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Integral representation of vector measures and linear operations

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1. Introduction. Let T be a locally compact space and ν a positive Radon measure on T. Let F be a Banach space, $(E(t))_{t\in T}$ a family of Banach spaces and $\mathscr{A}$ a fundamental family of continuous vector fields.

In [6]-[9] and [11]-[13] we have given integral representations of the form

$$(1)\qquad \langle f(\boldsymbol{x}), z\rangle = \int \langle U_f(t)\boldsymbol{x}(t), z\rangle d\nu(t)$$

for certain linear mappings f of Lebesgue spaces $\mathscr{L}^p_{\mathscr{A}}(\nu)$, $1 \leqslant p < \infty$, or of the space $\mathscr{K}_{\mathscr{A}}(T)$ into F, and of the form

$$(2)\qquad \langle \int \boldsymbol{x}\, d\boldsymbol{m}, z\rangle = \int \langle U_{\boldsymbol{m}}(t)\boldsymbol{x}(t), z\rangle d\nu(t)$$

for certain vector measures $\boldsymbol{m}$ absolutely continuous with respect to ν.

The proof has used essentially the fact that the spaces $E(t)$ and F were of countable type and that F was the dual of a Banach space.

Using the existence of a lifting of $\mathscr{L}^\infty_R(\nu)$ [19], Alexandra and Cassius Ionescu Tulcea [20] have succeeded in dropping the countability hypotheses in formula (1) in the case where E and F are locally convex and $E(t) = E$ for every $t\in T$.

In this paper we use also the lifting of $\mathscr{L}^\infty_R(\nu)$ to prove formula (2) (theorems 2 and 3), without any countability hypotheses in the case where $E(t) = E$ for every $t\in T$. Using (2) we then prove (1) and we give supplementary information about U_f and $U_{\boldsymbol{m}}$. For simplicity we consider only the case of the Banach spaces E and F.

The linear mappings f on $\mathscr{L}^p_E(\nu)$ which can be represented by formula (1) can also be represented in the form

$$(3)\qquad f(\boldsymbol{x}) = \int \boldsymbol{x}\, d\boldsymbol{m},$$

where $\boldsymbol{m}$ is a suitable vector measure (Theorem 9).

We mention that the dominated linear mappings $f\colon \mathscr{K}_E(T) \to F$ can be identified with the dominated linear mappings $f'\colon \mathscr{K}(T) \to \mathscr{L}(E, F)$ by the formula

$$f(\varphi x) = f'(\varphi)x \quad \text{for} \quad \varphi \in \mathscr{K}(T) \text{ and } x \in E, \tag{4}$$

and that by the same formula we can identify the linear mappings $f\colon \mathscr{L}_E^p(\nu) \to F$ such that $|||f||| < \infty$ with the linear mappings $f'\colon \mathscr{L}^p(\nu) \to \mathscr{L}(E, F)$ such that $|||f'||| < \infty$ (Theorems 4 and 7).

If there exists a strong lifting of $\mathscr{L}_R^\infty(\nu)$ [21], then it is possible to drop the countability hypotheses also in the case of the spaces of vector fields.

With obvious modifications, the results of this paper (except for those of § 4) remain valid in the case where ν is a σ-finite measure on a σ-algebra of subsets of an abstract space T.

2. The lifting. Let T be a locally compact space, and μ a positive Radon measure on T. If two scalar functions f and g defined on T are equal locally μ-almost everywhere, we shall write $f \equiv g$. If A and B are two subsets of T, then $A \equiv B$ means that $\varphi_A \equiv \varphi_B$.

Consider the space $\mathscr{L}^\infty(\mu)$. There exists then (see [19]) a mapping $\varrho\colon \mathscr{L}^\infty(\mu) \to \mathscr{L}^\infty(\mu)$, called a *lifting* of $\mathscr{L}^\infty(\mu)$, having the following properties:

1. $\varrho(f) \equiv f$;
2. $f \equiv g$ implies $\varrho(f) = \varrho(g)$;
3. $\varrho(1) = 1$;
4. $f \geqslant 0$ implies $\varrho(f) \geqslant 0$;
5. $\varrho(\alpha f + \beta g) = \alpha\varrho(f) + \beta\varrho(g)$;
6. $\varrho(fg) = \varrho(f)\varrho(g)$.

A lifting ϱ of $\mathscr{L}^\infty(\mu)$ can be extended uniquely to a mapping of $\mathscr{L}_C^\infty(\mu)$ into itself by putting $\varrho(f+ig) = \varrho(f) + i\varrho(g)$ for $f, g \in \mathscr{L}^\infty(\mu)$ and the extension has all the properties 1-6.

A lifting ϱ of $\mathscr{L}^\infty(\mu)$ has also the following properties:

7. $|\varrho(f)| = \varrho(|f|)$;
8. $\varrho(f) = f$ implies $\sup_{t \in T} |f(t)| = N_\infty(f)$.

If ϱ is a lifting of $\mathscr{L}^\infty(\mu)$, for every μ-measurable set $A \subset T$, $\varrho(\varphi_A)$ is the characteristic function of a set denoted by $\varrho(A)$. Then

1'. $\varrho(A) \equiv A$;
2'. $A \equiv B$ implies $\varrho(A) = \varrho(B)$;
3'. $\varrho(E) = E$ and $\varrho(\varphi) = \varphi$;
4'. $\varrho(A \cup B) = \varrho(A) \cup \varrho(B)$;
5'. $\varrho(A \cap B) = \varrho(A) \cap \varrho(B)$.

In the rest of the paper we shall denote by E and F two Banach spaces and by Z a subspace of the dual F' of F. We shall suppose that Z is norming, i. e.

$$y = \sup_{z \in Z} \frac{|\langle y, z\rangle|}{|z|} \quad \text{for every } y \in F.$$

(Here, $|a|$ is the norm of an element a of one of the spaces E, F and Z). In this case we can plunge F isometrically in Z'.

Let $\mathscr{L}^*(E, F)$ be the space of the linear mappings of E into F and $\mathscr{L}(E, F)$ the space of the continuous linear mappings of E into F. For every $U \in \mathscr{L}^*(E, F)$ we put

$$|U| = \sup_{x \in E} |Ux| \leqslant \infty.$$

For every function $U\colon T \to \mathscr{L}^*(E, F)$, $\boldsymbol{x}\colon T \to E$ and $z \in Z$ we shall denote by $\langle U\boldsymbol{x}, z\rangle$ the function $t \to \langle U(t)x(t), z\rangle$. In particular, if $x \in E$ and $z \in Z$ the function $t \to \langle U(t)x, z\rangle$ will be denoted by $\langle Ux, z\rangle$.

For two functions $U, U'\colon T \to \mathscr{L}^*(E, F)$ we shall write $U \equiv U'$ if for each $x \in E$ and $z \in Z$ we have $\langle Ux, z\rangle = \langle U'x, z\rangle$ locally μ-almost everywhere.

A function $U\colon T \to \mathscr{L}^*(E, F)$ is said to be *Z-weakly μ-measurable* if for each $x \in E$ and $z \in Z$ the function $\langle Ux, z\rangle$ is μ-measurable. The function U is said to be *simply μ-measurable* if the function $t \to U(t)x$ is μ-measurable for every $x \in E$.

Let us denote by $\mathscr{C}(\mu)$ the set of all locally countable families $\mathscr{K} = (K_j)_{j \in J}$ of disjoint compact parts of T such that $T - \bigcup_{j \in J} K_j$ is locally μ-negligible.

We remark that if $\mathscr{K}_1 = (K_i)_{i \in I} \in \mathscr{C}(\mu)$ and $\mathscr{K}_2 = (K_j)_{j \in J} \in \mathscr{C}(\mu)$, then $\mathscr{K} = (K_i \cap K_j)_{(i,j) \in I \times J} \in \mathscr{C}(\mu)$. Hence $\mathscr{C}(\mu)$ is directed by the relation $\mathscr{K}' \succ \mathscr{K}''$, which means that every set of $\mathscr{K}'$ is contained in some set of $\mathscr{K}''$.

Let ϱ be a lifting of $\mathscr{L}^\infty(\mu)$ and $U\colon T \to \mathscr{L}^*(E, F)$. We shall write $\varrho[U] = U$ if there exists a family $\mathscr{K} = (K_j)_{j \in J} \in \mathscr{C}(\mu)$ such that for every $j \in J$, $x \in E$ and $z \in Z$ we have $\varphi_{K_j}\langle Ux, z\rangle \in \mathscr{L}_C^\infty(\mu)$ and

$$\varrho(\varphi_{K_j}\langle Ux, z\rangle) = \varphi_{\varrho(K_j)}\langle Ux, z\rangle.$$

We remark that if $\mathscr{K}' = (K_i)_{i \in I} \succ \mathscr{K}$, then we still have $\varphi_{K_i}\langle Ux, z\rangle \in \mathscr{L}_C^\infty(\mu)$ and

$$\varrho(\varphi_{K_i}\langle Ux, z\rangle) = \varphi_{\varrho(K_i)}\langle Ux, z\rangle$$

for every $i \in I$, $x \in E$ and $z \in Z$.

It follows that if $U_1, U_2\colon T\to\mathscr{L}^*(E,F)$ are two functions such that $\varrho[U_1]=U_1$ and $\varrho[U_2]=U_2$, then we can find a common family $\mathscr{K}=(K_j)_{j\in J}\in\mathscr{C}(\mu)$ with $\varphi_{K_j}\langle U_i x, z\rangle\in\mathscr{L}_c^\infty(\mu)$ and

$$\varrho(\varphi_{K_j}\langle U_i x, z\rangle)=\varphi_{\varrho(K_j)}\langle U_i x, z\rangle$$

for $j\in J$, $x\in E$, $z\in Z$ and $i=1,2$.

We shall write $\varrho(U)=U$ if for every $x\in E$ and $z\in Z$ we have $\langle Ux, z\rangle\in\mathscr{L}_c^\infty(\mu)$ and

$$\varrho(\langle Ux, z\rangle)=\langle Ux, z\rangle.$$

We immediately deduce the following properties:

1) $\varrho(U)=U$ implies $\varrho[U]=U$ (for every family $\mathscr{K}\in\mathscr{C}(\mu)$ in the definition of $\varrho[U]=U$).

2) If $\varrho[U]=U$, then U is Z-weakly μ-measurable.

3) If $\varrho(U)=U$, then $|U(t)|\leqslant N_\infty(U)$ for every $t\in T$.

4) If $U\equiv U'$, $\varrho[U]=U$ and $\varrho[U']=U'$, then $U(t)=U'(t)$ locally μ-almost everywhere.

5) If $\varrho[U]=U$ and $U'(t)=U(t)$ locally μ-almost everywhere, then $\varrho[U']=U'$.

6) If $U\equiv U'$, $\varrho(U)=U$ and $\varrho(U')=U'$, then $U(t)=U'(t)$ for every $t\in T$.

PROPOSITION 1. *If $U\colon T\to\mathscr{L}^*(E,F)$ is such that $\varrho[U]=U$, then the function $t\to|U(t)|$ is μ-measurable.*

For the proof, see [20], p. 782, remark 1.

PROPOSITION 2. *If $U, U'\colon T\to\mathscr{L}^*(E,F)$ are two functions such that $U'\equiv U$, $\varrho[U']=U'$ and if the function $t\to|U(t)|$ is μ-measurable, then $|U'(t)|\leqslant|U(t)|$ locally μ-almost everywhere.*

Let $\mathscr{K}=(K_j)_{j\in J}\in\mathscr{C}(\mu)$ be a family such that

$$\varphi_{K_j}\langle U'x, z\rangle\in\mathscr{L}_C^\infty(\mu)\quad\text{and}\quad\varrho(\varphi_{K_j}\langle U'x, z\rangle)=\varphi_{\varrho(K_j)}\langle U'x, z\rangle$$

for $j\in J$, $x\in E$ and $z\in Z$. We remark that $\varphi_{\varrho(K_j)}|\langle U'x, z\rangle|=\varrho(\varphi_{K_j}|\langle U'x, z\rangle|)$ and that $\varphi_{K_j}|\langle U'x, z\rangle|\equiv\varphi_{K_j}|\langle Ux, z\rangle|$; therefore

$$\varphi_{\varrho(K_j)}|\langle U'x, z\rangle|=\varrho(\varphi_{K_j}|\langle Ux, z\rangle|)\leqslant|x||z|\,\varrho(\varphi_{K_j}|U|).$$

It follows that $\varphi_{\varrho(K_j)}|U'|\leqslant\varrho(\varphi_{K_j}|U|)\equiv\varphi_{\varrho(K_j)}|U|$; hence $|U'(t)|\leqslant|U(t)|$ locally μ-almost everywhere on $\bigcup_{j\in J}\varrho(K_j)$, and therefore on T.

PROPOSITION 3. *Let $U\colon T\to\mathscr{L}^*(E,Z')$ be Z-weakly μ-measurable.*

a) *If there exists a family $(K_j)_{j\in J}\in\mathscr{C}(\mu)$ such that for each $j\in J$ and $x\in E$ we have $\sup_{t\in K_j}|U(t)x|<\infty$, then we can find a function $U'\colon T\to\mathscr{L}^*(E,Z')$ such that $U'\equiv U$ and $\varrho[U']=U'$.*

b) *If for each $x\in E$ we have $\sup_{t\in T}|U(t)x|<\infty$, then we can find a function $U'\colon T\to\mathscr{L}^*(E,Z')$ such that $U'\equiv U$ and $\varrho(U')=U'$.*

For the proof, see [20], proposition 1 and proposition 4.

PROPOSITION 4. *Let $U\colon T\to\mathscr{L}^*(E,Z')$ be Z-weakly μ-measurable, such that the function $t\to|U(t)|$ is μ-measurable and finite. Then there exists a function $U'\colon T\to\mathscr{L}^*(E,Z')$ such that $U'\equiv U$, $\varrho[U']=U'$ and $|U'(t)|\leqslant|U(t)|$ for every $t\in T$.*

If, in addition, $|U|\in\mathscr{L}^\infty(\mu)$, then the function U' can be taken such that $\varrho(U')=U'$ and $|U'(t)|\leqslant|U(t)|$ locally μ-almost everywhere.

In fact, since $|U|$ is μ-measurable, we can find a family $(K_j)_{j\in J}\in\mathscr{C}(\mu)$ such that the restriction of $|U|$ to each K_j is continuous, whence bounded. We can then apply Proposition 3 to find a function $U_1\colon T\to\mathscr{L}^*(E,Z')$ such that $U_1\equiv U$ and $\varrho[U_1]=U_1$. By proposition 2 we have $|U_1(t)|\leqslant|U(t)|$ locally μ-almost everywhere. Modifying U_1 on a locally μ-negligible set we can find a function U' such that $|U'(t)|\leqslant|U(t)|$ for every $t\in T$. Then we still have $U'\equiv U$ and $\varrho[U']=U'$.

The case $|U|\in\mathscr{L}^\infty(\mu)$ is proved in the same way.

3. Vector measures. We shall denote by $\mathscr{B}$ the clan of the relatively compact Borel subsets of T and by $\mathscr{S}_E(\mathscr{B})$ the set of the functions of the form $\sum\varphi_{A_i}x_i$ (finite sum) with $A_i\in\mathscr{B}$ and $x_i\in E$. We shall consider regular vector measures $\boldsymbol{m}\colon\mathscr{B}\to\mathscr{L}(E,F)$ with finite variation. For the definition of the integral $\int\boldsymbol{x}\,d\boldsymbol{m}$ of the functions $\boldsymbol{x}\colon T\to E$, we refer the reader to [13]. A positive regular measure μ will be identified with the corresponding Radon measure $f\to\int f\,d\mu$ defined for the continuous functions $f\colon T\to R$ with compact support. We shall write $\int^* f\,d\mu$ instead of $\overline{\int^*}f\,d\mu$ and we shall say "integrable function" instead of "essentially integrable function" [2].

The following theorem is essential for the proof of the other theorems of this paper:

THEOREM 1. *Let $\boldsymbol{m}\colon\mathscr{B}\to\mathscr{L}(E,F)$ be a measure with finite variation μ. There exists then a function $U_{\boldsymbol{m}}\colon T\to\mathscr{L}(E,Z')$ having the following properties:*

1) *$|U_{\boldsymbol{m}}(t)|=1$ locally μ-almost everywhere.*

2) *$U_{\boldsymbol{m}}$ is Z-weakly μ-measurable and we have*

$$\Big\langle\int\boldsymbol{x}\,d\boldsymbol{m}, z\Big\rangle=\int\langle U_{\boldsymbol{m}}(t)\boldsymbol{x}(t), z\rangle\,d\mu(t)\quad\text{for}\quad\boldsymbol{x}\in\mathscr{L}_E^1(\mu)\ \text{and}\ z\in Z.$$

3) *If ϱ is a lifting of $\mathscr{L}^\infty(\mu)$, we can choose $U_{\boldsymbol{m}}$ uniquely such that*

$$\varrho(U_{\boldsymbol{m}})=U_{\boldsymbol{m}}.$$

4) *If there exists a family $\mathscr{K}\in\mathscr{C}(\mu)$ such that for every $K\in\mathscr{K}$ and every $x\in E$ the convex equilibrated cover of the set $\{\int_K\varphi x\,d\boldsymbol{m};\ \varphi\in\mathscr{S}_E(\mathscr{B}),\ \int_K|\varphi|\,d\mu\leqslant 1\}$*

is relatively compact in F for the topology $\sigma(F,Z)$, then we can choose $U_{\boldsymbol{m}}(t)\in\mathscr{L}(E,F)$ locally μ-almost everywhere.

4′) *If for every $x\in E$ the convex equilibrated cover of the set $\{\int\varphi x\,d\boldsymbol{m};\ \varphi\in\mathscr{S}_E(\mathscr{B}),\ \int|\varphi|\,d\mu\leqslant 1\}$ is relatively compact in F for $\sigma(F,Z)$, then we can choose $U_{\boldsymbol{m}}(t)\in\mathscr{L}(E,F)$ for every $t\in T$.*

Proof. Let ϱ be a lifting of $\mathscr{L}^\infty(\mu)$. For every $x\in E$ and $z\in Z$ the set function $\boldsymbol{m}_{x,z}$ defined on $\mathscr{B}$ by the equality

$$\boldsymbol{m}_{x,z}(A)=\langle\boldsymbol{m}(A)x,z\rangle\quad\text{for}\quad A\in\mathscr{B},$$

is a regular scalar measure and we have

$$|\boldsymbol{m}_{x,z}|\leqslant|x|\,|z|\,\mu.$$

There exists then a bounded μ-measurable scalar function $g_{x,z}$ defined on T such that $\boldsymbol{m}_{x,z}=g_{x,z}\mu$; modifying $g_{x,z}$ on a locally μ-negligible set, we can take $g_{x,z}$ such that

$$\varrho(g_{x,z})=g_{x,z}.$$

From $|\boldsymbol{m}_{x,z}|=|g_{x,z}|\mu\leqslant|x|\,|z|\,\mu$, we deduce that

$$|g_{x,z}(t)|\leqslant|x|\,|z|\quad\text{for every}\quad t\in T.$$

Since the mapping $(x,z)\to\boldsymbol{m}_{x,z}$ of $E\times Z$ into the space of the scalar measures is bilinear and $\varrho(g_{x,z})=g_{x,z}$, we deduce that the mapping $(x,z)\to g_{x,z}$ of $E\times Z$ into the space of the bounded μ-measurable scalar functions is bilinear; therefore, for every $t\in T$, the mapping $(x,z)\to g_{x,z}(t)$ is a bilinear functional on $E\times Z$.

For fixed $t\in T$ and $x\in E$, the mapping $g_x(t)\colon z\to g_{x,z}(t)$ is a continuous linear functional on Z, whence $g_x(t)\in Z'$ and we have

$$|g_x(t)|\leqslant|x|.$$

For fixed $t\in T$, the mapping $U_{\boldsymbol{m}}(t)\colon x\to g_x(t)$ of E into Z' is linear and continuous, whence $U_{\boldsymbol{m}}(t)\in\mathscr{L}(E,Z')$ and we have

$$|U_{\boldsymbol{m}}(t)|\leqslant 1.$$

Then

$$\langle U_{\boldsymbol{m}}(t)x,z\rangle=g_{x,z}(t)\quad\text{for}\quad t\in T,\ x\in E\ \text{and}\ z\in Z.$$

It follows first that

$$\langle\boldsymbol{m}(A)x,z\rangle=\int_A\langle U_{\boldsymbol{m}}(t)x,z\rangle\,d\mu(t)\quad\text{for}\quad A\in\mathscr{B},\ x\in E\ \text{and}\ z\in Z,$$

and then that

$$\left\langle\int\boldsymbol{x}\,d\boldsymbol{m},z\right\rangle=\int\langle U_{\boldsymbol{m}}(t)\boldsymbol{x}(t),z\rangle\,d\mu(t)$$

for $x\in E$, $z\in Z$ and $\boldsymbol{x}$ step function of $\mathscr{S}_E(\mathscr{B})$; passing to the limit, the last equality remains valid for every $\boldsymbol{x}\in\mathscr{L}^1_E(\mu)$.

From the equalities $\langle U_{\boldsymbol{m}}x,z\rangle=g_{x,z}$ and $\varrho(g_{x,z})=g_{x,z}$ we deduce that $\varrho(U_{\boldsymbol{m}})=U_{\boldsymbol{m}}$.

Let us now prove that $|U_{\boldsymbol{m}}(t)|=1$ locally μ-almost everywhere. By proposition 1, the function $t\to|U_{\boldsymbol{m}}(t)|$ is μ-measurable; from the inequality $|U_{\boldsymbol{m}}(t)|\leqslant 1$ for every $t\in T$ we deduce that the function $|U_{\boldsymbol{m}}|$ is locally μ-integrable. For every $x\in E$, $z\in Z$ and $A\in\mathscr{B}$ we have

$$|\langle\boldsymbol{m}(A)x,z\rangle|\leqslant\int_A|x|\,|z|\,|U_{\boldsymbol{m}}(t)|\,d\mu(t);$$

therefore, putting $d\nu(t)=|U_{\boldsymbol{m}}(t)|\,d\mu(t)$, we have

$$|\boldsymbol{m}(A)|\leqslant\int_A|U_{\boldsymbol{m}}(t)|\,d\mu(t)=\nu(A).$$

Since μ is the least positive regular measure verifying the inequality $|\boldsymbol{m}(A)|\leqslant\mu(A)$ for every $A\in\mathscr{B}$, we deduce that $\mu\leqslant\nu$; hence

$$|U_{\boldsymbol{m}}(t)|\geqslant 1\quad\text{locally }\mu\text{-almost everywhere;}$$

consequently

$$|U_{\boldsymbol{m}}(t)|=1\quad\text{locally }\mu\text{-almost everywhere.}$$

Suppose now that there exists a family $\mathscr{K}\in\mathscr{C}(\mu)$ verifying condition 4, and let $K\in\mathscr{K}$ and $x\in E$. Denote by A the closure in $\sigma(F,Z)$ of the convex equilibrated cover of the set $\{\int_K\varphi x\,d\boldsymbol{m};\ \varphi\in\mathscr{S}_E(\mathscr{B}),\ \int_K|\varphi|\,d\mu\leqslant 1\}$. The set A is compact in F for $\sigma(F,Z)$, whence A is compact in the algebraic dual Z^*, for the topology $\sigma(Z^*,Z)$. There exists then a family $(z_i)_{i\in I}$ of elements of Z such that

$$A=\bigcap_{i\in I}\{y;\ y\in Z^*,\ |\langle y,z_i\rangle|\leqslant 1\}.$$

Then for every $i\in I$ and every $\varphi\in\mathscr{S}_E(\mathscr{B})$ with $\int_K|\varphi|\,d\mu\leqslant 1$ we have

$$\left|\left\langle\int_K\varphi x\,d\boldsymbol{m},z_i\right\rangle\right|\leqslant 1;$$

hence

$$\left|\int_K\langle U_{\boldsymbol{m}}(t)x,z_i\rangle\varphi(t)\,d\mu\right|\leqslant 1,$$

and consequently

$$|\langle U_{\boldsymbol{m}}(t)x,z_i\rangle|\leqslant 1\quad\mu\text{-almost everywhere on }K$$

for each $i\in I$. We then deduce that for every $z\in I$ and every $t\in\varrho(K)$

$$|\langle U_{\boldsymbol{m}}(t)x,z_i\rangle|\leqslant 1;$$

hence $U_{\boldsymbol{m}}(t)x\in A\subset F$ for every $t\in\bigcup\varrho(K)$.

Part 4' is proved in the same manner. Thus, the theorem is completely proved.

Remark. The function U_m depends not only on E and F, but also on the space Z. If we take $Z = F'$, then $U_m(t) = \mathscr{L}(E, F'')$ for every $t \in T$; if, in this case, Z_1 is an arbitrary norming subspace of F' and if $U_1\colon T \to \mathscr{L}(E, Z_1)$ is the corresponding function, then for every $t \in T$ and every $x \in E$, the functional $U_1(t)x \in Z_1'$ is the restriction to Z_1 of the functional $U_m(t)x \in F''$. In the sequel we shall denote U and U_1 by the same letter.

Corollary. *If μ is a scalar regular measure on T, then there exists a locally μ-integrable scalar function φ such that $|\varphi(t)| = 1$ and $\mu = \varphi|\mu|$. The function $1/\varphi$ is also locally μ-integrable and we have $|\mu| = \frac{1}{\varphi}\mu$.*

In fact,

$$\frac{1}{\varphi}\mu = \frac{1}{\varphi}(\varphi|\mu|) = \left(\frac{1}{\varphi}\varphi\right)|\mu| = |\mu|.$$

If m and n are two vector measures with finite variations μ and ν, we say that m is *absolutely continuous with respect to n* if μ is absolutely continuous with respect to ν.

We now give a generalization of the theorem of Lebesgue-Nikodym.

Theorem 2. *Let ν be a scalar regular Borel measure and $m\colon \mathscr{B} \to \mathscr{L}(E, F)$ a measure with finite variation μ and absolutely continuous with respect to ν. There exists then a function $V_m\colon T \to \mathscr{L}(E, Z')$ having the following properties:*

1) *The function V_m is locally ν-integrable and we have*

$$\int f d\mu = \int |V_m| f d|\nu| \quad \text{for} \quad f \in \mathscr{L}^1(\mu).$$

2) *V_m is Z-weakly ν-measurable and we have*

$$\langle \int x dm, z\rangle = \int \langle V_m(t)x(t), z\rangle d\nu \quad \text{for} \quad x \in \mathscr{L}^1_E(m) \text{ and } z \in Z.$$

3) *If ϱ is a lifting of $\mathscr{L}^\infty(\nu)$, we can choose V_m uniquely locally ν-almost everywhere, such that $\varrho[V_m] = V_m$.*

If, in addition, there exists $a > 0$ such that $\mu \leqslant a|\nu|$, then we can choose V_m uniquely such that $\varrho(V_m) = V_m$.

4) *If there exists a family $\mathscr{K} \in \mathscr{C}(\mu)$ (respectively $\mathscr{K} \in \mathscr{C}(\nu)$) such that for every $K \in \mathscr{K}$ and every $x \in E$ the convex equilibrated cover of the set $\{\int_K \varphi x dm$; $\varphi \in \mathscr{S}_E(\mathscr{B}), \int_K |\varphi| d\mu \leqslant 1$ (respectively $\int_K |\varphi| d|\nu| \leqslant 1)\}$ is relatively compact in F for the topology $\sigma(F, Z)$, then we can choose $V_m(t) \in \mathscr{L}(E, F)$ locally ν-almost everywhere.*

4') *If for every $x \in E$ the convex equilibrated cover of the set $\{\int \varphi x dm$; $\varphi \in \mathscr{S}_E(\mathscr{B}), \int |\varphi| d\mu \leqslant 1$ (respectively $\int |\varphi| d|\nu| \leqslant 1)\}$ is relatively compact in F for the topology $\sigma(F, Z)$, then we can choose $V_m(t) \in \mathscr{L}(E, F)$ for every $t \in T$.*

Proof. Since μ is absolutely continuous with respect to ν, there exists a locally ν-integrable function $g \geqslant 0$ such that $\mu = g|\nu|$; hence

$$\int f d\mu = \int g f d|\nu| \quad \text{for every} \quad f \in \mathscr{L}^1(\mu).$$

By the corollary of Theorem 1 there exists a locally ν-integrable scalar function φ such that $|\varphi(t)| = 1$, $\nu = \varphi|\nu|$ and $|\nu| = \frac{1}{\varphi}\nu$. The function $g\frac{1}{\varphi}$ is then locally ν-integrable and we have $\mu = g\frac{1}{\varphi}\mu$.

Let $U_m\colon T \to \mathscr{L}(E, Z')$ be the function corresponding to m by Theorem 1, and put $V = U_m g\frac{1}{\varphi}$. As $|U_m(t)| = 1$ locally μ-almost everywhere, the function $|U_m| - 1$ is locally μ-negligible, whence the function $(|U_m| - 1)g$ is locally ν-negligible; therefore $|U_m(t)|g(t) = g(t)$ locally ν-almost everywhere. It follows that $|V(t)| = g(t)$ locally ν-almost everywhere, whence $|V|$ is locally ν-integrable and

$$\int f d\mu = \int |V| f d|\nu| \quad \text{for} \quad f \in \mathscr{L}^1(\mu).$$

Since U_m is Z-weakly μ-measurable, we deduce that for every $y \in E$ and $z \in Z$ the function $\langle U_m y, z\rangle$ is μ-measurable; it follows that the function $\langle U_m y, z\rangle g$ is ν-measurable, whence the function $\langle Vy, z\rangle = \langle U_m y, z\rangle g\frac{1}{\varphi}$ is ν-measurable, i. e. V is Z-weakly ν-measurable.

For every $x \in \mathscr{L}^1_E(\mu)$ and $z \in Z$, the function $\langle Vx, z\rangle$ is then ν-measurable and

$$\int^* |\langle Vx, z\rangle| d|\nu| \leqslant |z| \int^* |V||x| d|\nu| = |z| \int^* |x| d\mu < \infty;$$

hence $\langle Vx, z\rangle$ is ν-integrable and we have

$$\langle \int x dm, z\rangle = \int \langle U_m x, z\rangle d\mu = \int \langle U_m g\frac{1}{\varphi}x, z\rangle d\nu = \int \langle Vx, z\rangle d\nu.$$

By taking $V_m = V$, the first two parts are proved.

Now let ϱ be a lifting of $\mathscr{L}^\infty(\nu)$. There exists a function $V_m\colon T \to \mathscr{L}^*(E, Z')$ with $\varrho[V_m] = V_m$, $V_m \equiv V$ and $|V_m(t)| \leqslant |V(t)|$ for every $t \in T$. By Proposition 1, the function $|V_m|$ is ν-measurable and from

$|V_m| \leqslant |V|$ we deduce that $|V_m|$ is locally ν-integrable. From $V_m \equiv V$ we deduce that

$$\langle \int x\,dm, z\rangle = \int \langle V_m x, z\rangle d\nu \quad \text{for} \quad x \in \mathcal{L}^1_E(m) \text{ and } z \in Z.$$

For $x = \varphi_A x$ with $A \in \mathcal{B}$ and $x \in E$ it follows that

$$|\langle m(A)x, z\rangle| \leqslant \int_A |V_m||x||z|\,d|\nu| \quad \text{for every} \quad z \in Z.$$

Taking the supremum for $z \in Z$ with $|z| \leqslant 1$ and $x \in E$ with $|x| \leqslant 1$ we obtain

$$|m(A)| \leqslant \int_A |V_m|\,d|\nu| \quad \text{for every} \quad A \in \mathcal{B}.$$

It follows that $|m| \leqslant |V_m||\nu| \leqslant |V||\nu| = \mu$, whence $\mu = |V_m||\nu|$; therefore

$$\int f\,d\mu = \int f|V_m|\,d|\nu| \quad \text{for} \quad f \in \mathcal{L}^1(\mu).$$

Suppose now that $\mu \leqslant \alpha|\nu|$ for some $\alpha > 0$. Then the function g belongs to $\mathcal{L}^\infty(\nu)$, whence $|V| \in \mathcal{L}^\infty(\nu)$. From Proposition 4 we deduce that there exists a function $V_m\colon T \to \mathcal{L}^*(E, Z')$ with $\varrho(V_m) = V_m$, $V_m \equiv V$ and $|V_m(t)| \leqslant |V(t)|$ locally ν-almost everywhere.

We then prove that V_m has all the required properties as in the case of $\varrho[V_m] = V_m$.

If condition 4 is fulfilled with respect to μ, then we can choose $U_m(t) \in \mathcal{L}(E, F)$ locally μ-almost everywhere; therefore $V_m(t) \in \mathcal{L}(E, F)$ locally ν-almost everywhere. If condition 4 is fulfilled with respect to ν, then we prove as in Theorem 1 that $V_m(t) \in \mathcal{L}(E, F)$ locally ν-almost everywhere. Part 4′ is proved in the same way. Thus the theorem is completely proved.

Remark. The function V_m depends on E, F and Z.

The following theorem is, in a certain sense, converse to Theorem 2:

THEOREM 3. *Let ν be a scalar regular measure on $\mathcal{B}$ and let $U\colon T \to \mathcal{L}^*(E, F)$ be a Z-weakly ν-measurable function such that the function $|U|$ is locally ν-integrable. There exists then a regular measure $m\colon \mathcal{B} \to \mathcal{L}(E, Z')$ with finite variation μ such that*

$$\langle \int x\,dm, z\rangle = \int \langle U(t)x(t), z\rangle d\nu \quad \textit{for} \quad x \in \mathcal{L}^1_E(|U||\nu|) \textit{ and } z \in Z,$$

and

$$\int |f|\,d\mu \leqslant \int |U(t)||f(t)|\,d|\nu| \quad \textit{for} \quad f \in \mathcal{L}^1(|U||\nu|).$$

If, in addition, there exists a lifting ϱ of $\mathcal{L}^\infty(\nu)$ such that $\varrho[U] = U$, then we have

$$\int f\,d\mu = \int |U(t)|f(t)\,d|\nu| \quad \textit{for} \quad f \in \mathcal{L}^1(\mu).$$

The measure m has values in $\mathcal{L}(E, F)$ in each of the following cases:

a) *U is simply ν-measurable; in particular F is of countable type.*

b) *For every $x \in E$ there exists a family $\mathcal{K} \in \mathcal{C}(\nu)$ such that for every $K \in \mathcal{K}$ the convex equilibrated cover of the set $\{U(t)x;\ t \in K\}$ is relatively compact in F for the topology $\sigma(F, Z)$.*

Proof. For every $x \in E$ and $z \in Z$ the function $\langle Ux, z\rangle$ is ν-measurable and $|\langle Ux, z\rangle| \leqslant |U||x||z|$. Since $|U|$ is locally ν-integrable, we deduce that $\langle Ux, z\rangle$ is locally ν-integrable. Put

$$m_{x,z}(A) = \int_A \langle Ux, z\rangle d\nu \quad \text{for} \quad A \in \mathcal{B},\ x \in E \text{ and } z \in Z.$$

For $A \in \mathcal{B}$ and $x \in E$ fixed, the mapping $m_x(A)\colon z \to m_{x,z}(A)$ is a continuous linear functional on Z:

$$|m_{x,z}(A)| \leqslant |x||z| \int_A |U|\,d|\nu|;$$

therefore $m_x(A) \in Z'$ and

$$|m_x(A)| \leqslant |x| \int_A |U|\,d|\nu|.$$

For $A \in \mathcal{B}$ fixed, the mapping $m(A)\colon x \to m_x(A)$ of E into Z' is linear and continuous; therefore $m(A) \in \mathcal{L}(E, Z')$ and

$$|m(A)| \leqslant \int_A |U|\,d|\nu| = \int_A d(|U||\nu|).$$

The measure $\lambda = |U||\nu|$ is positive and regular and we have

$$|m(A)| \leqslant \lambda(A) \quad \text{for} \quad A \in \mathcal{B}.$$

We have also

$$\langle m(A)x, z\rangle = m_{x,z}(A) = \int_A \langle Ux, z\rangle d\nu \quad \text{for} \quad A \in \mathcal{B},\ x \in E \text{ and } z \in Z.$$

From the last equality we easily deduce that the mapping $m\colon \mathcal{B} \to \mathcal{L}(E, Z')$ is additive. From the inequality $|m(A)| \leqslant \lambda(A)$ for $A \in \mathcal{B}$ we deduce that m is countably additive, regular and with finite variation μ, and that $\mu(A) \leqslant \lambda(A)$ for $A \in \mathcal{B}$, i. e. $\mu \leqslant \lambda$. Then $\mathcal{L}^1(\lambda) \subset \mathcal{L}^1(\mu)$ and

$$\int |f|\,d\mu \leqslant \int |U||f|\,d|\nu| \quad \text{for} \quad f \in \mathcal{L}^1(\lambda).$$

We have also $\mathcal{L}^1_E(\lambda) \subset \mathcal{L}^1_E(\mu)$. For every step function $x = \sum \varphi_{A_i} x_i$ with $A_i \in \mathcal{B}$ and $x_i \in E$ we have

$$\langle \int x\,dm, z\rangle = \sum \langle m(A_i)x_i, z\rangle = \sum \int_{A_i} \langle U(t)x_i, z\rangle d\nu = \int \langle U(t)x(t), z\rangle d\nu$$

for every $z \in Z$. Now let $x \in \mathcal{L}_E^1(\lambda)$ and let (x_n) be a sequence of step functions of the preceding form, converging to x λ-almost everywhere and in the topology of $\mathcal{L}_E^1(\lambda)$.

The function $|x_n - x|$ converges to 0 locally almost everywhere with respect to $\lambda = |U||\nu|$; therefore the function $|U||x_n - x|$ converges to 0 locally almost everywhere with respect to $|\nu|$. From the inequality

$$|\langle Ux_n, z\rangle - \langle Ux, z\rangle| \leqslant |U||z||x_n - x|$$

we deduce that $\lim\limits_{n\to\infty}\langle U(t)x_n(t), z\rangle = \langle U(t)x(t, z)\rangle$ locally ν-almost everywhere for each $z \in Z$. On the other hand, from the inequality

$$\int |\langle U(x_n - x_m), z\rangle| \, d|\nu| \leqslant |z| \int |x_n - x_m| \, d\lambda$$

we deduce that $(\langle Ux_n, z\rangle)$ is a Cauchy sequence in $\mathcal{L}^1(\nu)$. It follows that the function $\langle Ux, z\rangle$ is ν-integrable for each $z \in Z$ and that

$$\lim_{n\to\infty} \int \langle Ux_n, z\rangle d\nu = \int \langle Ux, z\rangle d\nu.$$

From the inequality $\mu \leqslant \lambda$ we deduce also that x is μ-integrable and that (x_n) converges to x in $\mathcal{L}_E^1(\mu)$; therefore

$$\lim_{n\to\infty} \int x_n dm = \int x\, dm$$

and

$$\lim_{n\to\infty} \langle \int x_n dm, z\rangle = \langle \int x\, dm, z\rangle \quad \text{for} \quad z \in Z.$$

For each step function x_n we have

$$\langle \int x_n dm, z\rangle = \int \langle Ux_n, z\rangle d\nu \quad \text{for} \quad z \in Z.$$

Passing to the limit we obtain

$$\langle \int x\, dm, z\rangle = \int \langle Ux, z\rangle d\nu \quad \text{for} \quad z \in Z.$$

Suppose now that $\varrho[U] = U$ for some lifting ϱ of $\mathcal{L}^\infty(\nu)$. Let V_m be the function corresponding to m by Theorem 2, such that $\varrho[V_m] = V_m$. Also let φ be a locally ν-integrable scalar function such that $|\varphi(t)| = 1$ and $\nu = \varphi|\nu|$. Then

$$\langle m(A)x, z\rangle = \int_A \langle U(t)x, z\rangle \varphi(t)\, d|\nu| = \int_A \langle V_m(t)x, z\rangle \varphi(t)\, d|\nu|.$$

It follows that for every $x \in E$ and $z \in Z$ we have

$$\langle U(t)x, z\rangle \varphi(t) = \langle V_m(t)x, z\rangle \varphi(t)$$

locally ν-almost everywhere, whence

$$\langle U(t)x, z\rangle = \langle V_m(t)x, z\rangle \quad \text{locally } \nu\text{-almost everywhere},$$

i. e. $U \equiv V_m$. Then $U(t) = V_m(t)$ locally ν-almost everywhere; therefore

$$\int f d\mu = \int |V_m| f d|\nu| = \int |U| f d|\nu| \quad \text{for} \quad f \in \mathcal{L}^1(\mu).$$

Suppose now that U is simply ν-measurable. Then, for every $A \in \mathcal{B}$ and $x \in E$, the function $\varphi_A x$ is ν-integrable and we have

$$\langle m(A)x, z\rangle = \int_A \langle U(t)x, z\rangle d\nu = \langle \int_A U(t)x\, d\nu, z\rangle$$

for every $z \in Z$, whence

$$m(A)x = \int_A U(t)x\, d\nu \in F,$$

and consequently $m(A) \in \mathcal{L}(E, F)$.

Suppose finally that U verifies the condition b. Let $x \in E$ and let $\mathcal{K} \in \mathcal{C}(\nu)$ such that for every $K \in \mathcal{K}$ the closed (in $\sigma(F, Z)$) convex equilibrated covers $A(K)$ of the set $\{U(t)x: t \in K\}$ is compact for the topology $\sigma(F, Z)$.

For every $K \in \mathcal{K}$ the set $A(K)$ is closed in Z^* for the topology $\sigma(Z^*, Z)$. There exists then a family $(z_i)_{i\in I}$ of elements of Z such that

$$A(K) = \bigcap_{i \in I} \{y;\, y \in Z^*,\ |\langle y, z_i\rangle| \leqslant 1\}.$$

Then

$$|\langle U(t)x, z_i\rangle| \leqslant 1 \quad \text{for} \quad i \in I \text{ and } t \in K,$$

whence

$$|\langle m(K)x, z_i\rangle| \leqslant \int_K |\langle U(t)x, z_i\rangle| \, d|\nu| \leqslant |\nu|(K),$$

and consequently

$$m(K)x \in |\nu|(K)A(K) \subset F.$$

It follows then that $m(A)x \in F$ for every $A \in \mathcal{B}$ and every $x \in E$, whence $m(A) \in \mathcal{L}(E, F)$ for every $A \in \mathcal{B}$. The theorem is completely proved.

Remarks. (i) The measure m depends on the space Z. If Z_1 and Z_2 are two norming subspaces of F' such that $Z_1 \subset Z_2$ and if $m_1: \mathcal{B} \to \mathcal{L}(E, Z_1')$ and $m_2: \mathcal{B} \to \mathcal{L}(E, Z_2')$ are the corresponding measures, then for every $A \in \mathcal{B}$ and every $x \in E$, the functional $m_1(A)x \in Z_1'$ is the restriction to Z_1 of the functional $m_2(A)x \in Z_2'$.

(ii) One can prove that we have

$$\int f d\mu = \int |U| f d|\nu| \quad \text{for} \quad f \in \mathcal{L}^1(\mu)$$

in each of the following cases:

1) E is of countable type and there exists a countable norming set $S \subset Z$.

2) E is of countable type and U is simply ν-measurable.

3) U is ν-measurable.

4. Linear operations on $\mathscr{K}_E(T)$. Let $\mathscr{K}_E(T)$ be the space of the continuous functions $x\colon T \to E$ with a compact carrier. For every set $A \subset T$ we denote by $\mathscr{K}_E(T, A)$ the set of the functions $x \in \mathscr{K}_E(T)$ with the carrier contained in A.

We say that a linear mapping $f\colon \mathscr{K}_E(T) \to F$ is *dominated* if there exists a positive Radon measure ν such that

$$|f(x)| \leqslant \int |x|\,d\nu \quad \text{for every} \quad x \in \mathscr{K}_E(T).$$

If f is dominated, there exists a smallest positive Radon measure μ_f dominating f [7].

THEOREM 4. *There exists an isomorphism $f \leftrightarrow \boldsymbol{m}$ between the set of the dominated linear mappings $f\colon \mathscr{K}_E(T) \to F$ and the set of the regular measures $\boldsymbol{m}\colon \mathscr{B} \to \mathscr{L}(E, F)$ with finite variation μ, given by the equality*

$$f(x) = \int x\,d\boldsymbol{m}, \quad \text{for} \quad x \in \mathscr{K}_E(T).$$

If f and $\boldsymbol{m}$ are in correspondence, then $\mu_f = \mu$.

For the proof, see [8] and [13].

Remark. If f and $\boldsymbol{m}$ are in correspondence, we can extend f to the space $\mathscr{L}_E^1(\mu_f) = \mathscr{L}_E^1(\boldsymbol{m})$ by the equality

$$f(x) = \int x\,d\boldsymbol{m} \quad \text{for} \quad x \in \mathscr{L}_E^1(\boldsymbol{m}).$$

THEOREM 5. *There exists an isomorphism $f \leftrightarrow f'$ between the set of the dominated linear mappings $f\colon \mathscr{K}_E(T) \to F$ and the set of the dominated linear mappings $f'\colon \mathscr{K}(T) \to \mathscr{L}(E, F)$, given by the equality*

$$f(\varphi x) = f'(\varphi)x \quad \textit{for every} \quad \varphi \in \mathscr{K}(T) \ \textit{and} \ x \in E.$$

Moreover, we have $\mu_f = \mu_{f'}$.

The isomorphism $f \leftrightarrow f'$ is realized by the aid of the isomorphisms $f \leftrightarrow \boldsymbol{m}$ and $f' \leftrightarrow \boldsymbol{m}$, with the set of the regular measures $\boldsymbol{m}\colon \mathscr{B} \to \mathscr{L}(E, F)$ with finite variation μ:

$$f(x) = \int x\,d\boldsymbol{m} \quad \text{for} \quad x \in \mathscr{K}_E(T),$$
$$f'(\varphi) = \int \varphi\,d\boldsymbol{m} \quad \text{for} \quad \varphi \in \mathscr{K}(T)$$

and $\mu_f = \mu = \mu_{f'}$. For $\varphi \in \mathscr{K}(T)$ and $x \in E$ we have $\varphi x \in \mathscr{K}_E(T)$ and

$$\int \varphi\,d\boldsymbol{m} \cdot x = \int \varphi x\,d\boldsymbol{m};$$

therefore $f'(\varphi)x = f(\varphi x)$.

Theorem 5 allows us to identify f and f' and to write f instead of f'. With this convention we have

$$f(\varphi x) = f(\varphi)x \quad \text{for} \quad \varphi \in \mathscr{K}(T) \ \text{and} \ x \in E.$$

Using the isomorphism given in Theorem 4, we obtain from Theorems 1 and 3 the following two theorems proved in [7] (under some countability conditions) and in [20] in a different way.

THEOREM 6. *Let $f\colon \mathscr{K}_E(T) \to F$ be a dominated linear mapping. There exists then a function $U_f\colon T \to \mathscr{L}(E, Z')$ having the following properties:*

1) *$|U_f(t)| = 1$ locally μ_f-almost everywhere.*

2) *U_f is Z-weakly μ_f-measurable and we have*

$$\langle f(x), z\rangle = \int \langle U_f(t)x(t), z\rangle\,d\mu_f \quad \textit{for} \quad x \in \mathscr{L}_E^1(\mu_f) \ \textit{and} \ z \in Z.$$

3) *If ϱ is a lifting of $\mathscr{L}^\infty(\mu_f)$ we can take U_f uniquely such that*

$$\varrho(U_f) = U_f.$$

4) *If there exists a family $\mathscr{K} \in \mathscr{C}(\mu_f)$ such that for every $K \in \mathscr{K}$ and every $x \in E$ the convex equilibrated cover of the set $\{f(\varphi x); \varphi \in \mathscr{K}(T, K), \mu(|\varphi|) \leqslant 1\}$ is relatively compact in F for the topology $\sigma(F, Z)$, then we can choose $U_f(t) \in \mathscr{L}(E, F)$ locally μ_f-almost everywhere.*

4′) *If for every $x \in E$ the convex equilibrated cover of the set $\{f(\varphi x); \varphi \in \mathscr{K}(T), \mu_f(|\varphi|) \leqslant 1\}$ is relatively compact in F for the topology $\sigma(F, Z)$, then we can choose $U_f(t) \in \mathscr{L}(E, F)$ for every $t \in T$.*

In fact, let $\boldsymbol{m}\colon \mathscr{B} \to \mathscr{L}(E, F)$ be the measure corresponding to f by Theorem 5 and let $U_{\boldsymbol{m}}$ be the function corresponding to $\boldsymbol{m}$ by Theorem 1. If we take $U_f = U_{\boldsymbol{m}}$, the theorem is proved.

Remark. The function U_f depends on the space Z.

Here is the converse of the preceding theorem:

THEOREM 7. *Let μ be a scalar regular Borel measure and let $U\colon T \to \mathscr{L}^*(E, F)$ be a Z-weakly μ-measurable function such that $|U(t)| = 1$ locally μ-almost everywhere. There exists then a dominated linear mapping $f\colon \mathscr{K}_E(T) \to Z'$ such that*

$$\langle f(x), z\rangle = \int \langle U(t)x(t), z\rangle\,d\mu \quad \textit{for} \quad x \in \mathscr{K}_E(T) \ \textit{and} \ z \in Z$$

and $\mu_f \leqslant |\mu|$.

If, in addition, there exists a lifting ϱ of $\mathscr{L}^\infty(\mu)$ such that $\varrho[U] = U$, then we have $\mu_f = |\mu|$.

The mapping f has values in F in each of the following cases:

a) *U is simply μ-measurable; in particular F is of countable type.*

b) *For every $x \in E$ there exists a family $\mathscr{K} \in \mathscr{C}(\mu)$ such that for every $K \in \mathscr{K}$ the convex equilibrated cover of the set $\{U(t)x; t \in K\}$ is relatively compact in F for the topology $\sigma(F, Z)$.*

In fact, if $\boldsymbol{m}: \mathscr{B} \to \mathscr{L}(E, Z')$ is the measure corresponding to μ and U by Theorem 3, then the mapping $f: \mathscr{K}_E(T) \to F$ corresponding to $\boldsymbol{m}$ by Theorem 5 has all the required properties.

Remarks. (i) The mapping f depends on Z.

(ii) We also have $\mu_f = |\mu|$ in each of the following cases:

1) E is of countable type and there exists a countable norming subset $S \subset Z$.

2) E is of countable type and U is simply μ-measurable.

3) U is μ-measurable.

5. Linear operations on $\mathscr{L}_E^p$. Let ν be a positive regular measure on T. Consider the space $\mathscr{L}_E^p(\nu)$ with $1 \leqslant p < \infty$. For every linear mapping $f: \mathscr{L}_E^p(\nu) \to F$ we put

$$|||f||| = \sup \sum |f(\varphi_{A_i} x_i)|$$

where the supremum is taken for all the step functions $\boldsymbol{x} = \sum \varphi_{A_i} x_i$ such that A_i are disjoint sets of $\mathscr{B}$, $x_i \in E$ and $N_p(\boldsymbol{x}, \nu) \leqslant 1$. We have $\|f\| \leqslant |||f||| \leqslant \infty$. If $p = 1$, or if $F = C$, then $\|f\| = |||f|||$, [13].

THEOREM 8. *There exists an isomorphism $f \leftrightarrow f'$ between the set of the linear mappings $f: \mathscr{L}_E^p(\nu) \to F$ with $|||f||| < \infty$ and the set of the linear mappings $f': \mathscr{L}^p(\nu) \to \mathscr{L}(E, F)$ with $|||f'||| < \infty$ given by the equality*

$$f(\varphi x) = f'(\varphi)x \quad \text{for} \quad \varphi \in \mathscr{L}^p(\nu) \text{ and } x \in E.$$

Moreover $|||f||| = |||f'|||$.

Let $f: \mathscr{L}_E^p(\nu) \to F$ be a linear mapping with $|||f||| < \infty$. Let $\varphi \in \mathscr{L}^p(\nu)$. For every $x \in E$ we have $\varphi x \in \mathscr{L}_E^p(\nu)$. The mapping $f'(\varphi)$: $x \to f(\varphi_x)$ of E into F is linear and continuous:

$$|f'(\varphi)x| = |f(\varphi x)| \leqslant \|f\| N_p(\varphi x, \nu) \leqslant |x| \, \|f\| N_p(\varphi, \nu) < \infty;$$

hence $f'(\varphi) \in \mathscr{L}(E, F)$ and

$$|f'(\varphi)| \leqslant \|f\| N_p(\varphi, \nu).$$

The mapping $f': \varphi \to f'(\varphi)$ of $\mathscr{L}^p(\nu)$ into $\mathscr{L}(E, F)$ is linear and $\|f'\| \leqslant \|f\|$. We also have $|||f'||| \leqslant |||f|||$. In fact, let $\varphi = \sum_{i=1}^{n} \varphi_{A_i} \alpha_i$ be a real step function with A_i disjoint sets of $\mathscr{B}$ and $N_p(\varphi, \nu) \leqslant 1$, and let $\varepsilon > 0$. For each i there exists $x_i \in E$ with $|x_i| = 1$ such that

$$|f'(\varphi_{A_i})| < |f'(\varphi_{A_i}) x_i| + \varepsilon/n.$$

Then

$$N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} \alpha_i x_i, \nu\Big) = N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} \alpha_i, \nu\Big) \leqslant 1$$

and

$$\sum_{i=1}^{n} |f'(\varphi_{A_i}) \alpha_i| - \varepsilon < \sum_{i=1}^{n} |f'(\varphi_{A_i}) \alpha_i x_i| = \sum_{i=1}^{n} |f(\varphi_{A_i} \alpha_i x_i)| \leqslant |||f|||,$$

whence

$$|||f'||| \leqslant |||f|||.$$

The correspondence $f \to f'$ is, evidently, linear. It is also one-to-one. In fact, if $f' = 0$, then $f(\sum \varphi_{A_i} x_i) = 0$ for every step function $\sum \varphi_{A_i} x_i$ with $A_i \in \mathscr{B}$ and $x_i \in E$. Since the step functions are dense in $\mathscr{L}_E^p(\nu)$, we deduce that $f = 0$.

Now let $f': \mathscr{L}^p(\nu) \to \mathscr{L}(E, F)$ be a linear mapping with $|||f'||| < \infty$. We shall prove that there exists a linear mapping $f: \mathscr{L}_E^p(\nu) \to F$ with $|||f||| < \infty$ such that $f(\varphi x) = f'(\varphi)x$ for $\varphi \in \mathscr{L}^p(\nu)$ and $x \in E$.

For every set function $\boldsymbol{x} = \sum \varphi_{A_i} x_i$ with $A_i \in \mathscr{B}$ and $x_i \in E$, put

$$f(\boldsymbol{x}) = \sum f'(\varphi_{A_i}) x_i.$$

The definition of $f(\boldsymbol{x})$ does not depend on the particular form in which $\boldsymbol{x}$ is written as a step function. If we take the sets A_i disjoint, we have

$$|f(\boldsymbol{x})| = \Big|\sum f'(\varphi_{A_i}) x_i\Big| \leqslant \sum |f'(\varphi_{A_i})| \, |x_i|$$
$$\leqslant |||f'||| N_p\Big(\sum \varphi_{A_i} |x_i|, \nu\Big) = |||f'||| N_p(\boldsymbol{x}, \nu).$$

It follows that f is continuous on the step functions for the semi-norm N_p, whence f can be extended to a continuous linear mapping of $\mathscr{L}_E^p(\nu)$ into F, denoted also by f.

Let $\boldsymbol{x} = \sum \varphi_{A_i} x_i$ be a step function with A_i disjoint sets of $\mathscr{B}$, $x_i \in E$ and $N_p(\boldsymbol{x}, \nu) \leqslant 1$. Then $|\boldsymbol{x}| = \sum \varphi_{A_i} |x_i|$; hence $N_p(|\boldsymbol{x}|, \nu) \leqslant 1$ and

$$\sum |f(\varphi_{A_i} x_i)| = \sum |f'(\varphi_{A_i}) x_i| \leqslant \sum |f'(\varphi_{A_i})| \, |x_i| \leqslant |||f'|||;$$

therefore $|||f||| \leqslant |||f'||| < \infty$. From the first part of the proof we deduce that $|||f||| = |||f'|||$ and thus the theorem is completely proved.

COROLLARY 1. *There exists an isomorphism $f \leftrightarrow f'$ between the set of the linear continuous mappings f: $\mathscr{L}_E^1(\nu) \to F$ and the set of the linear continuous mappings f': $\mathscr{L}^1(\nu) \to \mathscr{L}(E, F)$ given by the equality*

$$f(\varphi x) = f'(\varphi)x \quad \text{for} \quad \varphi \in \mathscr{L}^1(\nu) \text{ and } x \in E$$

and we have $\|f\| = \|f'\|$.

In fact, in this case we have $\|f\| = |||f|||$ and $\|f'\| = |||f'|||$.

COROLLARY 2. *There exists an isomorphism $f \leftrightarrow f'$ between the set of the continuous linear functionals $f: \mathcal{L}^p_E(\nu) \to C$ and the set of the linear mappings $f': \mathcal{L}^p(\nu) \to E' = \mathcal{L}(E, C)$ with $|||f'||| < \infty$, given by*

$$f(\varphi x) = f'(\varphi) x \quad \text{for} \quad \varphi \in \mathcal{L}^p(\nu) \text{ and } x \in E,$$

and we have $\|f\| = |||f'|||$.

In fact, in this case we have $\|f\| = |||f|||$.

Remark. Theorem 8 gives rise to the identification of the corresponding mappings f and f'. We shall write f instead of f'. Then we can write

$$f(\varphi x) = f(\varphi) x \quad \text{for} \quad \varphi \in \mathcal{L}^p(\nu) \text{ and } x \in E.$$

THEOREM 9. *Let X be a Banach space. There exists an isomorphism $f \leftrightarrow \boldsymbol{m}$ between the set of the linear mappings $f: \mathcal{L}^p(\nu) \to X$ with $|||f||| < \infty$ and the set of the regular measures $\boldsymbol{m}: \mathcal{B} \to X$ with finite variation μ absolutely continuous with respect to ν such that*

$$\int |\varphi| \, d\mu \leqslant a N_p(\varphi, \nu)$$

for some $a > 0$ and every step function $\varphi = \sum \varphi_{A_i} a_i$ with $A_i \in \mathcal{B}$. The correspondence is given by the equality $f(\varphi_A) = \boldsymbol{m}(A)$ for $A \in \mathcal{B}$.

If f and $\boldsymbol{m}$ are in correspondence, then we have $\mu = g\nu$ with $N_q(g, \nu) = |||f|||$, $1/p + 1/q = 1$, and for every pair of Banach spaces E and F such that $X \subset \mathcal{L}(E, F)$ we have $\mathcal{L}^p_E(\nu) \subset \mathcal{L}^1_E(\boldsymbol{m})$ and

$$f(\boldsymbol{x}) = \int \boldsymbol{x} \, d\boldsymbol{m} \quad \text{for} \quad \boldsymbol{x} \in \mathcal{L}^p_E(\nu),$$

$$\int |\boldsymbol{x}| \, d\mu \leqslant |||f||| N_p(\boldsymbol{x}, \nu) \quad \text{for} \quad \boldsymbol{x} \in \mathcal{L}^p_E(\nu).$$

a) Let $f: \mathcal{L}^p(\nu) \to X$ be a linear mapping with $|||f||| < \infty$. For every set $A \in \mathcal{B}$ we put

$$\boldsymbol{m}(A) = f(\varphi_A).$$

The set function $\boldsymbol{m}: \mathcal{B} \to X$ is additive, and we shall prove that it is with finite variation. Let $A \in \mathcal{B}$ and let (A_i) be a finite family of disjoint sets of $\mathcal{B}$ contained in A. Then

$$\sum |\boldsymbol{m}(A_i)| = \sum |f(\varphi_{A_i})| \leqslant |||f||| N_p\Big(\sum \varphi_{A_i}, \nu\Big) \leqslant |||f||| N_p(\varphi_A, \nu) < \infty;$$

therefore $\boldsymbol{m}$ is with finite variation μ and

$$\mu(A) \leqslant |||f||| N_p(\varphi_A, \nu).$$

From this inequality we deduce that μ is regular, countably additive and absolutely continuous with respect to ν; therefore $\boldsymbol{m}$ also has these properties.

Now let $\varphi = \sum_{i=1}^{n} \varphi_{A_i} a_i$ be a step function with $A_i \in \mathcal{B}$ and prove that $\int |\varphi| \, d\mu \leqslant |||f||| N_p(\varphi, \nu)$. We can take the sets A_i disjoint. Let $\varepsilon > 0$. For each i there exists a finite family (B_{ij}) of disjoint sets of $\mathcal{B}$ contained in A_i such that

$$\mu(A_i) \leqslant \sum_j |\boldsymbol{m}(B_{ij})| + \varepsilon / |a_i| n, \quad \text{if} \quad a_i \neq 0,$$

$$\mu(A_i) \leqslant \sum_j |\boldsymbol{m}(B_{ij})| + \varepsilon / n, \quad \text{if} \quad a_i = 0.$$

Then, for every i we have

$$\mu(A_i) |a_i| \leqslant \sum_j |\boldsymbol{m}(B_{ij})| \, |a_i| + \varepsilon / n;$$

therefore

$$\sum_i \mu(A_i) |a_i| \leqslant \sum_{i,j} |\boldsymbol{m}(B_{ij})| |a_i| + \varepsilon = \sum_{i,j} |f(\varphi_{B_{ij}}) a_i| + \varepsilon$$
$$\leqslant |||f||| N_p\Big(\sum_{i,j} \varphi_{B_{ij}} a_i, \nu\Big) \leqslant |||f||| N_p\Big(\sum_i \varphi_{A_i} a_i, \nu\Big) + \varepsilon;$$

t being arbitrary, we deduce that

$$\int |\varphi| \, d\mu = \sum \mu(A_i) |a_i| \leqslant |||f||| N_p(\varphi, \nu).$$

From the definition of $\boldsymbol{m}$ we immediately deduce that

$$\int \varphi \, d\boldsymbol{m} = \sum \boldsymbol{m}(A_i) a_i = \sum f(\varphi_{A_i}) a_i = f\Big(\sum \varphi_{A_i} a_i\Big) = f(\varphi).$$

The correspondence $f \to \boldsymbol{m}$ is evidently linear. It is also one-to-one, because if $\boldsymbol{m} = 0$, then $f(\varphi) = 0$ for every step function $\varphi \in \mathcal{L}^p(\nu)$, whence $f = 0$.

b) Conversely, let $\boldsymbol{m}: \mathcal{B} \to X$ be a measure with finite variation μ such that for some $a > 0$ we have

$$\int |\varphi| \, d\mu \leqslant a N_p(\varphi, \nu)$$

for every step function $\varphi = \sum \varphi_{A_i} a_i$ with $A_i \in \mathcal{B}$.

It follows that μ is absolutely continuous with respect to ν; consequently there exists a locally ν-integrable function $g \geqslant 0$ such that $\mu = g\nu$. For every step function $\varphi = \sum \varphi_{A_i} a_i$ with $A_i \in \mathcal{B}$ we have

$$\int |\varphi| \, g \, d\nu = \int |\varphi| \, d\mu \leqslant a N_p(\varphi, \nu);$$

therefore $N_q(g, \nu) \leqslant a < \infty$, where $1/p + 1/q = 1$.

It follows that if $\varphi \in \mathscr{L}^p(\nu)$, then $\varphi g \in \mathscr{L}^1(\nu)$; hence $\varphi \in \mathscr{L}^1(\mu)$, and therefore $\mathscr{L}^p(\nu) \subset \mathscr{L}^1(\mu) = \mathscr{L}^1(\boldsymbol{m})$. We then put

$$f(\varphi) = \int \varphi\, d\boldsymbol{m} \quad \text{for} \quad \varphi \in \mathscr{L}^p(\nu).$$

The mapping $f: \mathscr{L}^p(\nu) \to X$ is linear and we show that $|||f||| < \infty$. Let $\varphi = \sum \varphi_{A_i} a_i$ be a step function where A_i are disjoint sets of $\mathscr{B}$. Then

$$\sum |f(\varphi_{A_i} a_i)| = \sum \Big| \int \varphi_{A_i} a_i\, d\boldsymbol{m} \Big| \leqslant \sum \int \varphi_{A_i} |a_i|\, d\mu$$

$$= \int \sum \varphi_{A_i} |a_i|\, d\mu = \int |\varphi|\, d\mu \leqslant a N_p(\varphi, \nu),$$

hence $|||f||| \leqslant a < \infty$.

c) Now let $\boldsymbol{m}$ and f be in correspondence and let E, F be two Banach spaces such that $X \subset \mathscr{L}(E, F)$. From the first part of the proof we deduce that $\int |\varphi|\, d\mu \leqslant |||f||| N_p(\varphi, \nu)$ for every step function $\varphi = \sum \varphi_{A_i} a_i$ with $A_i \in \mathscr{B}$, and from the second part of the proof, for $a = |||f|||$, we deduce that there exists a locally ν-integrable function $g \geqslant 0$ such that $\mu = g\nu$ and $N_q(g, \nu) \leqslant |||f||| < \infty$. Let $\varphi = \sum \varphi_{A_i} a_i$ be a step function with A_i disjoint sets of $\mathscr{B}$ and $N_p(\varphi, \nu) \leqslant 1$. Then

$$\sum |f(\varphi_{A_i} a_i)| = \sum \Big| \int \varphi_{A_i} a_i\, d\boldsymbol{m} \Big| \leqslant \sum \int \varphi_{A_i} |a_i|\, d\mu$$

$$= \int \sum \varphi_{A_i} |a_i|\, d\mu = \int |\varphi|\, d\mu = \int |\varphi|\, g\, d\nu \leqslant N_q(g, \nu),$$

hence $|||f||| \leqslant N_q(g, \nu)$, and therefore

$$N_q(g, \nu) = |||f|||.$$

If $\boldsymbol{x} \in \mathscr{L}^p_E(\nu)$, then $\boldsymbol{x} g \in \mathscr{L}^1_E(\nu)$, hence $\boldsymbol{x} \in \mathscr{L}^1_E(\mu) = \mathscr{L}^1_E(\boldsymbol{m})$; therefore $\mathscr{L}^p_E(\nu) \subset \mathscr{L}^1_E(\boldsymbol{m})$ and we have

$$\int |\boldsymbol{x}|\, d\mu = \int |\boldsymbol{x}|\, g\, d\nu \leqslant N_p(\boldsymbol{x}, \nu) N_q(g, \nu) = |||f||| N_p(\boldsymbol{x}, \nu).$$

It follows that

$$\Big| \int \boldsymbol{x}\, d\boldsymbol{m} \Big| \leqslant \int |\boldsymbol{x}|\, d\mu \leqslant |||f||| N_p(\boldsymbol{x}, \nu) \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^p_E(\nu);$$

therefore the mapping $\boldsymbol{x} \to \int \boldsymbol{x}\, d\boldsymbol{m}$ is continuous on $\mathscr{L}^p_E(\nu)$. Since

$$f(\boldsymbol{x}) = \int \boldsymbol{x}\, d\boldsymbol{m} \text{ for every step function } \boldsymbol{x} = \sum \varphi_{A_i} x_i \text{ of } \mathscr{S}_E(\mathscr{B}),$$

and since the two members of this equality are continuous functions of $\boldsymbol{x}$ in $\mathscr{L}^p_E(\nu)$, we deduce that

$$f(\boldsymbol{x}) = \int \boldsymbol{x}\, d\boldsymbol{m} \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^p_E(\nu),$$

and thus the theorem is completely proved.

Corollary 1. *Let X be a Banach space. There exists an isomorphism $f \leftrightarrow \boldsymbol{m}$ between the set of the continuous linear mappings $f: \mathscr{L}^1(\nu) \to X$ and the set of the regular measures $\boldsymbol{m}: \mathscr{B} \to X$ with finite variation μ such that $\mu \leqslant a\nu$ for some $a > 0$.*

The correspondence is given by the equality

$$f(\varphi_A) = \boldsymbol{m}(A) \quad \text{for every} \quad A \in \mathscr{B}.$$

If f and $\boldsymbol{m}$ are in correspondence, then $\mu = g\nu$ with $N_\infty(g, \nu) = \|f\|$ and for every pair of Banach spaces E, F with $X \subset \mathscr{L}(E, F)$ we have

$$f(\boldsymbol{x}) = \int \boldsymbol{x}\, d\boldsymbol{m} \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^1_E(\nu)$$

and

$$\int |\boldsymbol{x}|\, d\mu \leqslant \|f\| \int |\boldsymbol{x}|\, d\nu \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^1_E(\nu).$$

Corollary 2. *Let X be a Banach space. For every linear mapping $f: \mathscr{L}^p(\nu) \to X$ we denote by f^0 the restriction of f to $\mathscr{K}(T)$. Then the correspondence $f \to f^0$ is an isomorphism between the set of the linear mappings $f: \mathscr{L}^p(\nu) \to X$ with $|||f||| < \infty$ and the set of the dominated linear mappings $f^0: \mathscr{K}(T) \to X$ such that*

$$\int |\varphi|\, d\mu_{f^0} \leqslant a N_p(\varphi, \nu) \quad \text{for some } a > 0 \text{ and every } \varphi \in \mathscr{K}(T).$$

Then $\mu_{f^0} = g\nu$ with $N_q(g, \nu) = |||f|||$, $1/p + 1/q = 1$, and for every pair of Banach spaces E and F with $X \subset \mathscr{L}(E, F)$ we have $\mathscr{L}^p_E(\nu) \subset \mathscr{L}^1_E(\mu_{f^0})$ and

$$\int |\boldsymbol{x}|\, d\mu_{f^0} \leqslant |||f||| N_p(\boldsymbol{x}, \nu) \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^p_E(\nu).$$

The following two theorems are proved in [3] (under some countability hypotheses) and in [20] in a different way (for E and F locally convex spaces).

Theorem 10. *Let ν be a scalar regular Borel measure and $f: \mathscr{L}^p_E(\nu) \to F$ a linear mapping with $|||f||| < \infty$. There exists then a function $U_f: T \to \mathscr{L}(E, Z')$ having the following properties:*

1) *The function $|U_f|$ belongs to $\mathscr{L}^q(\nu)$ and we have*

$$N_q(U_f, \nu) = |||f|||, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

2) *The function U_f is Z-weakly ν-measurable and we have*

$$\langle f(\boldsymbol{x}), z\rangle = \int \langle U_f(t)\boldsymbol{x}(t), z\rangle\, d\nu(t) \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}^p_E(\nu) \text{ and } z \in Z.$$

3) *If ϱ is a lifting of $\mathscr{L}^\infty(\nu)$ we can choose U_f uniquely locally ν-almost everywhere such that $\varrho[U_f] = U_f$.*

If, in addition, $p = 1$, we can take U_f uniquely such that $\varrho(U_f) = U_f$.

4) *If there exists a family* $\mathscr{K} \in \mathscr{C}(\nu)$ *such that for every* $K \in \mathscr{K}$ *and every* $x \in E$ *the convex equilibrated cover of the set* $\{f(\varphi x); \varphi \in \mathscr{K}(T, K), \int |\varphi| d|\nu| \leqslant 1\}$ *is relatively compact in* F *for the topology* $\sigma(F, Z)$, *then we can choose* $U_f(t) \in \mathscr{L}(E, F)$ *locally* μ-*almost everywhere.*

4') *If for every* $x \in E$ *the convex equilibrated cover of the set* $\{f(\varphi x); \varphi \in \mathscr{K}(T), \int |\varphi| d|\nu| \leqslant 1\}$ *is relatively compact in* $\sigma(F, Z)$ *for the topology* $\sigma(F, Z)$, *then we can choose* $U_f(t) \in \mathscr{L}(E, F)$ *for every* $t \in T$.

Let $\boldsymbol{m}: \mathscr{B} \to \mathscr{L}(E, F)$ be the measure with finite variation μ, corresponding to f and $|\nu|$ by Theorem 9.

We have $\mu = g\nu$ with $N_q(g, \nu) = |||f|||$; if $p = 1$, then $\mu \leqslant |||f|||\nu$. We have $\mathscr{L}_E^p(\nu) \subset \mathscr{L}_E^1(\boldsymbol{m})$ and

$$f(\boldsymbol{x}) = \int \boldsymbol{x}\, d\boldsymbol{m} \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}_E^p(\nu).$$

Then let $V_{\boldsymbol{m}}: T \to \mathscr{L}(E, Z')$ be the function corresponding to $\boldsymbol{m}$ and ν by Theorem 2.

We have $\mu = |V_m| |\nu|$, whence $|V_m(t)| = g(t)$ locally ν-almost everywhere and therefore

$$N_q(V_{\boldsymbol{m}}, \nu) = N_q(g, \nu) = |||f|||.$$

If we take $U_f = U_{\boldsymbol{m}}$, then from Theorem 2 we deduce that U_f has all the required properties.

The following theorem is in a certain sense the converse of Theorem 10:

THEOREM 11. *Let* ν *be a regular scalar measure,* $1 \leqslant p < \infty$ *and* $U: T \to \mathscr{L}^*(E, F)$ *a* Z-*weakly* ν-*measurable function such that*

(i) $N_q(U, \nu) < \infty$, $1/p + 1/q = 1$.

There exists a linear mapping $f: \mathscr{L}_E^p(\nu) \to Z'$ *such that*

$$|||f||| \leqslant N_q(U, \nu)$$

and

$$\langle f(\boldsymbol{x}), z\rangle = \int \langle U(t)\boldsymbol{x}(t), z\rangle d\nu \quad \textit{for} \quad \boldsymbol{x} \in \mathscr{L}_E^p(\nu) \textit{ and } z \in Z.$$

(ii) *If, in addition, there exists a lifting* ϱ *of* $\mathscr{L}^\infty(\nu)$ *such that* $\varrho[U] = U$, *then we have*

$$|||f||| = N_q(U, \nu).$$

(iii) *We have* $f(\boldsymbol{x}) \in F$ *for every* $\boldsymbol{x} \in \mathscr{L}_E^p(\nu)$ *in each of the following cases:*

a) V *is simply* ν-*measurable; in particular* F *is of countable type;*

b) *For every* $x \in E$ *there exists a family* $\mathscr{K} \in \mathscr{C}(\nu)$ *such that for every* $K \in \mathscr{K}$ *the convex equilibrated cover of the set* $\{U(t)x; t \in K\}$ *is relatively compact in* F *for the topology* $\sigma(F, Z)$.

Proof. We remark first that for each $\boldsymbol{x} \in \mathscr{L}_E^p(\nu)$ and each $z \in Z$ the function $\langle U\boldsymbol{x}, z\rangle$ is ν-measurable and

$$\int^* |\langle U\boldsymbol{x}, z\rangle| d|\nu| \leqslant |z| \int^* |U| |\boldsymbol{x}| d|\nu| \leqslant |z| N_p(\boldsymbol{x}, \nu) N_q(U, \nu) < \infty;$$

therefore $\langle U\boldsymbol{x}, z\rangle$ is ν-integrable. Put

$$f_z(\boldsymbol{x}) = \int \langle U\boldsymbol{x}, z\rangle d\nu.$$

The mapping $f(\boldsymbol{x}): z \to f_z(\boldsymbol{x})$ is a linear and continuous functional on Z; therefore $f(\boldsymbol{x}) \in Z'$ and

$$|f(\boldsymbol{x})| \leqslant N_q(U, \nu) N_p(\boldsymbol{x}, \nu).$$

The mapping $f: \boldsymbol{x} \to f(\boldsymbol{x})$ of $\mathscr{L}_E^p(\nu)$ into Z' is linear and continuous and we have

$$\|f\| \leqslant N_q(U, \nu)$$

and

$$\langle f(\boldsymbol{x}), z\rangle = f_z(\boldsymbol{x}) = \int \langle U\boldsymbol{x}, z\rangle d\nu \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}_E^p(\nu) \text{ and } z \in Z.$$

We now prove that $|||f||| < N_q(U, \nu)$. Let $\boldsymbol{x} = \sum \varphi_{A_i} x_i$ be a step function such that A_i are disjoint sets of $\mathscr{B}$ and $N_q(\boldsymbol{x}, \nu) \leqslant 1$. Then (see [2], chap. V, § 2, lemme 2, pp. 11-12)

$$\sum |f(\varphi_{A_i} x_i)| \leqslant \sum \int^* |U\varphi_{A_i} x_i| d|\nu| \leqslant \sum \int^* |U| \varphi_{A_i} |x_i| d|\nu|$$
$$= \int^* |U| \Big(\sum \varphi_{A_i} |x_i|\Big) d|\nu| \leqslant \int^* |U| \Big|\sum \varphi_{A_i} x_i\Big| d|\nu|$$
$$\leqslant N_q(U, \nu) N_p(\boldsymbol{x}, \nu) \leqslant N_q(U, \nu);$$

therefore $|||f||| \leqslant N_q(U, \nu) < \infty$.

Suppose now that $\varrho[U] = U$ for some lifting ϱ of $\mathscr{L}^\infty(\nu)$. Let U_f be the function corresponding to f by Theorem 10, such that $\varrho[U_f] = U_f$, $|||f||| = N_q(U, \nu)$ and

$$\langle f(\boldsymbol{x}), z\rangle = \int \langle U_f \boldsymbol{x}, z\rangle d\nu \quad \text{for} \quad \boldsymbol{x} \in \mathscr{L}_E^p(\nu) \text{ and } z \in Z.$$

We then deduce that

$$\int \langle Ux, z\rangle \varphi\, d\nu = \int \langle U_f x, z\rangle \varphi\, d\nu \quad \text{for} \quad \varphi \in \mathscr{K}(T),\ x \in E \text{ and } z \in Z;$$

therefore $\langle U(t)x, z\rangle = \langle U_f(z)x, z\rangle$ locally ν-almost everywhere for each $x \in E$ and $z \in Z$, i. e. $U \equiv U_f$. Then $U(t) = U_f(t)$ locally ν-almost everywhere, and consequently

$$|||f||| = N_q(U_f, \nu) = N_q(U, \nu).$$

Let $m: \mathscr{B} \to \mathscr{L}(E, Z')$ be the measure corresponding to f by Theorem 5:

$$f(x) = \int x\,dm \quad \text{for} \quad x \in \mathscr{K}_E(T).$$

If U verifies one of the conditions (iii), then by Theorem 3, m is with values in $\mathscr{L}(E, F)$; therefore $f(x) \in \mathscr{L}(E, F)$ for every $x \in \mathscr{K}_E(T)$ and then, passing to the limit, for every $x \in \mathscr{L}^p_E(\nu)$.

The theorem is completely proved.

Remarks. 1 If we consider the condition

$$\text{(i')} \qquad \int^* |U(t)x(t)|\,d|\nu| < \infty \quad \text{for every} \quad x \in \mathscr{L}^p_E(\nu)$$

instead of (i), we can deduce as in the proof of Theorem 11 that there exists a linear mapping $f: \mathscr{L}^p_E(\nu) \to Z'$ such that

$$\langle f(x), z\rangle = \int \langle Ux, z\rangle\,d\nu \quad \text{for} \quad x \in \mathscr{L}^p_E(\nu) \text{ and } z \in Z,$$

and $|||f||| \leqslant N_q(U, \nu) \leqslant \infty$.

We remark that (i) implies (i')

2. Consider the conditions (i') and (ii). Then we have $|||f||| = N_q(U, \nu)$.

In fact, if $|||f||| = \infty$, it follows that $N_q(U, \nu) = \infty$, and if $|||f||| < \infty$, we reason as in the proof of the theorem.

3. If condition (i') is verified, one can prove that we have the equality $|||f||| = N_q(U, \nu)$ also in each of the following cases:

1) E is of countable type and there exists a countable norming set $S \subset Z$.

2) E is of countable type and U is simply ν-measurable. In this case we deduce from (i') that the function $t \to U(z)x$ is locally ν-integrable and then that the function $t \to |U(t)|$ is locally ν-integrable, and we have

$$f(x) = \int U(t)x(t)\,d\nu(t) \quad \text{for} \quad x \in \mathscr{L}^p_E(\nu).$$

3) U is locally ν-integrable. In this case we have

$$f(\varphi) = \int U(t)\varphi(t)\,d\nu(t) \quad \text{for} \quad \varphi \in \mathscr{L}^p(\nu).$$

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