# On an irreducible absolute retract

by

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In 1950 K. Borsuk constructed [1] in the 3-dimensional Euclidean space $E^3$ a 2-dimensional absolute retract such that every 2-dimensional closed set of it has an infinite 1-dimensional Betti number. The purpose of this paper is to show that Borsuk's construction can be built in an $(n+1)$-dimensional space, and with a slight modification also in a Hilbert space, so that in the first case we obtain an $n$-dimensional absolute retract such that every $n$-dimensional closed proper subset of it has an infinite $(n-1)$-dimensional Betti number; in the second case we obtain an infinite-dimensional absolute retract such that the $k$-dimensional Betti number of every closed proper subset of it containing an inner point of it, converges to an infinity with $k$.

The constructions and the proofs are suitably adapted constructions and proofs of [1].

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**1. Irreducible cuttings.** A subcompactum $C$ of the $(n+1)$-dimensional space $E^{n+1}$ is said to be an *irreducible cutting of* $E^{n+1}$ provided that $E^{n+1}-C$ is not connected but for every closed proper subset $A$ of $C$ the set $E^{n+1}-A$ is connected. Any irreducible cutting of $E^{n+1}$ is an $n$-dimensional Cantor-manifold.

If the irreducible cutting $C$ of $E^{n+1}$ is a polytope, then $E^{n+1}-C$ contains exactly two regions for which $C$ is a common boundary. One of these regions has a finite diameter; it will be called the *interior region* and denoted by $\Gamma$. The other region, with an infinite diameter, will be called the *exterior region* and denoted by $\Lambda$.

Let $a \in E^{n+1}$. Let us put the space $E^{n+1}$ isometrically in the space $E^{n+2}$, and let $L$ be the line in $E^{n+2}$ orthogonal to $E^{n+1}$ and passing through $a$. Let $b^+$ and $b^-$ be two different points in $L$ at a distance 1 from $a$. Let $F$ be an arbitrary subset of $E^{n+1}$. The subset of $E^{n+2}$ consisting of all segments $\overline{xb^+}$ and all segments $\overline{xb^-}$ where $x \in F$ is called the *suspension* of $F$ and denoted by $S(F)$. If $C$ is a polyhedral irreducible cutting of $E^{n+1}$ and $a \in \Gamma$, then the suspension $S(C)$ is a polyhedral irreducible cutting of $E^{n+2}$ which disconnects $E^{n+2}$ into exactly two regions, $S(\Gamma)$ and $E^{n+2}-S(\Gamma\cup C)$.

Let $\tau(C)$ be a triangulation of a polyhedral irreducible cutting $C$ of $E^{n+1}$. For every simplex $T\in\tau(C)$, let us denote its barycentre by $b_T$ and let $|T|$ denote the space of $T$. For every line $L$ in $E^{n+1}$ such that $L\cap U_T=b_T$ for each neighbourhood $U_T$ of $b_T$ in $C$, there exist on $L$ two points, $b'$ and $b''$, different from $b_T$ and such that the interior of the simplex $|Tb'|$ lies in the interior region $\Gamma$ and the interior of the simplex $|Tb''|$ lies in the exterior region $\Lambda$. We shall say that every point belonging to the interior of $|Tb'|$ lies on the *interior side* of the simplex $T$. By an *inner ray* of $T$ we understand any ray starting from $b_T$ and containining at least one point $b'$ lying on the interior side of $T$.

Two $n$-dimensional simplexes $T'$ and $T''$ of $\tau(C)$ are said *to be adjoined* in the dimension $k$ if they have a common $k$-dimensional face $J$ but they do not have any common face of dimension $>k$, and for every two points $b'$ and $b''$ such that $b'$ lies on the interior side of $T'$ and $b''$ on the interior side of $T''$, there exists in every neighbourhood of the barycentre of $J$ a point $b$ such that the polygonal line $\overline{bb'}+\overline{bb''}$ lies in the interior region $\Gamma$. It can easily be seen that to every $(n-1)$-dimensional face $J$ of every $n$-dimensional simplex $T'\in\tau(C)$ there exists exactly one simplex $T''\in\tau(C)$ such that $|T'|\cap|T''|=|J|$, and the simplexes $T'$ and $T''$ adjoin in the dimension $n-1$.

Let $T'$ and $T''$ be two $n$-dimensional simplexes of the triangulation $\tau(C)$ adjoining in the dimension $k$, and let $J$ denote their common $k$-dimensional face. Let $\pi'$ and $\pi''$ denote two $n$-dimensional half-hyperplanes containing respectively $|T'|$ and $|T''|$ and such that $|J|\subseteq\pi'\cap\pi''$. There exists exactly one $n$-dimensional hyperplane $\pi$ passing by $|J|$ and such that $\pi'$ and $\pi''$ lie symmetrically to $\pi$. This hyperplane $\pi$ will be called the *hyperplane separating the simplexes $T'$ and $T''$*.

**2. Zones.** Let $E^{n+1}$ be the hyperplane of $E^{n+m}$, where $m$ is a finite number or infinity. Let $E^{*m-1}$ denote the orthogonal complementary space to $E^{n+1}$ in $E^{n+m}$ and let $l_2,\ldots,l_m$ denote the orthogonal basis of $E^{*m-1}$. Let $\tau(C)$ be a triangulation of a polyhedral irreducible cutting $C$ of $E^{n+1}$ and $T$ a simplex of $\tau(C)$. Giving a sequence of non-negative numbers $\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m\}$ let us denote by an $m$-dimensional $k$-zone $Z_k^m(T,\{\varepsilon_i\})$ of $T$ the set defined in the following manner:

If $\dim T<k$ then $Z_k^m(T,\{\varepsilon_i\})=|T|$; if $\dim T=k$ then by $Z_k^m(T,\{\varepsilon_i\})$ we shall understand the minimal convex subset of $E^{n+m}$ containing the simplex $|T|$ and the segments $L_1,L_2,\ldots,L_m$ where $L_1$ is an inner ray with length $\varepsilon_1$, and $L_i$ $(i=2,\ldots,m)$ are the segments with length $2\varepsilon_i$, centre $b_T$ and direction of the vector $l_i$. We see at once that for an arbitrary choice of inner rays and for $\varepsilon_i$ sufficiently small the common part of the $m$-dimensional $k$-zones of different $k$-dimensional simplexes of $\tau$ coincide with the common part of the boundaries of those simplexes. The sequence $\{\varepsilon_i\}$ satisfying this condition is said to be suitable for the triangulation $\tau$.

If $\dim T>k$, then by $Z_k^m(T,\{\varepsilon_i\})$ we shall understand the minimal convex subset of $E^{n+m}$ containing the simplex $|T|$ and the sets $Z_k^m(T',\{\varepsilon_i\})$ of all $k$-dimensional faces $T'$ of $T$. Clearly, if the sequence $\{\varepsilon_i\}$ is suitable for all $k$-dimensional simplexes of $\tau$, then it is also suitable for all simplexes of $\tau$. If $K$ is a subcomplex of triangulation $\tau(C)$, then by the *$m$-dimensional $k$-zone* of $K$ we mean the polytope $Z_k^m(K,\{\varepsilon_i\})$ which is the sum of $m$-dimensional $k$-zones of all simplexes constituting the complex $K$ and satisfying the following conditions: (i) $Z_k^m(T',\{\varepsilon_i\})\cap Z_k^m(T,\{\varepsilon_i\})=Z_k^m(T\cap T',\{\varepsilon_i\})$ for $T,T'\in K$, (ii) the sequence $\{\varepsilon_i\}$ is suitable.

In this paper we use the following cases of zones:

1. 1-dimensional $(n-1)$-zone $Z_{n-1}^1(K,\varepsilon)$;
2. infinite dimensional $k$-zone $Z_k^\infty(K,\{\varepsilon_i\})$.

In case 2. we assume that the sequence $\{\varepsilon_i\}$ converges to 0.

3. 2-dimensional $k$-zone $Z_k^2(K,\{0,\varepsilon\})$;
4. 1-dimensional 2-zone $Z_2^1(K,\{\varepsilon\})$.

LEMMA 1. *If $C$ is a polyhedral irreducible cutting of a hyperspace $E^{n+1}$ in $E^{n+m}$ and $K$ is a subcomplex of a triangulation $\tau(C)$ of $C$, then for every sequence $\{\varepsilon_i\}$ $(i=1,2,\ldots,m)$ suitable for $\tau$ there exists a mapping $r(x,t)$ retracting by deformation the $m$-dimensional $k$-zone $Z_k^m(K,\{\varepsilon_i\})$ to $|K|$ in such a manner that $r(x,t)=x$ for every $x\in|K|$ and $0\leqslant t\leqslant 1$, and $r(x,t)\in Z_k^m(T,\{\varepsilon_i\})$ for every simplex $T\in\tau(C)$, every $x\in Z_k^m(T,\{\varepsilon_i\})$ and every $0\leqslant t\leqslant 1$.*

Proof. If $x\in Z_k^m(K^{(s)},\{\varepsilon_i\})$, $s<k$, where $K^{(s)}$ denotes the $s$-dimensional skeleton of $K$, then $Z_k^m(K^{(s)},\{\varepsilon_i\})=|K^{(s)}|$ and we put $r(x,t)=x$ for every $x\in Z_k^m(K^{(s)},\{\varepsilon_i\})$, $0\leqslant t\leqslant 1$.

If $\dim T=k$, then each point $x\in Z_k^m(T,\{\varepsilon_i\})$ is an element of the set $|T|\times I^m$ that is $x=(a,x_1,x_2,\ldots,x_m)$, where $a\in|T|$ and $x_i$ are real numbers and $x=(a,0,0,\ldots,0)$ for $x\in|T|$.

We put $r(x,t)=\big(a,(1-t)x_1,\ldots,(1-t)x_m\big)$ for every $x\in Z_k^m(T,\{\varepsilon_i\})$, $T\in K^{(k)}$, $0\leqslant t\leqslant 1$.

Suppose that we have defined the mapping $r(x,t)$ for $x\in Z_k^m(K^{(s)},\{\varepsilon_i\})$, $s\geqslant k$, and let $\dim T=s+1$. Consider the set $A=I\times|T|\cup\big(\bigcup_j I\times Z_k^m(T_j,\{\varepsilon_i\})\big)$ where the summing is over all $s$-dimensional face $T_j$ of $T$. Let us define the mapping $r_A(x,t)$ on $A$ by

$$r_A(x,t)=\begin{cases} x & \text{for}\quad x\in|T|,\ 0\leqslant t\leqslant 1,\\ r(x,t) & \text{for}\quad (x,t)\in\bigcup_j\big(I\times Z_k^m(T_j\{\varepsilon_i\})\big).\end{cases}$$

Using the homotopy extension theorem for polytopes we can extend this mapping to the mapping $r_T(x,t)$ defined on $I\times Z_k^m(T,\{\varepsilon_i\})$ so that $r_T(x,0)=x$ and $r_T(x,1)\in|T|$. Putting $r(x,t)=r_T(x,t)$ for $x\in Z_k^m(T,\{\varepsilon_i\})$, $0\leqslant t\leqslant 1$ for $T\in K^{(s+1)}$, we obtain the retraction by deformation $r(x,t)$ defined on $Z_k^m(K^{(s+1)},\{\varepsilon_i\})$ so that the conditions of the lemma are satisfied. We infer that we can define the mapping $r(x,t)$ retracting $Z_k^m(K,\{\varepsilon_i\})$ to $|K|$ satisfying the conditions of the lemma.

It follows that $|K|$ is an absolute retract if and only if $Z_k^m(K,\{\varepsilon_i\})$ is an absolute retract.

**3. Subpolytopes smoothly connected in dimension** $k$. Let $P$ be a homogenously $n$-dimensional subpolytope of a polyhedral irreducible cutting $C$ of the space $E^{n+1}$. Let $\tau(C)$ be a triangulation of $C$ such that $P$ is representable as a subcomplex $K$ of $\tau(C)$. The polytope $P$ is said to be *smoothly connected* in the dimension $k$ on $C$ if for every two $n$-dimensional simplexes $T$ and $T'$ of it $K$ there exists in $K$ a finite sequence of $n$-dimensional simplexes $T=T_0,T_1,\ldots,T_l,T_{l+1}=T'$ such that $T_i$ and $T_{l+1}$ adjoin in the dimension $\geqslant k$ for every $i=0,1,\ldots,l$. Obviously this property is independent of the choice of the triangulation $\tau(C)$; it depends only upon the polytopes $P$ and $C$.

If $C$ is a polyhedral irreducible cutting of $E^{n+1}$, let us denote by $\bar{S}(C)$ a set homeomorphic to the suspension $S(C)$ defined in the following manner. Consider the set $W(C)$ of pairs $(c,t)\in E^{n+2}$ where $c\in C$ and $t\in[-1,1]$. Let $a$ be a point of an interior region $\Gamma$ of $E^{n+1}-C$. Denote by $\Delta^+(C)$ the cone in $E^{n+2}$ with the base consisting of points of the form $(c,1)$ and with the vertex at the point $(a,2)$, and by $\Delta^-(C)$ the cone in $E^{n+2}$ with the base consisting of points of the form $(c,-1)$ and with the vertex at the point $(a,-2)$. We put $\bar{S}(C)=W(C)\cup\Delta^+(C)\cup\Delta^-(C)$. It is easy to see that if $\varepsilon<1$, then $Z_k^2(\tau(C),\{0,\varepsilon\})$ is a subpolytope of $\bar{S}(C)$ and that if the subpolytope $|K|$ is smoothly connected in the dimension $l$ in $C$, then the polytope $Z_k^2(K,\{0,\varepsilon\})$ is also smoothly connected in the dimension $l$ in $\bar{S}(C)$.

**4. Base of a rosary.** Let $C$ be a polyhedral irreducible cutting of the space $E^{n+1}$ $(n\geqslant 3)$ and let $P$ be a subpolytope of $C$ smoothly connected in the dimension $\geqslant 2$ in $C$. Consider the triangulation $\tau(C)$ such that $P$ constitutes a subcomplex $K$ of $\tau(C)$. Let $R$ denote the sum of all simplexes of $\tau(C)$ not belonging to $K$, $R=\overline{C-P}$. From the smooth connectivity in the dimension $\geqslant 2$ of $P$ we infer that there exists for every $n$-dimensional simplex $T\in K$ a polygonal simple arc $L_T\subset P-R$ such that:

1. $L_T$ has as its starting point $a'_T$ an interior point of the simplex $T$ and its end-point $a''_T$ lies in the interior of an $n$-dimensional simplex of $K$.



2. If $T$ and $T'$ are two different simplexes of $K$, then $L_T\cap L_{T'}=\emptyset$ and $L_T\cap|T'|\neq\emptyset$.

3. If $a$ is a point of $L_T$ lying on the $k$-dimensional simplex $J$, $k<n$, then there exists a neighbourhood $U$ of $a$ such that $U\cap L_T$ consists of two segments lying in the interiors of two $n$-simplexes adjoining in the dimension $k$. It follows that the arc $L_T$ does not pass through 1-dimensional simplexes or through the vertices of $K$.

We can assume that all arcs $L_T$, for each $T$, consist of $(s+2)$ segments, $L_T=L_{T,0}\cup L_{T,1}\cup\ldots\cup L_{T,s+1}$, having disjoint interiors and such that the interior of every segment $L_{T,i}$ lies in the interior of one of the $n$-dimensional simplexes of $K$. Denote the end-points of $L_{T,i}$ by $a_{T,i}$ and $a_{T,i+1}$, $a_{T,0}=a'_T$, $a_{T,s+2}=a''_T$, and the centre of $L_{T,i}$ by $b_{T,i}$. Denote by $Q_{T,i}$ the $n$-dimensional cube lying in $P$ with centre in $b_{T,i}$ one of its faces parallel to $L_{T,i}$, and $\alpha$ as the length of its edges. Obviously, if the number $\alpha$ is sufficiently small, then the cubes $Q_{T,i}$ are disjoint, lie in the interiors of $n$-simplexes of $K$, and the common part of $Q_{T,i}$ with the polygonal line $L_T$ is a subsegment of $L_{T,i}$ having $b_{T,i}$ as its centre and $\alpha$ as its length. Let us denote the end-points of this segment (ordered as they appear on the oriented segment $L_{T,i}$ from $a_{T,i}$ to $a_{T,i+1}$) by $a'_{T,i}$ and $a''_{T,i}$.

Let us put:

$$M_{T,i}=Q_{T,i}\cup\overline{a''_{T,i}a_{T,i+1}}\cup\overline{a_{T,i+1}a'_{T,i+1}}\cup\dot{Q}_{T,i+1}$$

(where $\dot{Q}_{T,i+1}$ denotes the boundary of the cube $Q_{T,i+1}$),

$$M_T=\bigcup_i M_{T,i}\,,\qquad M=\bigcup_T M_T\,.$$

We shall say that $M$ is a *base of a rosary* for the polytope $P$. The polytopes $M_T$ will be called *components* of $M$, and the polytopes $M_{T,i}$ *links* of $M$. The segments of the form $\overline{a_{T,i+1}a'_{T,i+1}}$ will be called the *entrance segments* and the segments of the form $\overline{b_{T,i}a_{T,i+1}}$ the *exit segments* of the base $M$ of the rosary.

**5. Rosary.** Let $\varepsilon$ be a positive number suitable to the triangulation $\tau(C)$. Consider the 1-dimensional 2-zone $Z_2^1(K,\varepsilon)$ of the complex $K$ and choose a positive number $\beta$ so small that, if $E$ is an arbitrary cube or segment of the base $M$ of the rosary contained in the simplex $|T_0|$ and $x$ is a point lying on the interior side of $T_0$ at a distance $\leqslant 2\beta$ from $E$, then $x\in Z_2^1(K,\varepsilon)$. In particular, (i) if $E$ lies in the interior of $|T_0|$ then $x\in Z_2^1(K,\varepsilon)$, (ii) if $x,y$ are two points belonging to two disjoint segments of the base of the rosary, then $\varrho(x,y)>2\beta$; (iii) the length of the edges of the cubes of $M$ is $>4\beta$.

Let $|T_0|$ be the $n$-simplex containing the cube $Q_{T,i}$ and $|T_0'|$ the $n$-simplex containing the entrance segment $J' = \overline{a_{T,i+1}a'_{T,i+1}}$ ($|T_0'|$ may be equal to $|T_0|$). The exit segment $J = \overline{b_{T,i}a_{T,i+1}}$ lies in $|T_0|$. Let us denote by $\Delta(Q_{T,i})$ the pyramid with base $Q_{T,i}$ lying on the interior side of $T_0$ and of height $\beta$. Denote its vertex by $b'_{T,i}$. Consider the $n$-dimensional hyperplane $\pi$ in $E^{n+1}$ parallel to $|T_0|$, lying on the interior side of $T_0$ at a distance $\beta$ from $|T_0|$ and the $n$-dimensional hyperplane $\pi'$ parallel to $|T_0'|$, lying on the interior side of $T'$ at a distance $\beta$ from $|T_0'|$. Denote by $Q^{\sim}_{T,i}$, $J^{\sim}_{\mu}$, $J^{\sim}$ the cube, the $(n-1)$ faces of this cube and the line lying on $\pi$, which are the results of a parallel translation of, respectively, the cube $Q_{T,i}$, its $(n-1)$ faces $J_\mu$ and the line containing the segment $J$, in the direction of the vector $\overline{b_{T,i}b'_{T,i}}$, and by $J^{\sim\prime}$ the line lying on $\pi'$ which is the result of a parallel translation of the line containing the segment $J'$ in the direction of the vector $\overline{b_{T,i+1}b'_{T,i+1}}$. Clearly, the edges of $Q^{\sim}_{T,i}$ are of length $\alpha$. Consider the cube $Q^{\sim\prime}_{T,i}$ lying in $\pi$ with the same centre as $Q^{\sim}_{T,i}$ with the edges parallel to the edges of $Q^{\sim}_{T,i}$ but of length $\alpha/2$. The set $\overline{Q^{\sim}_{T,i}-Q^{\sim\prime}_{T,i}}$ is the sum of $2n$ sets $\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_{2n}$, where $\bar{J}_1 = J^{\sim}_{\mu}\times I_\mu$, $\mu = 1, 2, \ldots, 2n$, and $I_\mu$ denotes a segment of length $\alpha/4$ lying in $Q^{\sim}_{T,i}$, orthogonal to $J^{\sim}_{\mu}$ with one end-point in $J^{\sim}_{\mu}$. Let us denote by $\nabla(J_\mu)$ the minimal convex set containing $J_\mu$ and $\bar{J}_\mu$. $\nabla(J_\mu)$ is a polytope homeomorphic to $I^{n-1}\times I^2$ lying in the zone $Z_2^1(T_0, \varepsilon)$ which forms a neighbourhood of it in $C \cup \Gamma$, and its common part with $|K|$ is equal to $J_\mu$. The set $\nabla(\dot{Q}_{T,i}) = \bigcup_{\mu=1}^{2n}(J_\mu)$ is homeomorphic to the set $S^{n-1}\times I^2$ ($S^{n-1}$ denotes the $(n-1)$-dimensional sphere).

The common part of the hyperplanes $\pi$ and $\pi'$ is a $(n-1)$-dimensional hyperplane parallel to the common face of the simplexes $T_0$ and $T_0'$. Let $d$ denote the common point of two lines $\tilde{J}$ and $\tilde{J}'$ and let $D$ denote a regular $(n-1)$-dimensional simplex lying in $\pi \cap \pi'$ with centre at $d$ and with edges of length $\alpha/8$. For each point $b$ of the ray $\overline{db'_{T,i}}$ denote by $D_b$ the result of a parallel translation of the simplex $D$ in the direction of the vector $\overline{db}$ and let $J$ be the closed part of $A$ bounded by the hyperplane $\pi \cap \pi'$ and by the $n$-hyperplane passing through $b_{T,i}$ orthogonal to the exit segment $J$ and to $|T_0|$. For each point $b'$ of the ray $\overline{db'_{T,i+1}}$ denote by $D'_{b'}$ the result of parallel translation of the simplex $D$ in the direction of the vector $\overline{db'}$. Denote by $A'$ the union of the $D'_{b'}$'s for each $b' \in \overline{db'_{T,i+1}}$ and let $\bar{J}'$ be the closed part of $A'$ bounded by the hyperplane $\pi \cap \pi'$ and by the $n$-hyperplane passing through $a'_{T,i+1}$ and orthogonal to $|T_0'|$ and to the entrance segment $J'$.

Denote by $\nabla(J)$ and $\nabla(J')$ the minimal convex sets containing, respectively, $\nabla(J)$ the sets $J$ and $\bar{J}$, $\nabla(J')$ the sets $J'$ and $\bar{J}'$. $\nabla(J)$ and $\nabla(J')$ are the polytopes contained: the first in the zone $Z_2^1(T_0, \varepsilon)$, the second in the zone $Z_2^1(T_0', \varepsilon)$ (these zones are their neighbourhoods in $C \cup \Gamma$), and their common parts with $|T_0|$ respective with $|T_0'|$ are $J$ and $J'$.

Now let $D^{\vee}$ denote a regular $(n-1)$-simplex in $\pi \cap \pi'$ with centre at $d$, with edges parallel to the edges of $D$ but of length $\alpha/4$. Denote by $J^{\vee}$ the set built by means of the set $D^{\vee}$ in the same manner as the set $\bar{J}$ was built by means of $D$, and let $\Delta(J^{\vee})$ be the minimal convex set containing $J$ and $J^{\vee}$.



Let us put

$$\nabla^*(\dot{Q}_{T,i}) = \overline{\nabla(\dot{Q}_{T,i}) - \nabla(J^{\vee})},$$
$$N_{T,i} = \Delta(Q_{T,i}) \cup \nabla(J) \cup \nabla(J') \cup \nabla^*(\dot{Q}_{T,i+1}),$$
$$N_T = \bigcup_i N_{T,i}, \qquad N = \bigcup_T N_T.$$

The polytope $N$ will be called the *rosary* for the polytope $P$. The polytopes $N_T$ will be called the *components*, and the polytopes $N_{T,i}$ the *links* of this rosary. The common part of the rosary $N$ and the polytope $C$ is the base $M$ of the rosary. The rosary $N$ lies in the zone $Z_2^1(K, \varepsilon)$ which constitutes a neighbourhood of $N$ in the set $C \cup \Gamma$.

In the same manner as in [1] we can prove the following

LEMMA 2. *There exists a mapping $r_1(x, t)$ retracting by deformation the polytope $P \cup N$ to the polytope $P$ in such a manner that*

$$\varrho\big(r_1(x, t), x\big) < \varepsilon \quad \textit{for every } x \in P \cup N \textit{ and } 0 \leqslant t \leqslant 1.$$

It follows that $P$ is an absolute retract if and only if $P \cup N$ is an absolute retract and hence the set $(P \cup N) \cap Z_2^1(T, \varepsilon)$, where $T \in K$, is an absolute retract.

LEMMA 3. *The link $N_{T,i}$ is homeomorphic to the $(n+1)$-dimensional ball in which an $(n-2)$-dimensional sphere belonging to the surface of this ball is reduced to a point.*

Proof. Denote by $J_0$ the $(n-1)$-dimensional face of $Q_{T,i+1}$ containing the point $a''_{T,i+1}$. It is easy to see that the set $\nabla(\dot{Q}_{T,i+1}) \cap \nabla(J^{\vee})$ is a con having as its base the convex set $\bar{J}_\nu \cap J^{\vee}$ and as its vertex the point $a''_{T,i+1}$. Consider the set $\dot{Q}_{T,i+1}$ homeomorphic to the $(n-1)$-dimensional sphere and an $(n-1)$-dimensional ball $Q^{n-1}$ constituting a neighbourhood of point $a''_{T,i+1}$ in $\dot{Q}_{T,i+1}$. Let us denote by $V$ the minimal convex set containing $Q^{n+1}$ and the set $\bar{J}_\nu \cap J^{\vee}$. Obviously, the set $Q^{n+1} = \overline{\nabla(\dot{Q}_{T,i+1}) - V}$ is homeomorphic to an $(n+1)$-dimensional ball. The boundary $\dot{Q}^{n-1}$ of $Q^{n-1}$, which is an $(n-2)$-dimensional sphere, lies on the boundary $\dot{Q}^{n+1}$ of $Q^{n+1}$. By identifying $\dot{Q}^{n-1}$ with a point we obtain from the set $\overline{\nabla(\dot{Q}_{T,i+1}) - V} = Q^{n+1}$ a new set homeomorphic to the set

$$\overline{\nabla(\dot{Q}_{T,i+1}) - [\nabla(\dot{Q}_{T,i+1}) \cap \nabla(J^{\vee})]} = \overline{\nabla(\dot{Q}_{T,i+1}) - \nabla(J^{\vee})} = \nabla^*(\dot{Q}_{T,i+1}).$$

But the set $\nabla^*(\dot{Q}_{T,i+1})$ is obviously homeomorphic to the link $N_{T,i}$ and thus the proof is finished.

From lemma 3 we infer the following

COROLLARY. *The set* $\dot{N}_{T,i}-(Q_{T,i}-\dot{Q}_{T,i})$, *where* $\dot{N}_{T,i}$ *denotes the boundary of* $N_{T,i}$, *is a deformation retract of* $N_{T,i}$.

**6. Subordinate polytope and subordinate zone.** Let $N^{\cdot}$ denote the boundary of the rosary $N$. The polytope

$$P' = P \cup N^{\cdot} - \bigcup_{T} \bigcup_{i=0}^{s+1} (Q_{T,i}-\dot{Q}_{T,i})$$

is said to be a *subordinate polytope* to $P$ corresponding to the triangulation and to the zone $Z_2^1(\tau, \varepsilon)$. We see at once that the polytope

$$C' = P' \cup R$$

is an irreducible cutting of the space $E^{n+1}$ with the interior region $\Gamma' = \Gamma - N$ and the exterior region $\Lambda' = \Lambda \cup (N-N^{\cdot}) \cup \bigcup_{T} \bigcup_{i=0}^{s-1} (Q_{T,i}-\dot{Q}_{T,i})$.

The polytope $P'$ is smoothly connected in a dimension $\geqslant 2$ on $C'$ and the $(n-1)$-dimensional skeleton of $P$ corresponding to the triangulation $\tau$ is a subpolytope of $P'$.

Using the same reasoning as in [1] we can prove the following

LEMMA 4. *There exists a retraction by deformation* $r_2(x, t)$ *of the polytope* $P \cup N$ *to the subordinate polytope* $P'$ *such that for* $x \in P \cup N$

$$\varrho\big(r_2(x, t), x\big) < 2(\varepsilon+\eta),$$

*where the diameters of the simplexes of the triangulation* $\tau$ *are all* $\leqslant \eta$.

It follows that $P'$ is an absolute retract if and only if $P \cup N$ is an absolute retract and consequently if and only if $P$ is an absolute retract.

Consider the triangulation $\tau'$ of the polytope $C'$ such that the sets $P'$, $P \cap P'$, $\overline{P'-P} \cap P$ and $P' \cap Z_2^1(T, \varepsilon)$ for each $T \in \tau$ are representable in the form of subcomplexes of $\tau$ and let $K'$ be the subcomplex correspondig to $P'$. We can assume that the diameters of all simplexes of the triangulation $\tau'$ are $\leqslant \eta$, and that for the $(n-1)$-skeleton of $K$ the triangulation $\tau'$ is a subdivision of the triangulation $\tau$.

LEMMA 5. *For every sufficiently small number* $\varepsilon' > 0$ *there exists a retraction* $r_4$ *of the zone* $Z_2^1(K, \varepsilon)$ *to the zone* $Z_2^1(K', \varepsilon')$ *satisfying the condition*

$$\varrho\big(r_4(x), x\big) < 4\varepsilon+2\eta \quad \textit{for every } x \in Z^1(K, \varepsilon).$$

Proof. Let $T'$ be a simplex of the triangulation $\tau'$. There exists a simplex $T \in K$ such that $|T'| \subset Z_2^1(T, \varepsilon)$ but $|T'|$ is not included in the 2-zone of any proper face of $T$. For simplexes $T'$ of dimension $<2$ we have $Z_2^1(T', \varepsilon') = |T'| \subset Z_2^1(T, \varepsilon)$; if $\dim T' = 2$ then there exists a positive number $\varepsilon'$ and an inner ray $L$ of $T'$ such that $Z_2^1(T', \varepsilon') \subset Z_2^1(T, \varepsilon)$. From the definition of the 1-dimensional 2-zone we infer that $Z_2^1(T', \varepsilon') \subset Z_2^1(T, \varepsilon)$ for all simplexes $T'$ of $K'$.



Let us define a retraction $r_3$ of $Z_2^1(K, \varepsilon)$ to the set $Z_2^1(K', \varepsilon') \cup N$. Since $Z_2^1(K^{(1)}, \varepsilon) = |K^{(1)}|$ then the retraction is defined for the 2-zone $Z_2^1(K^{(1)}, \varepsilon)$ of the 1-skeleton $K^{(1)}$ of $K$. Suppose that we have defined the mapping $r_3$ for $Z_2^1(K^{(m)}, \varepsilon)$ and let $T$ be an $(m+1)$-dimensional simplex of $K$. The set

$$P' \cap Z_2^1(T, \varepsilon) \cup N \cap Z_2^1(T, \varepsilon) = P' \cup N \cap Z_2^1(T, \varepsilon)$$

is an absolute retract. Consequently the set

$$W(T) = Z_2^1\Big(\tau'\big(P' \cap Z_2^1(T, \varepsilon)\big), \varepsilon'\Big) \cup N \cap Z_2^1(T, \varepsilon)$$

is an absolute retract. The mapping $r_3$ is defined on the zone $Z_2^1(T^{\cdot}, \varepsilon)$ and retracts this zone to the set

$$Z_2^1\Big(\tau'\big(P' \cap Z_2^1(T^{\cdot}, \varepsilon)\big), \varepsilon'\Big) \cup N \cap Z_2^1(T^{\cdot}, \varepsilon) \subset W(T).$$

Hence putting $r_3(x) = x$ for every $x \in W(T)$ we obtain a mapping $r_3$ which can be extended over the set $Z_2^1(T, \varepsilon)$ in such a manner that its values lie in $W(T)$. If we extend $r_3$ in this manner over all zones $Z_2^1(T, \varepsilon)$ of $(m+1)$-simplexes $T \in K$, then we obtain a retraction of the zone $Z_2^1(K^{(m+1)}, \varepsilon)$ to the set

$$W(K^{(m+1)}) = Z_2^1\Big(\tau'\big(P' \cap Z_2^1(K^{(m+1)}, \varepsilon)\big), \varepsilon'\Big) \cup N \cap Z_2^1(K^{(m+1)}, \varepsilon).$$

Thus we can define the retraction $r_3$ over $Z_2^1(K, \varepsilon)$ to the set $W(K) = Z_2^1(K', \varepsilon') \cup N$.

Now consider the retraction by deformation $r_2$ defined in 6. Putting

$$\begin{aligned} g(x) &= x \quad &&\text{for} \quad x \in Z_2^1(K', \varepsilon'), \\ g(x) &= r_2(x, 1) \quad &&\text{for} \quad x \in N, \\ r_4(x) &= g r_3(x) \quad &&\text{for} \quad x \in Z_2^1(K, \varepsilon), \end{aligned}$$

we obtain a retraction $r_4$ of $Z_2^1(K, \varepsilon)$ to $Z_2^1(K', \varepsilon')$ such that for every point $x \in Z_2^1(T, \varepsilon)$ the point $r_4(x)$ belongs to $Z_2^1(T, \varepsilon)$ or to $Z_2^1(T_1, \varepsilon)$, where $T_1$ is an $n$-simplex adjoining in a dimension $\geqslant 2$ to the $n$-simplex $T$. Since the diameters of $Z_2^1(T, \varepsilon)$ and of $Z_2^1(T_1, \varepsilon)$ are $\leqslant 2\varepsilon+\eta$, then

$$\varrho\big(r_4(x), x\big) \leqslant 4\varepsilon+2\eta \quad \text{for every } x \in Z_2^1(K, \varepsilon).$$

The set $Z_2^1(K', \varepsilon')$ will be called the 1-dimensional 2-zone of the polytope $P'$ *subordinate* to the 1-dimensional 2-zone $Z_2^1(K, \varepsilon)$.

**7. Construction of the set $P_\infty$ in the finite-dimensional case.** Let $H$ be an $(n+1)$-dimensional simplex in the space $E^{n+1}$ with the edges of length 1. Let $C$ denote the boundary of $H$, $P$ one of its $n$ faces and $R$ the sum of all the other $n$ faces.

We shall define two sequences of polytopes, $\{P_k\}$ and $\{D_k\}$, satisfying the following conditions:

$(1_k)$ *$C_k = P_k \cup R$ is a polyhedral irreducible cutting of $E^{n+1}$. The interior region of the set $E^{n+1} - C_k$ will be denoted by $\Gamma_k$, and the exterior region by $\Lambda_k$.*

$(2_k)$ $P_k \cap R = R^{\cdot}$.

$(3_k)$ *The polytope $P_k$ is smoothly connected in the dimension $(n-1)$ in $C_k$.*

$(4_k)$ *$D_k$ is the $(n-1)$-zone $Z^1_{n-1}\big(\tau_k(P_k), \varepsilon^{(k)}\big)$ of the polytope $P_k$ corresponding to the $\eta_k$-triangulation $\tau_k$ of $C_k$, where $\eta_k \leqslant 1/2^{2k-1}$ and $\varepsilon^{(k)}$ is suitable for the triangulation $\tau_k$ and $\varepsilon^{(k)} < 1/2^{2k}$.*

$(5_k)$ *$D_{k+1} \subset D_k$ and there exists a retraction $r_k$ of the set $D_k$ to the set $D_{k+1}$ such that $\varrho\big(r_k(x), x\big) \leqslant 1/2^{k-3}$ for $x \in D_k$.*

The sequences $\{P_k\}$ and $\{D_k\}$ will be defined by induction. We put $P_1 = P$ and denote by $D_1$ the $(n-1)$-zone $Z^1_{n-1}\big(\tau_1(P_1), \frac{1}{4}\big)$ corresponding to the arbitrary triangulation $\tau_1$ of $C_1$.

Assume that the polytopes $P_k$ and $D_k$ and the triangulation $\tau_k$ satisfying conditions $(1_k), \ldots, (5_k)$ are already defined. We shall define the polytopes $P_{k+1}$ and $D_{k+1}$ in the following manner:

Let $P_{k+1}$ denote a subordinate polytope to $P_k$ corresponding to the triangulation $\tau_k$ and to the zone $Z^1_{n-1}\big(\tau_k(P_k), \varepsilon^{(k)}\big)$. Let $\tau_{k+1}$ be a triangulation of $P_{k+1}$ such that the polytopes $P_k \cap P_{k+1}$, $\overline{P_{k+1} - P_k} \cap P_k$ and $P_{k+1} \cap Z^1_{n-1}(T, \varepsilon^{(k)})$ for each $T \in \tau_k$ are representable by the subcomplexes of the triangulation $\tau_{k+1}$.

By the same reasoning as in 6. there exists a positive number $\varepsilon^{(k+1)}$ suitable for the triangulation $\tau_{k+1}$ and such that the set $D_{k+1} = Z^1_{n-1}\big(\tau_k(P_{k+1}), \varepsilon^{(k+1)}\big)$ corresponding to the triangulation $\tau_{k+1}(P_{k+1})$ satisfies the conditions $(4_{k+1})$ and $(5_{k+1})$.

Now consider the sequence of mappings $\{f_k\}$ defined on the polytope $D_1$ by the formula

$$f_k(x) = r_k r_{k-1} \ldots r_2 r_1(x) \quad \text{for} \quad x \in D_1 .$$

By $(5_k)$ the mapping $f_k$ is a retraction of $D_1$ to the polytope $D_{k+1}$ and $\varrho\big(f_k(x), f_{k+1}(x)\big) < 1/2^{k-2}$ for $x \in D_1$.

It follows that the sequence $\{f_k\}$ uniformly converges in $D_1$ to a mapping $f$ which retracts the set $D_1$ to the set $P_\infty = f(D_1) \subset D_k$ for every $k = 1, 2, \ldots$, and hence $P_\infty$ is an absolute retract.



In exactly the same manner as in [1] we can prove that

1. *$P_\infty$ is the limit of the sequence of absolute retracts $\{P_k\}$.*
2. *$\overline{P_{k+1} - P_k} \cap P_k \subset P_\infty$ for every $k = 1, 2, \ldots$*
3. *The $n$-dimensional Betti-number $p^n(P_\infty \cup R) = 1$.*
4. *$P_\infty \cup R$ cuts $E^{n+1}$ into exactly two regions $\Gamma_\infty$ and $\Lambda_\infty$ where $\Lambda_\infty = \bigcup_k \Lambda_k$ and $\Gamma_\infty = \bigcap_k \Gamma_k$.*
5. *$P_\infty$ is an $n$-dimensional Cantor manifold.*

**8. Construction of the set $P'_\infty$ in the infinite-dimensional case.** We shall define the sequences of polytopes $\{P'_k\}$, $\{D'_k\}$ and $\{R'_k\}$ satisfying the following conditions:

$(1'_k)$ *$C'_k = P'_k \cup R'_k$ is a $(k+1)$-dimensional polyhedral irreducible cutting of $E^{k+2}$. The interior region of the set $E^{n+2} - C'_k$ will be denoted by $\Gamma'_k$ and the exterior region by $\Lambda'_k$.*

$(2'_k)$ $P'_k \cap R'_k = R'^{\cdot}_k$.

$(3'_k)$ *The polytope $P'_k$ is smoothly connected in the dimension 2 in $C'_k$.*

$(4'_k)$ *$D'_k = Z^\infty_2\big(\tau_k(P'_k), \{\varepsilon_i^{(k)}\}\big)$ where $\tau_k$ is a $\eta_k$-triangulation of the polytope $C'_k$, $\eta_k \leqslant 1/2^{2k-1}$, and $\{\varepsilon_i^{(k)}\}$ is a sequence of numbers suitable for the triangulation $\tau_k(C'_k)$, $\varepsilon_i^k < 1/i$, $\varepsilon_i^k < 1/2^{2k}$, $i = 1, 2, \ldots$*

$(5'_k)$ *$D'_{k+1} \subset D'_k$ and there exists a retraction $r_k$ of $D'_k$ to $D'_{k+1}$ such that $\varrho\big(r_k(x), x\big) < 1/2^{k-3}$ for $x \in D'_k$.*

The sequences $\{P'_k\}$, $\{D'_k\}$ and $\{R'_k\}$ will be defined by induction. Let $H$ be a regular 4-dimensional simplex lying in the space $E^4$ with sides of length 1. We put $C'_1 = H^{\cdot}$; let $P'_1$ denote one of its 3-dimensional faces and $R'_1$ the sum of all the other faces. Let $D'_1 = Z^\infty_2\big(\tau_1(P'_1), \{\varepsilon_i^{(1)}\}\big)$ where $\tau_1$ is a $\frac{1}{2}$-triangulation of $C'_1$ and $\{\varepsilon_i^{(1)}\}$ is a sequence of numbers satisfying $(4'_k)$.

Assume that the polytopes $P'_k$, $D'_k$ and $R'_k$ and the triangulation $\tau_k$ satisfying the conditions $(1'_k), \ldots, (5'_k)$ are already defined. We shall define the polytopes $P'_{k+1}$, $D'_{k+1}$ and $R'_{k+1}$ in the following manner:

Let $P''_k$ denote the subordinate polytope to $P'_k$ corresponding to the triangulation $\tau_k$ and to the zone $D'_k$ and let $\tau_k$ be a triangulation of the polytope $P''_k$ such that the polytope $P''_k \cap P'_k$, $\overline{P''_k - P'_k} \cap P'_k$ and $P''_k \cap Z^\infty_2(T, \{\varepsilon_i^{(k)}\})$ for every $T \in \tau_k(P'_k)$ are representable by the subcomplexes of $\tau'_k$. Let $C'_{k+1}$ denote the suspension $\overline{S}(P''_k \cup R'_k)$. We put

$$P'_{k+1} = Z^2_2\big(\tau'_k(P''_k), \{0, \varepsilon\}\big) \subset \overline{S}(P''_k \cup R'_k) = C'_{k+1}, \qquad \varepsilon < \eta_k .$$

Let $\tau_{k+1}$ be an $\eta_{k+1}$-triangulation of the polytope $P'_{k+1}$ $(\eta_{k+1} < 1/2^{2k+1})$ such that for the simplexes of $P''_k$, $\tau_{k+1}$ is a *subdivision* of $\tau_k$. We put

$$R'_{k+1} = C'_{k+1} - P'_{k+1}, \qquad D'_{k+1} = Z^\infty_2\big(\tau_{k+1}(P'_{k+1}), \{\varepsilon^{(k+1)}\}\big)$$

where the sequence $\{\varepsilon_i^{(k+1)}\}$ satisfies conditions $(4'_{k+1})$ and $(5'_{k+1})$. Just as in the finite dimensional case using the remark at end of 3. and generalizing Lemma 5 we can prove that the sequences $\{P'_k\}$, $\{D'_k\}$ and $\{R'_k\}$ defined as above satisfy conditions $(1'_k), \ldots, (5'_k)$, and that the sequence $\{f_k\}$ of the retractions $f_k = r_k r_{k-1} \ldots r_1$ of $D'_1$ to $D'_k$ uniformly converges to a mapping $f$. The mapping $f$ retracts $D'_1$ to the set $f(D'_1)$ denoted by $P'_\infty$. It is an absolute retract of infinite dimension because it contains a $k$-dimensional skeleton of the complex $\tau_k(P'_k)$ for every $k = 1, 2, \ldots$

### 9. Main theorem.

LEMMA 6. *If $A$ is an $n$-dimensional closed proper subset of $P_\infty$, then there exists a natural number $k_0$ such that for every $k > k_0$ there exists in the triangulation $\tau_k(P_k)$ an $n$-dimensional simplex $T$ such that $|T| \cap P_\infty \subset A$. If $A$ is an open non-void subset of $P'_\infty$, then there exists a natural number $k_0$ such that for every $k > k_0$ there exists in the triangulation $\tau_k(P'_k)$ a $(k+1)$-dimensional simplex $T$ such that $|T| \cap P'_\infty \subset A$.*

The proof of this lemma in the finite-dimensional case is a repetition of the proof of an analogous lemma for dimension 2 in [1]. In the infinite-dimensional case the lemma follows at once from the fact that the diameters of simplexes of the triangulation $\tau_k(P'_k)$ tends to 0.

THEOREM. *If $A$ is an $n$-dimensional proper closed subset of $P_\infty$, then the $(n-1)$-dimensional Betti-number $p^{n-1}(A) = \infty$. If $A$ is a closed proper subset of $P'_\infty$ containing an inner point of $P'_\infty$, then the $k$-dimensional Betti-number $p^k(A)$ converges to an infinity with $k$.*

We shall prove this theorem only in the infinite-dimensional case. The proof in the finite-dimensional case is the same as in [1].

Let $m$ be an arbitrary natural number. Since $A$ is a subset of $P'_\infty$, containing an inner point of $P'_\infty$, there exist $m$ disjointed closed subsets $A_1, A_2, \ldots, A_m$ of $A$ and a closed subset $A_0$ of $P'_\infty$ contained in $P'_\infty - A$. Each of the sets $A_i$ $(i = 0, 1, \ldots, m)$ contains an inner point of $P'_\infty$. By the preceding lemma there exists a natural number $k_0$ such that for $k > k_0$ there exist in the triangulation $\tau_k(P'_k)$ such $(k+1)$-dimensional simplexes $T_0, T_1, \ldots, T_m$, that $|T_\nu| \cap P'_\infty \subset A_\nu$, $\nu = 0, 1, \ldots, m$.

Consider the component $M_{T_\nu}$ of the base of the rosary of the polytope $P'_k$. The boundary $\dot{Q}_{T_\nu}$ of the first cube of the component $M_{T_\nu}$ lie in the simplex $|T_\nu|$ and in $P'_\infty$ and hence in $A$. Among the cubes $Q_{T_\nu, i}$, $i = 0, 1, \ldots, s+1$, of $M_{T_\nu}$ there exists one lying on $|T_0| \subset P'_k - A$. We infer that there exists an index $i_\nu$ such that $\dot{Q}_{T_\nu, i_\nu}$ lies on $A$ and $\dot{Q}_{T_\nu, i_\nu+1}$ does not lie on $A$. Hence the point $a''_{T_\nu, i_\nu+1} \in \dot{Q}_{T, i_\nu+1} - A$. Let us denote this point by $a''_\nu$. By the construction of the rosary of the polytope $P'_k$ there exists point $a'_\nu \in A'_k$ such that

(i) if $L'$ denotes the segment $\overline{a'_\nu a''_\nu}$ then $L'_\nu \cap A = \emptyset$,



(ii) there exists a simple arc $L''_\nu$ joining the points $a'_\nu$ and $a''_\nu$ and lying in the set $A'_k$ arbitrarily near the link $N_{T_\nu, i_\nu}$,

(iii) the arcs $L'_\nu$ and $L''_\nu$ have disjoint interiors.

Consider the set $F = Z_2^\infty(\tau_k(P'_k), \{\varepsilon_\nu^{(k)}\}) - \{a''_1, a''_2, \ldots, a''_m\}$. The set $F$ contains $A$ and there exists a retraction $r$ of $F$ to the set $P''_k - \{a''_1, a''_2, \ldots, a''_m\}$. There exists a subdivision $\tau'_k$ of the triangulation $\tau_k$ and a subpolytope $P'$ of the triangulation $\tau'_k$ such that $r(A) \subset P' \subset P'_k - \{a''_1, \ldots, a''_m\}$. The simple closed curve $\Omega_\nu = L'_\nu \cup L''_\nu \subset E^{k+2} - A$ has the absolute linking number with the $k$-dimensional topological sphere $\dot{Q}_{T_\nu, i_\nu}$ equal to 1 and with each sphere $\dot{Q}_{T_{\nu'}, i_{\nu'}}$, $\nu \neq \nu'$, equal to 0. Thus we obtain in $A$ a system of $m$ $k$-dimensional cycles linearly independent in $r(A)$ and therefore those cycles are independent also in $A$. Hence $p^k(A) \geqslant m$, and since $m$ is an arbitrary number, the proof is finished.

The set $P_\infty$ is an irreducible $n$-dimensional absolute retract and also an irreducible $n$-dimensional locally contractible compactum, because every locally contractible closed subset of $P_\infty^n$ has a dimension $< n$.

The set $P'_\infty$ can be called an *infinite-dimensional irreducible absolute retract*, and also—an *infinite-dimensional irreducible locally contractible compactum* in the sense that no locally contractible closed subset of $P'_\infty$ contains an inner point of it.

**Reference**

[1] K. Borsuk, *On a irreducible 2-dimensional absolute retract*, Fund. Math. 37 (1950), pp. 138-160.

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