar{A}$ (the closure of A) is a congruence ideal of L.*

(4) *A is closed $\Rightarrow$ each $\theta_A(x)$ is closed.*

(5) $\overline{\theta_A(x)} = \theta_{\bar{A}}(x)$.

If A is a congruence ideal of a $*$-lattice for simplicity of notation we shall sometimes denote the residue class $\theta_A(x)$ by $A(x)$.

We shall now briefly recall a few results from [2] and [8] which will be made use of in the sequel.

(2.7) *Let $\theta_1, \ldots, \theta_n$ be permutable maximal congruences of a lattice L. Then $L/\bigwedge_{i=1}^{n} \theta_i$ is isomorphic to the direct product of L/θ_i $(i = 1, \ldots, n)$.*

A continuous complemented modular lattice is called a *generalized continuous geometry*. If it is further irreducible then it is said to be a *continuous geometry*.

(2.8) *Let Z be the centre of an upper continuous complemented modular lattice L. Then for any arbitrary element a of L, $[z \wedge a/z \in Z]$ is the centre of $L(0, a)$* ([8], p. 89, Satz 1.4).

(2.9) *Let L be a lattice with 0 and 1. Then L is irreducible if and only if the centre of L is the two-element Boolean algebra* $(0, 1)$.

(2.10) *In a generalized continuous geometry irreducibility and simplicity are equivalent.* ([8], p. 124, Hilfssatz 3.2.)

(2.11) *The intersection of all maximal neutral ideals of a generalized continuous geometry is zero.* ([8], p. 124, Anmerkung 3.1.)

(2.12) *Let L_i $(i \in I)$ be lattices with 0_i and 1_i respectively and let L be their direct product. Then the centre of L is the direct product of the centres of the L_i $(i \in I)$.*

(2.13) *If there exist central elements (z_α) $(\alpha \in I)$ in an upper continuous lattice L so that $\dot{\bigvee}_{\alpha \in I} z_\alpha = 1$ then L can be decomposed into the direct sum $L = \dot{\bigvee}_{\alpha \in I} L(0, z_\alpha)$ $(L(0, z_\alpha) =$ the closed interval $[0, z_\alpha])$* ([8], p. 30, Satz 3.8).

(2.14) *Suppose that a continuous lattice L can be represented as the direct sum $L = \dot{\bigvee}_{\alpha \in I} S_\alpha$. Then L can be represented as the direct product $\prod_{\alpha \in I} S_\alpha$* ([8], p. 24, Satz 2.4 and Definition 2.5).

3. Linear compact C*-lattices.

DEFINITION. A congruence θ on a C-lattice (L, T) is said to be C-*closed* if each congruence class $\theta(x)$, $x \in L$, is closed in (L, T).

From (2.6) (4) it follows that in a C$*$-lattice the C-closedness of θ implies and is implied by the closedness of the corresponding congruence ideal.

DEFINITION. A residue class corresponding to a C-closed congruence of a C-lattice is called a *linear-variety*. A C-lattice (L, T) is said to be *linear-compact* in case every collection of linear varieties of (L, T) with the finite intersection property (i.e. any finite number of them has a non-null intersection) has a non-null intersection.

It can be seen that in a C$*$-lattice (L, T) every closed residue class is a linear variety.

The following lemma will be used often in the sequel:

(3.1) Lemma. *Let L be a *-lattice and L_1 be a homomorphic image of L by the homomorphism f. Then*

(1) *If P is a congruence ideal of L then $f(P)$ is a congruence ideal of L_1.*

(2) $f\big(P(x)\big) = f(P)\big(f(x)\big)$.

(3) *If P^* is a congruence ideal of L_1 then $f^{-1}(P^*)$ is a congruence ideal of L, and*

(4) $f^{-1}\big(P^*(x^*)\big) = f^{-1}(P^*)(x)$, *where* $f(x) = x^*$.

The proof of the following proposition can easily be verified.

(3.2) Proposition. *Let (L, T) be a linear compact C*-lattice and let f be a continuous homomorphism of (L, T) on another C*-lattice (L_1, T_1). Then (L_1, T_1) is linear-compact.*

Corollary i. *Let (L, T) be a linear-compact C*-lattice. Then for any congruence ideal A of L the quotient space L/A is a linear-compact C*-lattice*

Proof. Since L, being a *-lattice, the congruences on L are permutable, the quotient space L/A is a C*-lattice (cf. 2.3). Since the natural homomorphism $L \to L/A$ is a continuous mapping of (L, T) on the quotient space L/A, it follows that L/A is linear-compact.

Corollary ii. *Let (L, T) be a linear-compact C*-lattice. If T_1 is any congruence topology on L coarser than T then (L, T_1) is linear-compact.*

(3.3) Proposition. *The direct product of linear-compact relatively complemented C-lattices with zero is a linear-compact relative complemented C-lattice with zero.*

The direct product of relative complemented lattices with zero is also a relative complemented lattice with zero and hence a *-lattice. Further the product topology is a congruence topology (cf. (2.1)). Thus the direct product is a C*-lattice. Using the properties of C*-lattices the linear compactness can be established as in the theorem of Tichonow.

Using (3.3) we can construct examples of C*-lattices which are neither discrete nor compact. For instance, let L be any infinite projective geometry. Then L is a *-lattice, and being simple, is a linear-compact in the discrete topology. Let N be any infinite cardinal and let (P, T) be the direct product of N copies of L (with the product topology). L being complemented and modular P is also complemented and modular. Therefore by (3.3), (P, T) is a linear-compact C*-lattice. (P, T) is not discrete as N is infinite and is not compact as L is not finite. Thus (P, T) is a linear-compact C*-lattice which is neither compact nor discrete.

Now we shall prove

(3.4) Proposition. *Any linear-compact Hausdorff C-*lattice (L, T) is (topologically) complete in its congruence uniformity V.*

Proof. Let P_i $(i \in I)$ be congruence ideals of (L, T) corresponding to a base of nuclear congruences $[\theta_i]$ $(i \in I)$ of (L, T). Then it suffices to show that every Cauchy-I-net of (L, V) converges (where I is directed as follows: $i \geqslant j \leftrightarrow V_i \subseteq V_j$ where $V_i = [(x, y)/x\theta_i y]$).

Let (x_p) $(p \in I)$ be any Cauchy-I-net of (L, V). Then, given any member V_r $(r \in I)$ in the base $[V_r]$ $(r \in I)$ for V there exists an index $r_0 \in I$ such that $(x_p, x_q) \in V_r$ for all $p, q \geqslant r_0$, i.e.

$$\text{(A)} \qquad x_p\,\theta_r\,x_q \quad \text{for all} \quad p, q \geqslant r_0 .$$

In particular $x_{r_0}\theta_r x_q$ for all $q \geqslant r_0$, i.e. $P_r(x_{r_0}) = P_r(x_q)$ for all $q \geqslant r_0$. Thus given V_r there exists $r_0 \in I$ such that

$$\text{(B)} \qquad P_r(x_{r_0}) = P_r(x_q) \quad \text{for all} \quad q \geqslant r_0 .$$

Consider the system of residue classes $[P_r(x_{r_0})]$ $(r \in I)$ (r_0 chosen for r as in (B)). Now each P_r is closed since it is the zero class corresponding to a congruence in the base of nuclear congruences $[\theta_i]$ $(i \in I)$. Since (L, T) is a C*-lattice, it follows that each $P_r(x_{r_0})$ is a linear variety of (L, T). Further, given $P_{r_i}(x_{r_{i_0}})$ $(i = 1, 2, \ldots, n)$, $P_{r_i}(x_{r_{i_0}}) = P_{r_i}(x_q)$ for all $q \geqslant r_{i_0}$. This is true for each $i = 1, \ldots, n$. Since I is directed above given r_{i_0} $(i = 1, 2, \ldots, n)$, there exists an index $p \in I$ such that $p \geqslant r_{i_0}$ $(i = 1, \ldots, n)$. Hence $P_{r_i}(x_{r_{i_0}}) = P_{r_i}(x_p)$, $(i = 1, \ldots, n)$.

Therefore

$$x_p \in P_{r_i}(x_{r_{i_0}}) \ (i = 1, 2, \ldots, n), \qquad \text{i.e.} \quad \bigcap_{i=1}^{n} P_{r_i}(x_{r_{i_0}}) \neq \emptyset .$$

Hence the system $[P_r(x_{r_0})]$ $(r \in I)$ of linear varieties satisfies the finite intersection property. Since (L, T) is linear-compact, it follows that $\bigcap_{i \in I} P_r(x_{r_0}) \neq \emptyset$. Hence there exists an element $x \in L$ such that $x \in$ each $P_r(x_{r_0})$, i.e.

$$\text{(C)} \qquad P_r(x) = P_r(x_{r_0}) \quad \text{for each} \quad r \in I .$$

We shall now show that $x = \lim\limits_{p \in I} x_p$.

Now from (B) we infer $P_r(x_{r_0}) = P_r(x_q)$ for all $q \geqslant r_0$. From (C), $P_r(x) = P_r(x_{r_0}) = P_r(x_q)$ for all $q \geqslant r_0$. Hence $x\theta_r x_q$ for all $q \geqslant r_0$, i.e. $(x, x_q) \in V_r$ for all $q \geqslant r_0$. Therefore $\lim\limits_{q \in I} x_q = x$. Thus every Cauchy-$I$-net in (L, V) converges and hence (L, V) is topologically complete.

As a corollary to (3.4) we have

(3.5) Proposition. *Any linear-compact sublattice L (with the relative topology), which is also a $*$-lattice, of the Hausdorff* C*-*lattice (L, T) is closed in (L, T).*

Proof. Let T_1 be the relative topology on L_1. Then (L_1, T_1) is a C^*-lattice. Further the congruence uniformity V_1 of T_1 on L_1 is its relative uniformity as a sublattice of (L, V) where V is the congruence uniformity on L with respect to T. Since (L_1, T_1) is linear-compact, by (3.4) (L_1, V_1) is complete, and being a complete subspace of the Hausdorff uniform space (L, V), it is closed in (L, T).

As a particular case of (3.5) we have the

Corollary. *Any linear-compact congruence ideal of a relatively complemented Hausdorff* C*-*lattice is closed.*

We shall now prove the converse of (3.5) for relative complemented modular lattices with zero.

(3.6) Proposition. *Let (L, T) be a relatively complemented modular* C*-*lattice which is linear-compact and let A be any closed congruence ideal of (L, T). Then A is linear-compact (in its relative topology).*

Proof. Now A is a relatively complemented modular lattice with zero. Hence its congruence ideals are precisely its neutral ideals. Let $[A_i(x_i)]$ $(i \in I)$ be a system of linear varieties of A (corresponding to the closed neutral ideals A_i of A) satisfying the finite intersection property. Since L is also relatively complemented modular and has a zero, the congruence ideals are precisely its neutral ideals and hence A is a neutral ideal of L. Since A is neutral in L and A_i is neutral in A, it follows that A_i is a neutral ideal of L (cf. [3]). Again since A_i is closed in A and A is closed in L, A_i is closed in L. Further $y \equiv x_i (\operatorname{mod} A_i)$ in $L \Rightarrow y \vee a_i = x_i \vee a_i$ for some $a_i \in A_i \subseteq A$. Since $x_i \in A$, it follows that $x_i \vee a_i \in A$. As $y \leqslant x_i \vee a_i$, $y \in A$, i.e. $y \in A_i(x_i)$. Thus each $A_i(x_i)$ is also a residue class of L and, as A_i closed, is a linear variety of L. Since the system $[A_i(x_i)]$ $(i \in I)$ of linear varieties of L satisfies the finite intersection property and (L, T) is linear-compact, it follows that $\bigcap_{i \in I} A_i(x_i) \neq O$. Hence there exists some element $x \in L$, $x \in \bigcap_{i \in I} A_i(x_i)$. Since each $A_i(x_i) \subseteq A$, it follows that $x \in A$ and therefore A is linear-compact.

Now we shall prove the following proposition.

(3.7) Proposition. *Any linear-compact Hausdorff* C*-*lattice (L, T) is a projective limit of discrete linear-compact $*$-lattices.*

Proof. Let V be the congruence uniformity of (L, T). Then (L, V) is by (3.4) complete. Hence from (2.5) we can deduce that (L, V) is the projective limit of the discrete quotient lattices L/θ_i $(i \in I)$ (where $[\theta_i]$ $(i \in I)$ is a base of nuclear congruences for (L, T)). Further each L/θ_i is by the corollary to (3.2) a linear-compact C*-lattice and hence the result.

Thus we see that the study of Hausdorff linear-compact C^*-lattices can be reduced, in some sense, to the study of discrete linear-compact $*$-lattices. We shall now proceed to study the structure of discrete linear-compact $*$-lattices and characterize them for certain complemented modular lattices. We begin with

(3.8) Lemma. *Let L be a $*$-lattice with 1 and B the centre of L. Then the ideal $I(A)$ of L generated by any ideal A of B in L is a congruence ideal of L. If A is proper then so is $I(A)$.*

Proof. Let A be any ideal of B. Then $I(A) = [b \in L / b \leqslant \text{some } a \in A]$. We shall now show that $I(A)$ is a congruence ideal of L. This is verified if we show that for $x, y \in L$, $x \vee t = y \vee t$ for any $t \in I(A) \Rightarrow$ for any arbitrary element $z \in L$ there exists an element $t_1 \in I(A)$ such that $(x \wedge z) \vee t_1 = (y \wedge z) \vee t_1$. Let $x \vee t = y \vee t$ for some $t \in I(A)$ and let $z \in L$. Since $t \in I(A)$, $t \leqslant \text{some } a \in A \subseteq B$. Hence $x \vee a = y \vee a$ and as a is a central element we have $(x \wedge z) \vee a = (x \vee a) \wedge (z \vee a) = (y \vee a) \wedge \wedge (z \vee a) = (y \wedge z) \vee a$. Since $a \in I(A)$, this proves that $I(A)$ is a congruence ideal of L.

If A is a proper ideal of B, then $I(A) \neq 0$ as $A \neq 0$. Now $I(A) \neq L$ for if it were equal to L then $1 \in I(A)$. Hence $1 \leqslant \text{some } a \in A$, i.e. $1 = a$ for some $a \in A$, i.e. $A = B$—a contradiction and this proves the result.

Now we are in a position to prove

(3.9) The fundamental lemma. *Let L be a $*$-lattice (with 1) which is linear-compact in the discrete topology. Then L has finite centre.*

Proof. Suppose that the centre B of L is infinite and L is linear compact in the discrete topology. Let S be the set of all (proper) maximal ideals of the centre B. For any $A \in S$, let $I(A)$ be the ideal generated by A in L. Then from (3.8) it follows that $I(A)$ is a congruence ideal of L. Let θ_A be the congruence corresponding to $I(A)$ in L.

Since B is an infinite Boolean algebra, B contains a maximal non-principal ideal M (cf. [1]). Consider the set C of residue classes of L which consists of $I(M)$ (i.e. $\theta_M(0)$) and $\theta_{A_i}(1)$'s, where A_i runs through all the elements of S not equal to M. Since L has the discrete topology, any residue class of L is a linear variety. The set C can easily be verified to satisfy the finite intersection property. Since L is linear-compact, it follows that there exists an element $x \in L$ such that $x \in \bigcap_{M \neq A_i (\in S)} \theta_{A_i}(1) \cap \theta_M(0)$. Hence there exists an element $x \in L$ such that $x \in I(M)$ and $x \notin \text{any } I(A_i)$ (as $I(A_i) \neq L$ by (3.8)), i.e. x lies in precisely one $I(A)$, $A \in S$, viz. $I(M)$. We shall now show that this leads to a contradiction.

Since $x \in I(M)$, there exists an element $m_0 \in M (\subseteq B)$ such that $x \leqslant m_0$. Since x is not in any $I(A_i)$ and each $I(A_i)$ is an ideal, it follows that m_0 is not in any $I(A_i)$, $A_i \neq M$. Hence

(1) $$m_0 \in I(M), \quad \text{and} \quad m_0 \notin \text{any } I(A_i), \quad A_i \neq M, \ A_i \in S.$$

Since M is a non principal ideal of B, given this $m_0 \in M$ there exists an element $m_1 \in M (\subseteq B)$ such that $m_0 < m_1$. Since $m_0, m_1 \in B$ and $m_0 < m_1$, there exists a maximal ideal P of B (therefore $P \in S$) containing m_0 and not containing m_1. Consider $I(P)$. Since $m_0 \in P \subseteq I(P)$, it follows by (1) that $I(P) = I(M)$. Hence $m_1 \in M \subseteq I(M) = I(P)$. Therefore $m_1 \leqslant$ some $p \in P$. Since $m_1 \in B$, this implies $m_1 \in P$ contrary to the choice of m_1. This contradiction arose from that assumption that B is infinite. Therefore B is finite and hence the result.

As an immediate consequence of (3.9) we have

Corollary. *If a Boolean algebra B is linear-compact in the discrete topology then it is finite.*

(3.10) Proposition. *Let (L, T) be a Hausdorff linear-compact C*-lattice with 1. Then the centre of (L, T) is compact.*

Proof. By (3.7), L is the projective limit of L_α $(\alpha \in A)$ where the L_α's are discrete linear compact *-lattices. Now by (3.9), the centre B_α of L_α is finite. It is easy to see that the centre B of L is the projective limit of B_α, $\alpha \in A$. As the projective limit of finite discrete spaces is compact, the result follows.

Corollary. *Let (B, T) be a Boolean algebra with the Hausdorff congruence topology T. Then (B, T) is linear-compact if and only if it is compact.*

The proof of the following lemma can easily be verified.

(3.11) Lemma. *Let (L_1, T_1), (L_2, T_2) be C*-lattices and let f be a homomorphism of L_1 on L_2 which is continuous at the zero of L_1. Then f is uniformly continuous with respect to the congruence uniformities of (L_1, T_1) and (L_2, T_2).*

As an immediate consequence of this we have the

Corollary. *Let (L_1, T_1) and (L_2, T_2) be C*-lattices and let f be a homomorphism of L_1 on L_2. Then f is a continuous mapping of (L_1, T_1) on (L_2, T_2) if and only if it is continuous at the zero of L_1.*

Now we shall prove

(3.12) Proposition. *The compact Hausdorff complemented modular C*-lattices are precisely the direct products of finite complemented modular lattices (each with the discrete topology), the decomposition being both algebraic and topological.*

Proof. Any direct product of finite complemented modular lattices is easily seen to be a compact Hausdorff complemented modular lattice. To prove the converse let $[\theta_i]$ $(i \in I)$ be a base of nuclear congruences of (L, T) and let $\theta_i(0) = N_i$ for each $i \in I$. Then the quotient space L/N_i is discrete and is also compact being a continuous image of L. Hence it is finite. Also each L/N_i being a homomorphic image of the complemented modular lattice L is complemented modular. Thus each $L/N_i = L_i$ is a finite (and hence continuous) complemented modular lattice. Therefore it follows that the intersection $\bigcap_{k \in K} M_{ik}^*$ of all the maximal congruence ideals of L_i (which are precisely its maximal neutral ideals) is the zero of L_i (cf. (2.11)). Let f_i be the natural homomorphism $L \to L_i$. Since there exists a 1-1 correspondence between the congruence ideals of L_i and the congruence ideals of L containing N_i, $M_{ik} = f_i^{-1}(M_{ik}^*)$ is a proper maximal congruence ideal of L and $\bigcap_{k \in K} M_{ik} = N_i$. Hence

(I) $$\bigcap_{i,k} M_{ik} = \bigcap_{i \in I} N_i = 0 \quad \text{as } (L, T) \text{ is Hausdorff.}$$

Further since L_i is discrete and f_i is continuous, it follows that $M_{ik} = f_i^{-1}(M_{ik}^*)$ is open in (L, T) for every i, k. Hence we have proved the existence of maximal congruence ideals in L which are open in (L, T) and also established that the intersection of all the open maximal congruence ideals of L is zero (since the intersection of a subcollection of them is zero by (I)).

Let $[M_j]$ $(j \in J)$ be the set of all open maximal congruence ideals of (L, T). Then as $\bigcap_{j \in J} M_j = 0$ it follows that L is isomorphic to a sublattice of the direct product P of the lattices L/M_j $(j \in J)$, the isomorphism being defined by the correspondence $f\colon x \to [(x)_j]$ $(j \in J)$ (where $(x)_j$ is the congruence class containing x with respect to the congruence determined by M_j). Since M_j is an open congruence ideal of L, it is also closed. Hence each quotient space $L/M_j = L_j$ is discrete. Let T_1 be the topology of $f(L)$ as a subspace of P.

Since L is a *-lattice (being complemented modular), by the corollary to (3.11), in order to verify the continuity of f it suffices to verify its continuity at the zero of L.

Now any fundamental neighbourhood of zero of $(f(L), T_1)$ is of the form $V \cap f(L)$ where $V = \prod_{j \in J} V_j(0_j)$, $V_j(0_j) = L_j$ for $(j \neq j_i)$ $(i = 1, \ldots, n)$ and $V_{j_i}(0_{j_i}) = 0_{j_i}$ $(i = 1, \ldots, n)$. Given $V \cap f(L)$, let $M = \bigcap_{i=1}^{n} M_{j_i}$ (where M_{j_i} is the congruence ideal corresponding to $L_{j_i} = L/M_{j_i}$). M is open in (L, T) being the intersection of a finite number of open sets. Further if $m \in M$, then $f(m) = [(m)_j]$ $(j \in J)$. Since $m \in M$, $m \in M_{j_i}$ $(i = 1, \ldots, n)$.

Hence $(m)_{j_i} = 0_{j_i}$ $(i = 1, \ldots, n)$. Therefore $f(m) \in V \cap f(L)$ and hence f is continuous.

Now we shall show that $f(L)$ is dense in P. Let $p = (p_i)$, $(i \in I)$, $\in P$. Then any neighbourhood of p in P is of the form $U(p) = \prod_{j \in J} U_j(p_j)$. where $U_j(p_j) = L_j$ $(j \neq j_i)$ $(i = 1, \ldots, n)$ and $U_{j_i}(p_{j_i}) = p_{j_i}$ $(i = 1, \ldots, n)$, Now since M_{j_i} $(i = 1, \ldots, n)$ are maximal congruence ideals of L and the congruences on L are permutable, it follows that $L/\bigcap_{i=1}^{n} M_{j_i}$ is isomorphic to the direct product $\prod_{i=1}^{n} L/M_{j_i}$ (cf. (2.7)), the isomorphism being given by the mapping, φ: $M(a) \to \big((a)_{j_1}, \ldots (a)_{j_n}\big)$ (where $M = \bigcap_{i=1}^{n} M_{j_i}$ and $M(a)$ is the residue class containing a with respect to the congruence defined by M). Hence given $(p_{j_1}, \ldots, p_{j_n})$ there exists an element $Q \in L/M$ such that $Q = M(q)$ $(q \in L)$ and $\varphi\big(M(q)\big) = \big((q)_{j_1}, \ldots, (q)_{j_n}\big) = (p_{j_1}, \ldots, p_{j_n})$. Hence $(q)_{j_i} = p_{j_i}$ $(i = 1, \ldots, n)$, and hence $f(q) \in U(p) \cap f(L)$, i.e. $U(p) \cap$ $\cap f(L) \neq \emptyset$. This is true for each fundamental neighbourhood $U(p)$ of p. Therefore $p \in \overline{f(L)}$, i.e. $P \subseteq \overline{f(L)}$. Hence $P = \overline{f(L)}$ and, therefore, $f(L)$ is dense in P.

Since (L, T) is compact and f is continuous, it follows that $\big(f(L), T_1\big)$ is compact and is, therefore, closed in P as P is Hausdorff. Thus $f(L) = \overline{f(L)} = P$. Now (L, T) is compact and f is a (1-1) continuous mapping of (L, T) on the Hausdorff space P. Therefore f is a homeomorphism and this proves the result.

COROLLARY i. *A complemented modular lattice L which admits a compact Hausdorff congruence topology T is a generalized continuous geometry.*

Proof. Since (L, T) is the direct product of the finite complemented lattices L_j $(j \in J)$ and since each L_j being finite is continuous, it follows that L being the direct product of the L_j's is continuous. Therefore L is a generalized continuous geometry.

COROLLARY ii. *The compact Hausdorff* C*-Boolean algebras are precisely those of the form B_0^N, where B_0 is a two-element Boolean algebra and N is a cardinal.*

Proof. By (3.12), B is the direct product of B/M_j, $j \in J$. Since each M_j is a maximal ideal of B, each B/M_j is a two-element Boolean algebra and hence the result.

COROLLARY iii. *The centre of a linear compact* C*-*lattice with* 1 *is of the form B_0^N where B_0 is a two-element Boolean algebra.*

This is an immediate consequence of (3.10) and corollary ii (3.12).

Now we shall proceed to study the notion of linear-compactness in complemented C*-lattices. We begin with

(3.13) LEMMA. *If (L, T) is a Hausdorff* C*-*lattice with* 1 *and z is an element of the centre B of L then the principal ideal $(z)_\lambda$ generated by z in L is closed in (L, T).*

Proof. Let $\overline{(z)_\lambda}$ be the closure of $(z)_\lambda$ in (L, T). If $x \in \overline{(z)_\lambda}$, then for any ideal A_i corresponding to a congruence in the base of nuclear congruences of (L, T), there exists $a_i \in A_i$ such that $x \leqslant z \vee a_i$. Therefore $x \wedge z' \leqslant a_i \wedge z'$. Hence $x \wedge z' \in A_i$. This implies that $x \wedge z' \in \bigcap_{i \in I} A_i$. As (L, T) is Hausdorff we have that $x \wedge z' = 0$. But $x = x \wedge (z \vee z') = (x \wedge z) \vee (x \wedge z') = x \wedge z$, i.e. $x \leqslant z$. Hence $x \in (z)_\lambda$. Thus $(z)_\lambda$ is closed.

Now we shall prove

(3.14) PROPOSITION. *Let (L, T) be a Hausdorff linear-compact complemented* C*-*lattice. Then (L, T) is equivalent to the direct product $\prod_a L_a$, where L_a's are Hausdorff linear-compact irreducible* C-*lattices, the decomposition being both algebraic and topological.*

Proof. From corollary iii to (3.12) it follows that the centre B of L is of the form B_0^N (both algebraically and topologically). Let (z_a) be the set of all atoms of B and let L_a be $L/(z_a')_\lambda$, with the induced topology. Then L_a is a Hausdorff (see (3.13)) linear-compact complemented C*-lattice with two-element centre (and hence irreducible) and the canonical map f_a: $L \to L_a$ is continuous and open. Hence the map f: $L \to \prod_a L_a$ defined by $f(x) = \{f_a(x)\}$ is also continuous. It is easy to see that $f(L)$ is dense in $\prod_a L_a$. Therefore from (3.2), (3.3) and (3.5) we have $f(L) = \prod_a L_a$ (algebraically). We shall now show that for any ideal A_i which is the zero class of a congruence in the base of nuclear congruences of (L, T), $f(A_i)$ is open in $\prod_a L_a$. Since $A_i \cap B$ is open in B and $B = B_0^N$, we notice that A_i contains an element of the form $z_{a1}' \wedge \ldots \wedge z_{an}'$ where $z_{a1}, \ldots, z_{an}$ is a finite set of atoms of B. Now since $f(A_i)$ is an ideal in $\prod_a L_a$, we have $f(A_i) = f_{a1}(A_i) \times \ldots \times f_{an}(A_i) \times \prod_{a \neq ai} L_a$. Since $f_{ai}(A_i)$ is open in L_{ai} (as f_{ai} is an open map) for $i = 1, \ldots, n$, we get that $f(A_i)$ is open. Hence f is an open and continuous map of L onto $\prod L_a$. It remains to show that f is 1-1.

Since L is a *-lattice, it suffices to show that $f^{-1}(0) = 0$. Suppose that $f(x) = 0$. Then $x \leqslant z_a'$, for any atom z_a of B. Therefore $x \leqslant (z_{a1}') \wedge \ldots \wedge (z_{an}')$ for any finite sequence $z_{a1}, \ldots, z_{an}$ of atoms of B. If A is an open ideal in (L, T) then $A \cap B$ is open in B and hence it contains an element of the form $(z_{a1}') \wedge \ldots \wedge (z_{an}')$, where $z_{a1}, \ldots, z_{an}$ are as above. Thus $x \in A$ for any open ideal A, and as the space is Hausdorff this implies that $x = 0$ and hence the result.

Definition. A C*-lattice (L, T) is said to be a *PC*-lattice* (or a *C*-lattice with a principal congruence topology*) if it has a base of nuclear congruences whose congruence ideals are principal ideals.

Then we have:

(3.15) Proposition. *A Hausdorff linear-compact complemented* C**-lattice* (L, T) *is equivalent to* $\prod_a L_a$, *where* L_a *are discrete irreducible linear-compact complemented* C**-lattices, if and only if* (L, T) *is a* PC**-lattice.*

Proof. The "only if" part is obvious. The "if" part follows from (3.14) and the fact that each L_a being an irreducible PC*-lattice, its only principal congruence ideals are (0) and L_a.

4. Linear-compact C*-generalized continuous geometries.

In this section we shall study the notion of linear-compactness in the particular case of a generalized continuous geometry. We begin with the following

(4.1) Lemma. *Let L be a generalized continuous geometry and let $(p)_\lambda$ be a principal congruence ideal of L. Then $L/(p)_\lambda = L_p$ is a generalized continuous geometry.*

Proof. Since L is complemented modular L_p, being a homomorphic image of L, is also complemented modular.

Now we shall prove that L_p is a complete lattice.

Let f be the natural homomorphism $L \to L_p$, and X_i $(i \in I)$ be elements of L_p. Then as f is onto for each i, there exists $x_i \in L$ such that $f(x_i) = X_i$. Consider $\bigvee_{i \in I} x_i$ (this exists as L is a complete lattice). Since f preserves order, $f(\bigvee_{i \in I} x_i) \geqslant f(x_i)$ for each i. Let $Y \in L_p$ such that $Y \geqslant$ each $X_i = f(x_i)$ $(i \in I)$. Now $Y = f(y)$ for some $y \in L$ and $f(y \vee x_i) = f(y) \vee f(x_i) = f(y)$ for each i. Hence $y \vee x_i \vee p = y \vee p$ for each $i \in I$. Hence $y \vee (\bigvee_{i \in I} x_i) \vee p = y \vee p$ and therefore

$$f(y) \vee f(\bigvee_{i \in I} x_i) \vee f(p) = f(y) \vee f(p), \quad \text{i.e.} \quad f(y) \vee f(\bigvee_{i \in I} x_i) = f(y)$$

(as $f(p) = 0_{L_p}$). Therefore $Y = f(y) \geqslant f(\bigvee_{i \in I} x_i)$. Hence

$$\text{(I)} \qquad \bigvee_{i \in I} X_i \text{ exists in } L_p \text{ and is equal to } f(\bigvee x_i).$$

Since L contains a zero element, it follows that it is a complete lattice. We shall now show that $\bigwedge_{i \in I} X_i = f(\bigwedge_{i \in I} x_i)$. Clearly $f(\bigwedge_{i \in I} x_i) \leqslant f(x_i)$ for each $i \in I$. Let $Y \in L_p$ be such that $Y \leqslant f(x_i)$ for each $i \in I$, $Y = f(y)$ for some $y \in L$. Since $f(y \wedge x_i) = f(y) \wedge f(x_i)$, $Y = f(y)$; hence $y \wedge x_i \equiv y \pmod{(p)_\lambda}$, i.e. $(y \wedge x_i) \vee p = y \vee p$ for each i. Since $(p)_\lambda$ is a principal congruence ideal of L and L is complemented modular, it follows

that p is a central element of L. Hence $(y \vee p) \wedge (x_i \vee p) = (y \wedge x_i) \vee p = y \vee p$ for each $i \in I$. Therefore $\bigwedge_{i \in I} (x_i \vee p) \geqslant y \vee p$. Since p is a central element and L is a generalized continuous geometry, it follows that $\bigwedge_{i \in I} (x_i \vee p) = (\bigwedge_{i \in I} x_i) \vee p$ (cf. [8]). Hence we have $f[(\bigwedge_{i \in I} x_i) \vee p] \geqslant f(y \vee p)$, i.e. $f(\bigwedge_{i \in I} x_i) \geqslant f(y) = Y$. Therefore

$$\text{(II)} \qquad \bigwedge_{i \in I} X_i = f(\bigwedge_{i \in I} x_i).$$

Now we shall proceed to establish the continuity of the lattice L_p. Let $X_i \uparrow X$, X_i $(i \in I)$, $X \in L_p$ and let C be any arbitrary element of L_p. Let x_i $(i \in I)$, $c \in L$ be such that $f(x_i) = X_i$, $f(c) = C$. Consider the set $[x_i \vee p]$ $(i \in I)$. Let $i \leqslant j$. Then since $X_i \uparrow X$, we have $X_i \leqslant X_j$ for $i \leqslant j$ (i.e. $[X_i]$ is monotonic increasing). Hence $X_i \vee X_j = X_j$, i.e. $f(x_i \vee x_j) = f(x_j)$. Hence $x_i \vee x_j \vee p = x_j \vee p$ for $i \leqslant j$, i.e. $x_i \vee p \leqslant x_j \vee p$ for $i \leqslant j$ (and $f(x_i \vee p) = X_i$). Thus $(x_i \vee p)$ $(i \in I)$ is monotonic increasing. Let $f(x) = X = \bigvee_{i \in I} X_i$ $(x \in L)$. Then $f(x) = \bigvee_{i \in I} X_i = \bigvee_{i \in I} f(x_i \vee p) = f[\bigvee_{i \in I} (x_i \vee p)]$ (from (I)). Hence $x \vee p = [\bigvee_{i \in I} (x_i \vee p) \vee p] = \bigvee_{i \in I} (x_i \vee p)$, i.e. $x_i \vee p \uparrow x \vee p$. Since L is continuous, it follows that $(x_i \vee p) \wedge c \uparrow (x \vee p) \wedge c$. Hence $\bigvee_{i \in I} [(x_i \vee p) \wedge c] = (x \vee p) \wedge c$. Since p is a central element of L, $[\bigvee_{i \in I} (x_i \wedge c) \vee (p \wedge c)] = \bigvee_{i \in I} [(x_i \wedge c) \vee (p \wedge c)] = \bigvee_{i \in I} [(x_i \vee p) \wedge c] = (x \vee p) \wedge c = (x \wedge c) \vee (p \wedge c)$. Adding p on both sides we have $\big(\bigvee_{i \in I} (x_i \wedge c)\big) \vee p = (x \wedge c) \vee p$. Hence $f[\bigvee_{i \in I} (x_i \wedge c)] = f(x \wedge c)$. Now $\bigvee_{i \in I} (X_i \wedge C) = \bigvee_{i \in I} f(x_i \wedge c) = f\big(\bigvee_{i \in I} (x_i \wedge c)\big)$ (from (I)) $= f(x \wedge c) = f(x) \wedge f(c) = X \wedge C$. Hence $X_i \wedge C \uparrow X \wedge C$. Thus L_p is upper continuous.

Suppose that $X_i \downarrow X$ and $C = f(c) \in L_p$. As before we can show that $(x \vee p)$ $(i \in I)$ is monotonic decreasing. Since $\bigwedge_{i \in I} X_i = X$, $\bigwedge_{i \in I} f(x_i \vee p) = f(x \vee p)$. Hence from (II) $f\big(\bigwedge_{i \in I} (x_i \vee p)\big) = f(x \vee p)$, i.e. $\big(\bigwedge_{i \in I} (x_i \vee p)\big) \vee p = x \vee p$. Now $\bigwedge_{i \in I} (x_i \vee p) \geqslant p$. Hence $\bigwedge_{i \in I} (x_i \vee p) = x \vee p$. Thus $x_i \vee p \downarrow x \vee p$. The continuity of L implies that $x_i \vee p \vee c \downarrow x \vee p \vee c$. Hence $\bigwedge_{i \in I} (x_i \vee p \vee c) = (x \vee p \vee c)$. Since p is central, we have $\big(\bigwedge_{i \in I} (x_i \vee c)\big) \vee p = \bigwedge_{i \in I} (x_i \vee c \vee p) = (x \vee c) \vee p$. Hence $f\big(\bigwedge_{i \in I} (x_i \vee c)\big) = f(x \vee c)$. Now $\bigwedge_{i \in I} (X_i \vee C) = \bigwedge_{i \in I} f(x_i \vee c) = f\big(\bigwedge_{i \in I} (x_i \vee c)\big)$ (from (II)) $= f(x \vee c) = f(x) \vee f(c) = X \vee C$. Hence $X_i \vee C \downarrow X \vee C$. Thus L_p is also lower continuous and this proves the lemma.

Now we shall prove

(4.2) Proposition. *The Hausdorff linear-compact* C*-*generalized continuous geometries are precisely the direct products of discrete continuous geometries both algebraically and topologically.*

Proof. It is easy to see that a direct product of discrete continuous geometries is a Hausdorff linear-compact C*-generalized continuous geometry. To prove the converse, it follows from (3.14) that (L, T) is equivalent to the direct product $\prod L_a$ where each $L_a = L/(z'_a)_\lambda$ is irreducible. L_a is by (4.1) a generalized continuous geometry and is therefore simple (cf. (2.10)). Thus each L_a is a continuous geometry and consequently is also discrete and this proves the result.

Corollary. *Every Hausdorff linear-compact* C*-*generalized continuous geometry is a* PC*-*lattice.*

Now we shall give another characterization of the Hausdorff linear-compact PC*-generalized continuous geometries in terms of their centres. We have

(4.3) Proposition. *A Hausdorff* PC*-*generalized continuous geometry is linear-compact if and only if its centre is compact.*

Proof. The "only if" part follows from (3.10). To prove the converse let (L, T) be a Hausdorff PC*-generalized continuous geometry with compact centre B. Then B, being a compact Hausdorff C*-Boolean algebra is of the form B_0^N, where B_0 is a two-element Boolean algebra and N is an cardinal, and the topology of B is the Cartesian product topology of B_0^N (cf. Corollary iii (3.12)). Hence it follows that a base of neighbourhoods of zero of B can be taken as the principal ideals $(a)_B$ of B (generated by a) where $a =$ the complement of some finite sum of atoms of B.

We shall now show that the principal congruence ideals $(a)_\lambda$ generated by these elements a in L can be taken as the neighbourhoods of zero of L. Let a be one such element. Then given $(a)_B$ there exists some (principal) congruence ideal $(b)_\lambda$ (the zero class of some congruence in the base of nuclear congruences for (L, T)) such that $(a)_B \supseteq (b)_\lambda \cap B$. Since $(b)_\lambda$ is a principal congruence ideal of L, b is a central element of L. Hence $b \in B$. Since $b \in B$, $b \in B \cap (b)_\lambda \subseteq (a)_B$. Hence $b \leqslant a$ and therefore $(b)_\lambda \subseteq (a)_\lambda$. Conversely given $(b)_\lambda$, $(b)_\lambda \cap B \supseteq$ some $(a)_B$. Hence $a \in B$, $a \in (b)_\lambda$, i.e. $(a)_\lambda \supseteq (b)_\lambda$. Therefore we can take the congruence ideals $(a)_\lambda$ (a running through the complements of finite sums of atoms of B) as the neighbourhoods of zero of L. Thus we have proved that (1) the centre B of L is of the form B_0^N for some cardinal N and (2) the neighbourhoods of zero of (L, T) are the congruence ideals generated by the complements of finite sums of atoms of B. Therefore it follows from (2.13) and (2.14) that L is the direct product $L_a = L(0, z_a)$ $(a \in I)$ algebraically. From (2) it follows that T is equivalent to the Cartesian product topology of the discrete lattices L_a. As before, each L_a can easily be seen to be a continuous geometry. Therefore (L, T) is the direct product of discrete continuous geometries, the decomposition being both algebraic and topological and is therefore by (4.2) linear-compact. This completes the proof.

In the case of compact spaces it is well known that a 1-1 continuous map of a compact space into a Hausdorff space is a homeomorphism. Now with the help of (4.1) we shall obtain a similar property in the case of the linear-compact Hausdorff PC*-generalized continuous geometries. We have

(4.4) Proposition. *Let (L, T) be a linear-compact Hausdorff* PC*-*generalized continuous geometry and let f be a continuous algebraic isomorphism of (L, T) on another Hausdorff* C*-*lattice (L_1, T_1). Then f is a uni-morphism of (L, V) on (L_1, V_1) where V, V_1 are the congruence uniformities of (L, T) and (L_1, T_1), respectively.*

Proof. Since (L, T) is a PC*-lattice, it has a base of nuclear congruences $[\theta_i]$ $(i \in I)$ such that $\theta_i(0)$ are principal ideals $(p_i)_\lambda$ of L. It sufficies to show that each $f\big((p_i)_\lambda\big)$ is open. Consider $(p_i)_\lambda$. Since (L, T) is linear-compact, $(p_i)_\lambda$ is closed in (L, T) (being the zero class of a nuclear congruence) and L is complemented modular, it follows from (3.6) that $(p_i)_\lambda$ is linear-compact. Hence $f\big((p_i)_\lambda\big)$, being a continuous homomorphic image of the C*-lattice $(p_i)_\lambda$ is a linear-compact C*-lattice. Since (L_1, T_1) is Hausdorff, it follows that $f\big((p_i)_\lambda\big)$, being a congruence ideal of L_1 is closed in (L_1, T_1) (cf. corollary to (3.5)).

Next, as (L, T) is linear-compact the quotient space $L/(p_i)_\lambda$ is linear-compact and is also discrete. Further by (4.1) $L/(p_i)_\lambda$ is a generalized continuous geometry. Hence it follows from (4.2) that $L/(p_i)_\lambda$ is the direct product of a finite number of continuous geometries. Therefore the lattice of congruences of $L/(p_i)_\lambda$ is a finite Boolean algebra (cf. [3]). Since f is an isomorphism of L on L_1, it follows that $L/(p_i)_\lambda$ and $L_1/f\big((p_i)_\lambda\big)$ are also isomorphic. Hence $L_1/f\big((p_i)_\lambda\big)$ also has only a finite number of congruences. Now as $L_1/f\big((p_i)_\lambda\big)$ is a Hausdorff C*-lattice (as $f\big((p_i)_\lambda\big)$ is closed), this implies that $L_1/f\big((p_i)_\lambda\big)$ is discrete. Hence $f\big((p_i)_\lambda\big)$ is open in (L_1, T_1) and hence the result.

(4.2) shows that any linear-compact Hausdorff C*-generalized continuous geometry is the direct product of simple lattices and is hence a PC*-lattice. Further we have also shown (cf. (3.1)) that any Hausdorff compact complemented modular C*-lattice is also a PC*-lattice. Therefore the question naturally arises as to whether every Hausdorff linear-compact C*-lattice is a PC*-lattice. This problem remains open and an answer to the following questions will also aid us in its solution.

QUESTION i. *Does a Hausdorff linear-compact relatively complemented lattice with zero have a unit* 1?

QUESTION ii. *Is the lattice of congruences of a discrete linear-compact* C*-*lattice finite?*

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On a singular plane continuum *

by

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§ 1. Introduction. Using slightly extended Bernstein's argument on the decomposition of a plane into two disjoint and totally imperfect (1) subsets (cf. [3], p. 422), it is easy to decompose each complete separable space Y having the property:

(1) *if a set $A \subset Y$ separates Y, then A contains a perfect subset,*

into a countable sequence of disjoint, connected, ponctiform (2) and dense subsets. Such are, for instance, all manifolds (in particular, euclidean spaces) of dimension $n \geqslant 2$, the universal curve of Sierpiński ("a carpet"; see [4], p. 202) and many others. The points of these spaces are of continuumrange.

On the other hand, however, such a decomposition is impossible for a regular curve (3). Moreover, a regular curve even does not contain a countable sequence of disjoint and connected sets $\{S_k\}_{k=1,2,\ldots}$ of diameter $\delta(S_k) > \varepsilon > 0$ (4). Thus a natural question arises whether decomposition:

(2) $X = \bigcup_{k=1}^{\infty} S_k$, where S_k are mutually disjoint, connected, ponctiform and dense subsets of X

(hence of diameter $\delta(S_k) = \delta(X)$), is possible for a continuum X not possessing property (1)? Is it possible for a rational curve (5), which,

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(1) A subset A of a space Y is said to be *totally imperfect* provided that it does not contain any perfect subset of Y (cf. [3], p. 421).

(2) A set A is said to be *ponctiform* provided that each of its subcontinua consists of one point only (cf. [4], p. 130).

(3) A continuum Y is said to be *regular curve* provided that each its point is of finite or ω range or, in other words, that each its point has arbitrarily small neighbourhoods, the boundaries of which are finite ([4], p. 201). In particular, $\dim Y \leqslant 1$.

(4) For suppose that a regular curve Y does. As a compact, it contains then a point $p \in Y$ such that each neighbourhood G of p meets infinitely many S_k. Taking G of diameter $\delta(G) < \varepsilon$, we have, by our assumption and connectedness of S_k, $\mathrm{Fr}(G) \cap S_k \neq 0$ for infinitely many S_k, and therefore $\mathrm{Fr}(S_k)$ must be infinite (sets S_k are disjoint). A contradiction.

(5) A continuum Y is said to be a *rational curve* provided that each its point is of at most countable range or, in other words, that each point has arbitrarily small neighbourhoods, the boundaries of which are finite or countable ([4], p. 201).