A remark on my paper

Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces

by

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1. Let f denote a real-valued function defined for $-\infty < u < \infty$. We shall use the following notation:

$$C_f^0 = \{\mu : \lim_{u\to\infty}(f(u+\mu)-f(u)) = 0\},$$

$$C_f = \{\mu : \lim_{u\to\infty}(f(u+\mu)-f(u)) \text{ exists and is finite}\},$$

$$B_f = \{\mu : \overline{\lim_{u\to\infty}}|f(u+\mu)-f(u)| < \infty\}.$$

In [3] I dealt with theorems concerning the above-mentioned sets under the assumption that f is measurable or possesses Baire's property. In my proofs I considered sets of the form $E = \{\mu : |f(u+\mu)-f(u)| \leqslant \varepsilon$ for $\mu_1 \leqslant \mu \leqslant \mu_2$, $u \geqslant u_0\}$, making use of their measurability. J. Karamata kindly called my attention to the fact that this measurability is not quite evident. In fact, after strict examination of my proof I have found that the measurability of the set E is obvious only in some special cases, e. g. if we assume that f is continuous in $(-\infty, +\infty)$ or possesses discontinuities of a simple type. In the case of an arbitrary measurable function f, the measurability of E is not obvious: it is even rather doubtful if we take into account results of a recently published paper by Rubel [4]. Therefore in this paper I shall prove the theorems considered in 1.3, 1.4 of [3] by applying other arguments, consisting of a modification of the method used in [1]. Those arguments are a little more laborious than the previous one; however, if we assume f to be continuous (this may usually be assumed in various aplications), the arguments of [3] remain true, and the proof of the measurability of C_f, C_f^0 and B_f is easy. Moreover, the result of 1.3 [3] is even a little strengthened by replacing the assumption of the continuity of f by a more general assumption.

2. *Let f be a measurable function; then each of the sets C_f, C_f^0, B_f is either of measure 0 or identical with $(-\infty, \infty)$. If $C_f = (-\infty, \infty)$, then*

$$(*) \qquad f(u+\mu)-f(u)$$

converges to $a\mu$ uniformly in every bounded interval of values μ, where a is a constant. If $B_f = (-\infty, \infty)$, then $(*)$ is uniformly bounded in every bounded interval of values μ for sufficiently large u.

First we consider the case of C_f. Let $\varepsilon > 0$, $u_n \to \infty$, $v_n \to \infty$, $\mu_n \to 0$. Let us write

$$A_n^\varepsilon = \{\mu\colon |f(u_k+\mu)-f(u_k)-\big(f(v_k+\mu)-f(v_k)\big)| \leqslant \tfrac{1}{2}\varepsilon \text{ and}$$
$$|f(u_k+\mu_k+\mu)-f(u_k+\mu_k)-\big(f(v_k+\mu_k+\mu)-f(v_k+\mu_k)\big)| \leqslant \tfrac{1}{2}\varepsilon \text{ as } k \geqslant n\}.$$

Obviously, the sets A_n^ε are measurable, $C_f \subset \bigcap_l \bigcup_n A_n^{1/l} = M$, where M is a measurable set. We must show that $m_e(C_f) > 0$ implies $C_f = (-\infty, \infty)$. If $m_e(C_f) > 0$, then $m(M) > 0$, and so the set $\bigcup_n A_n^\varepsilon$ is of positive measure; hence at least one of the sets A_l^ε is of positive measure, where l depends on ε. Let us denote by $A_{l\mu}^\varepsilon$ the set obtained by translation by μ of the set A_l^ε. Then, by a known theorem of Steinhaus, there exists a $\mu_0 > 0$ such that $A_l^\varepsilon \cap A_{l\mu}^\varepsilon \neq 0$ for $|\mu| \leqslant \mu_0$. Let $|\mu_k| \leqslant \mu_0$ and $k \geqslant l$. There are $\mu', \mu'' \in A_l^\varepsilon$ such that $\mu' = \mu_k + \mu''$; hence

$$|f(u_k+\mu_k+\mu'')-f(u_k)-\big(f(v_k+\mu_k+\mu'')-f(v_k)\big)| \leqslant \tfrac{1}{2}\varepsilon,$$
$$|f(u_k+\mu_k+\mu'')-f(u_k+\mu_k)-\big(f(v_k+\mu_k+\mu'')-f(v_k+\mu_k)\big)| \leqslant \tfrac{1}{2}\varepsilon.$$

This implies

$$(+) \qquad |f(u_k+\mu_k)-f(u_k)-\big(f(v_k+\mu_k)-f(v_k)\big)| \leqslant \varepsilon \quad \text{for} \quad k \geqslant k_0.$$

Let us write

$$B_n^\varepsilon = \{\mu_0\colon |f(u+\mu)-f(u)-\big(f(v+\mu)-f(v)\big)| \leqslant \varepsilon \text{ as } u, v \geqslant n,\ |\mu-\mu_0| < \delta,$$
$$\text{where } \delta > 0 \text{ is independent of } u, v\}.$$

Let $\mu_0 \in C_f$; we shall show that $\mu_0 \in \bigcup_n B_n^\varepsilon$. Supposing $\mu_0 \notin \bigcup_n B_n^\varepsilon$, to every $n = 1, 2, \ldots$ there exist $\bar{\mu}_n$, u_n and v_n such that

$$|\bar{\mu}_n - \mu_0| < 1/n, \quad u_n, v_n \geqslant n, \quad |f(u_n+\bar{\mu}_n)-f(u_n)-\big(f(v_n+\bar{\mu}_n)-f(v_n)\big)| > \varepsilon.$$

Let $\mu_n = \bar{\mu}_n - \mu_0$. Since $\mu_n \to 0$, $u_n, v_n \to \infty$,

$$f(u_n+\bar{\mu}_n)-f(u_n)-\big(f(v_n+\bar{\mu}_n)-f(v_n)\big)$$
$$= [f(u_n+\mu_n+\mu_0)-f(u_n+\mu_n)-\big(f(v_n+\mu_n+\mu_0)-f(v_n+\mu_n)\big)]+$$
$$+[f(u_n+\mu_n)-f(u_n)-\big(f(v_n+\mu_n)-f(v_n)\big)],$$

and so by $\mu_0 \in C_f$ and $(+)$, we get

$$f(u_n+\bar{\mu}_n)-f(u_n)-\big(f(v_n+\bar{\mu}_n)-f(v_n)\big) \to 0,$$

which is a contradiction. We proved $C_f \subset C$, where $C = \bigcap_k \bigcup_n B_n^{1/k}$. On the other hand, it is clear that $C \subset C_f$; hence $C_f = C$. Since B_n^ε are open sets, C is G_δ, and so C_f is a measurable set. Thus the assumption $m_e(C_f) > 0$ gives $m(C_f) > 0$. Applying the already mentioned theorem of Steinhaus we see that C_f is a rational basis, or more precisely, every real number μ may be written in the form $\mu = m\mu' - m\mu''$, where m is an integer and $\mu', \mu'' \in C_f$. On the other hand, it is known that if $\mu', \mu'' \in C_f$, then every linear combination of μ', μ'' with integer coefficients belongs to C_f again; consequently, $\mu \in C_f$, $C_f = (-\infty, \infty)$.

Let us assume now that $C_f = (-\infty, \infty)$ and let $\langle \mu_1, \mu_2 \rangle$ be any closed interval. Since $C_f = C$, every point $\mu_0 \in \langle \mu_1, \mu_2 \rangle$ belongs to some B_n^ε. Hence the inequality

$$(++) \qquad |f(u+\mu)-f(u)-\big(f(v+\mu)-f(v)\big)| \leqslant \varepsilon$$

holds in a neighbourhood of μ_0 for $u, v \geqslant n$, where n depends on the neighbourhood. Since a finite number of these neighbourhoods cover $\langle \mu_1, \mu_2 \rangle$, inequality $(++)$ is satisfied for $\mu \in \langle \mu_1, \mu_2 \rangle$, $u, v \geqslant u_0$ where u_0 is sufficiently large.

Let

$$\lim_{u\to\infty} \big(f(u+\mu)-f(u)\big) = \varrho_f(\mu),$$

by $(++)$

$$|f(u+\mu)-f(u)-\varrho_f(\mu)| \leqslant \varepsilon$$

for $u \geqslant u_0$ and $\mu \in \langle \mu_1, \mu_2 \rangle$. We have yet to prove $\varrho_f(\mu) = a\mu$. It is easily shown that $\varrho_f(\mu_1+\mu_2) = \varrho_f(\mu_1)+\varrho_f(\mu_2)$ for arbitrary μ_1, μ_2 and that $\varrho_f(\mu)$ is a measurable function. Hence, by a well-known theorem of Fréchet, $\varrho_f(\mu)$ is linear. However, for the sake of completeness, we give the proof of this theorem by applying Steinhaus theorem once more. This proof may be applied without change also under the assumptions made in 2.3. The set $A_k = \{\mu\colon |\varrho_f(\mu)| \leqslant k\}$ is of positive measure for some k. Hence there exists a $\mu_0 > 0$ such that every number μ, $|\mu| \leqslant \mu_0$, is of the form $\mu = \mu' - \mu''$, where $\mu', \mu'' \in A_k$. Consequently, choosing as m an integer such that $1/\mu_0 + 1 \geqslant m > 1/\mu_0$, we can write an arbitrary number $\mu \in \langle -1, 1 \rangle$ in the form $\mu = m\mu' - m\mu''$ $(\mu', \mu'' \in A_k)$. Hence, by the additivity of $\varrho_f(\mu)$ we obtain $|\varrho_f(\mu)| \leqslant 2mk$ in $\langle -1, 1 \rangle$ and $\varrho_f(\mu) = a\mu$ by the classic elementary theorem.

Let us now consider the set C_f^0. Let $m_e(C_f^0) > 0$; since $C_f^0 \subset C_f$, we have $m(C_f) > 0$ and hence $C_f = (-\infty, \infty)$. But we know that $\varrho_f(\mu)$

$= a\mu$ for every μ, and, on the other hand, $\varrho_f(\mu) = 0$ for infinitely many μ. Consequently, $a = 0$ and $C_f^0 = (-\infty, \infty)$.

We now consider the set B_f. The proof is obtained in the same way as that for C_f, with slight changes only. Supposing $m_e(B_f) > 0$, the set $\bigcup_n A_n^\varepsilon$ is of positive measure for $\varepsilon \geqslant \varepsilon_0$, where ε_0 is sufficiently large. Hence we deduce the inequality

$$\overline{\lim_{k\to\infty}} |f(u_k+\mu_k)-f(u_k)-\big(f(v_k+\mu_k)-f(v_k)\big)| \leqslant \varepsilon_0 .$$

We define the set C as $C = \bigcup_n B_n^n$, and an argument analogous to the previous one gives $B_f \subset C$. On the other hand, it is clear that $C \subset B_f$. Hence $B_f = C$, and since B_n^n are measurable sets, so is B_f. Arguments analogous to those used in the case of C_f gives $B_f = (-\infty, \infty)$. Let $B_f = (-\infty, \infty)$ and let $\langle \mu_1, \mu_2\rangle$ be an arbitrary closed interval. Then, just as in the case of C_f, we find that inequality (++) is satisfied in $\langle \mu_1, \mu_2\rangle$ for sufficiently large $\varepsilon > 0$ and for $u, v \geqslant u_0$. Hence the inequality

$$(\overset{++}{+}) \qquad |f(u+\mu)-f(u)| \leqslant g(\mu)$$

holds for $\mu = \langle \mu_1, \mu_2\rangle$ and $u \geqslant u_0$, where $g(\mu) = \varepsilon + |f(u_0+\mu)-f(u_0)|$; evidently, $g(\mu)$ is a measurable function and $|g(\mu)| < \infty$. Since the set $A_k = \{\mu: g(\mu) \leqslant k, \mu \in \langle \mu_1, \mu_2\rangle\}$ is of positive measure for some constant $k > 0$ we deduce in the same way as in the case of $\varrho_f(\mu)$ that there exists a positive integer m such that any number $\mu_0 \in \langle -c, c\rangle$, where $c = \sup(|\mu_1|, |\mu_2|)$, may be written in the form $\mu_0 = m\mu' - m\mu''$ ($\mu', \mu'' \in A_k$). But ($\overset{++}{+}$) immediately implies $|f(u\pm m\mu)-f(u)| \leqslant mg(\mu)$ for $u \geqslant u_0 + mc$, $\mu \in \langle \mu_1, \mu_2\rangle$. Hence we have in $\langle -c, c\rangle$

$$|f(u+\mu_0)-f(u)| \leqslant mg(\mu') + mg(\mu'') \leqslant 2mk$$

for $u \geqslant u_0 + mc$.

2.1. A function f *possesses Baire's property* if there exists a set P of the first category in $R = (-\infty, \infty)$ such that f is continuous on $R \setminus P$ with respect to this set [2]. It is well known that every B-measurable function possesses Baire's property, but there are functions which posses this property and are not B-measurable.

2.2. *If f possesses Baire's property and the set $A = \{u: f(u) \leqslant k\}$ is of second category, then A is a rational basis; more precisely, to every $\mu_0 > 0$ there exists a positive integer m such that every μ satisfying the inequality $|\mu| \leqslant \mu_0$ may be written in the form $\mu = m\mu' - m\mu''$, where $\mu', \mu'' \in A$.*

As a set of the second category, $B = A \setminus P$ is dense in some interval $\delta = (a, b)$. Let $u_0 \in \delta \cap (R\setminus P)$; then there exists a sequence $u_n \in B$, $u_n \to u_0$; hence $u_0 \in A$. Thus $A \cap \delta = (R\setminus P) \cap \delta \cup A \cap P \cap \delta$, i. e. $A \cap \delta$ is a residual set in δ. We denote by A_μ the set obtained by translation of set A by the number μ. An arbitrary number μ, $|\mu| < \frac{1}{2}(b-a)$, may be written in the form $\mu = \mu' - \mu''$ ($\mu', \mu'' \in A$), since we have $A \cap A_\mu \neq 0$ for such μ. Consequently, choosing a positive integer m so that $1/m \leqslant (b-a)/2\mu_0$, we can write every μ, $|\mu| \leqslant \mu_0$, in the form $\mu = \mu' - \mu''$ ($\mu', \mu'' \in A$).

2.3. Theorem in section 2 remains true if we replace "measurable function" by "function possessing Baire's property" and "set of measure zero" by "set of the first category".

The proofs given in section 2 remain valid without changes if we again replace "measurable function" by "function possessing Baire's property", and "set of positive exterior measure" by "set of the second category", and apply lemma 2.2. In this case, measurable sets of values μ introduced above are replaced by sets of points of the type $A = \{\mu: f(\mu) \leqslant k\}$, where f possesses Baire's property.

References

[1] J. Korevaar, T. V. Ardenne-Ehrenfest and N. G. de Bruijn, *A note on slowly convergent oscillating functions*, Nieuw Archief voor Wiskunde 23 (1949-52), p. 77-86.

[2] C. Kuratowski, *Topologie I*, Warszawa 1958.

[3] W. Matuszewska, *Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces*, Studia Math. 21 (1962), p. 317-344.

[4] L. A. Rubel, *A pathological Lebesgue-measurable function*, J. London Math. Soc. 38 (1963), p. 1-4.

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