THEOREM 2. *Let X be a real Banach space of dimension not less than two. Conditions* (a), (b), (c) *of Theorem* 1 *are equivalent to each of the following conditions:*

(d) *for every rectifiable curve Γ in $X\setminus\{0\}$, $l(\operatorname{sgn}\Gamma)\leqslant l(\Gamma)/d(\Gamma)$;*

(e) *for every rectifiable curve Γ in X that contains no interior point of the unit sphere* (i. e., *with* $d(\Gamma)\geqslant 1$), $l(\operatorname{sgn}\Gamma)\leqslant l(\Gamma)$.

If the dimension of X is not less than three, X satisfies these equivalent conditions if and only if X is a Hilbert space.

Proof. (d) obviously implies (e); conversely, if Γ is any curve, and we set $\sigma = d(\Gamma)$, then $d(\delta^{-1}\Gamma)\geqslant 1$ and (e) implies $l(\operatorname{sgn}\Gamma) = l(\operatorname{sgn}\sigma^{-1}\Gamma)\leqslant l(\sigma^{-1}\Gamma) = l(\Gamma)/d(\Gamma)$, so that (e) implies (d). Now (a) implies (d) by Lemma 4. The conclusion will follow from Theorem 1 if we prove that (d) implies (b).

Let $u, v\in\partial\Sigma$, $u\pm v\neq 0$, be given. For each λ, $0<\lambda<1$, we consider the curve Γ_λ given by $f_\lambda(\tau) = u+\tau v$, $\tau\in[0,\lambda]$ (a line segment). Now $d(\Gamma_\lambda)\geqslant 1-\lambda$, $l(\Gamma_\lambda) = \lambda$, $l(\operatorname{sgn}\Gamma_\lambda)\geqslant \|\operatorname{sgn} f(\lambda)-\operatorname{sgn} f(0)\| = a[u, u+\lambda v]$. By (d),

$$\lambda/a[u, u+\lambda v]\geqslant l(\Gamma_\lambda)/l(\operatorname{sgn}\Gamma_\lambda)\geqslant d(\Gamma_\lambda)\geqslant 1-\lambda,$$

and (b) follows on taking the inferior limit as $\lambda\to +0$.

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Reçu par la Rédaction le 13. 5. 1964

Summability in $l(p_1, p_2, \ldots)$ spaces*

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A Banach space E will be said to have the BS-*property* provided every bounded sequence in E admits a subsequence z_a whose sequence of arithmetic means

$$z_1,\ \tfrac{1}{2}(z_1+z_2),\ \tfrac{1}{3}(z_1+z_2+z_3),\ \ldots$$

is norm-convergent to a point of E. This property was established by Banach and Saks [1] for the spaces L_p and l_p $(1<p<\infty)$, and by Kakutani [2] for all uniformly convex Banach spaces. Nishiura and Waterman [6] recently showed that the BS-property does not imply uniform convexifiability, that it does imply reflexivity, and that reflexivity is equivalent to a different summability property. In his review of [6], Sakai [7] asked for an example of a reflexive Banach space which lacks the BS-property. The purpose of this note is to supply such an example by means of the $l(p_1, p_2, \ldots)$ spaces of Nakano [5]. (I am indebted to Mr. K. Sundaresan for calling my attention to these spaces in a different connection.)

Let P denote the set of all sequences in $]1,\infty[$ and let s denote the linear space of all sequences of real numbers. For $p = (p_1, p_2, \ldots)\in P$ and $x = (x_1, x_2, \ldots)\in s$, let

$$\mu_p(x) = \sum_{i=1}^{\infty}|x_i|^{p_i}/p_i.$$

Let $l(p)$ denote the set of all points $x\in s$ such that $\|x\|_p<\infty$, where

$$\|x\|_p = \inf\left\{\lambda>0:\ \mu_p\left(\frac{1}{\lambda}x\right)\leqslant 1\right\}.$$

Then $l(p)$ is a linear subspace of s and $\|\ \|_p$ is a norm for $l(p)$. It follows from results of Nakano (or by direct reasoning analogous to that for the classical l_p spaces) that the spaces $l(p)$ are all reflexive Banach spaces (for $p\in P$), and that $l(p)$ is uniformly convex if and only if $1<\inf\{p_i\}$

* Research supported in part by the National Science Foundation, U. S. A. (NSF-GP-378).

$\leqslant \sup\{p_i\} < \infty$. (See especially Theorems 40.6, 40.9, 43.6, 44.8, and 54.14 of [3], Theorem 89.12 of [4], and the paper [5].) Hence the following result provides the example requested by Sakai [7]:

THEOREM. *For $1 < p_i < \infty$, the space $l(p_1, p_2, \ldots)$ has the BS-property if and only if $1 < \inf\{p_i\}$ and $\sup\{p_i\} < \infty$.*

Proof. The "if" part is immediate from Kakutani's theorem and the criterion for uniform convexity of $l(p)$. Using δ_i to denote the Kronecker delta (as a point of $l(p)$), we proceed to the "only if" part. Here the basic idea is that the sequence $\delta_1, \delta_2, \delta_3, \ldots$ is a bounded sequence in the space l_1 which does not admit any subsequence whose sequence of arithmetic means is norm-convergent, and the same behavior should be exhibited by the space $l(p_1, p_2, \ldots)$ if $\inf\{p_i\} = 1$. A similar remark applies to the sequence $\delta_1, \delta_1+\delta_2, \delta_1+\delta_2+\delta_3, \ldots$ in the space l_∞, and to the condition that $\sup\{p_i\} = \infty$.

Let B denote the open unit ball $\{x \in l(p): \|x\|_p < 1\}$. It follows from the definition of $\| \ \|_p$ that $B \subset \{x: \mu_p(x) < 1\}$, and then from the convexity of the function μ_p that

(1) $$\varepsilon B \subset \{x: \mu_p(x) < \varepsilon\} \quad \text{for} \quad 0 < \varepsilon < 1.$$

Now suppose first that $\inf\{p_i\} = 1$. For each $\tau \in]1, \infty[$ and each $t \in [0, \infty[$, let $\varphi(\tau, t) = t^\tau/t$. Let $m(1) = 1$, and having chosen $m(i)$ for $1 \leqslant i < j$, let $m(j)$ be such that $m(j-1) < m(j), p_{m(j-1)} > p_{m(j)}$, and

(2) $$\frac{\varphi(p_{m(j)}, \varepsilon)}{\varepsilon} > \frac{\varphi(p_{m(i)}, t+\eta) - \varphi(p_{m(i)}, t)}{\eta}$$

whenever $1/j \leqslant \varepsilon \leqslant 1$, $i < j$, $0 \leqslant t < t+\eta \leqslant 1$ and $t \leqslant 1/2$. (To achieve this it suffices to take $p_{m(j)}$ sufficiently close to 1.)

Consider the sequence $\delta_{m(1)}, \delta_{m(2)}, \ldots$ in $l(p)$. Since $\mu_p(\delta_i) = 1/p_i < 1$, $\|\delta_i\|_p < 1$ and the sequence $\delta_{m(a)}$ is bounded. Consider an arbitrary subsequence $\delta_{n(a)}$ of $\delta_{m(a)}$, and let u_a be the sequence of arithmetic means formed from $\delta_{n(a)}$. The sequence u_a converges coordinatewise to the origin 0 of $l(p)$, so it must be norm-convergent to 0 if it is norm-convergent to any point. If $\|u_a\| \to 0$, then $\mu_p(u_a) \to 0$ by (1). However, we show on the contrary that

(3) $$0 < \mu_p(u_1) < \mu_p(u_2) < \ldots,$$

and the contradiction yields the desired conclusion. To establish (3), observe that

$$\mu_p(u_k) - \mu_p(u_{k-1}) = \sum_{i=1}^{k} \varphi\left(p_{n(i)}, \frac{1}{k}\right) - \sum_{i=1}^{k-1} \varphi\left(p_{n(i)}, \frac{1}{k-1}\right)$$
$$= \frac{1}{k(k-1)} \sum_{i=1}^{k-1} \left(\frac{\varphi(p_{n(k)}, 1/k)}{1/k} - \frac{\varphi(p_{n(i)}, 1/(k-1)) - \varphi(p_{n(i)}, 1/k)}{1/(k-1) - 1/k}\right),$$

and then apply (2) to show that the individual summands are all positive, noting that $n(k) = m(j)$ with $j \geqslant k$ and hence $1/k \geqslant 1/j$.

Now suppose, finally, that $\sup\{p_i\} = \infty$, and let the sequence $m(1) < m(2) < \ldots$ be such that $p_{m(i)} > i$ for $i = 1, 2, \ldots$ For each j, let

$$y_j = \sum_{i=1}^{j} p_{m(i)}^{1/p_{m(i)}} \delta_{m(i)}.$$

The sequence y_a is bounded in $l(p)$, for

$$\mu_p\left(\frac{1}{2} y_j\right) = \sum_{i=1}^{j} \frac{1}{p_{m(i)}} \left(\frac{1}{2} p_{m(i)}^{1/p_{m(i)}}\right)^{p_{m(i)}} = \sum_{i=1}^{j} 2^{-p_{m(i)}} < \sum_{i=1}^{j} 2^{-1} < 1,$$

and consequently $\|y_j\|_p < 2$. The same reasoning shows that $l(p)$ includes a point w such that $w_{m(i)} = p_{m(i)}^{1/p_{m(i)}}$ for $i = 1, 2, \ldots$, while $w_j = 0$ for $j \notin \{m(1), m(2), \ldots\}$. Consider an arbitrary subsequence $y_{n(a)}$ of y_a, and let v_a be the sequence of arithmetic means formed from $y_{n(a)}$. The sequence v_a converges coordinatewise to the point w, so it is norm-convergent to w if it is norm-convergent to any point. But it cannot be norm-convergent to w, for with

$$v_j = \frac{1}{j}(y_{n(1)} + y_{n(2)} + \ldots + y_{n(j)})$$

and with $k = n(j+1)$, the k^{th} coordinate of v_j is equal to zero and the k^{th} coordinate of w is equal to p_k^{1/p_k}, whence we have

$$\mu_p(w - v_j) \geqslant \frac{1}{p_k}(p_k^{1/p_k})^{p_k} = 1$$

and consequently $\|w - v_j\|_p = 1$. The proof is complete.

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Reçu par la Rédaction le 24. 6. 1964

STUDIA MATHEMATICA, T. XXV. (1965)

On the theory of ($\mathcal{F}$)-sequences

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Introduction. It very often happens that considering an ($\mathcal{F}$)-space we are virtually confronted with an inverse sequence of pseudonormed spaces which yields the ($\mathcal{F}$)-space as its projective limit.

However, it may happen that together with restricting our attention to the inverse limit only, we are loosing some of important properties of the spaces from the initial sequence.

This paper suggests a method of handling an inverse sequence as a whole. The object introduced for such purposes we call an ($\mathcal{F}$)-sequence. The concept of ($\mathcal{F}$)-sequence, announced in [5] comes as a consequence of a careful analysis of results and methods of [2]-[4]. Applications to some essential points considered in [2]-[4] will appear separately.

Though the bare notion of ($\mathcal{F}$) and pre-($\mathcal{F}$)-sequences gives scarce intuition as to its most important applications, it is still a very natural thing to consider these notions and their elegant mathematical form should appeal even without any important applications at hand.

Terminology and notation. We denote by $\{x_n\}$ the set of elements of a sequence $x_1, x_2, \ldots$ of elements of some X which justifies writing the inclusion $\{x_n\} \subset X$.

An operation is said to be *linear* iff it is additive and homogeneous and we do not require any kind of continuity. This differs from the standpoint of [1].

Pseudonorms will always be understood as subadditive non-negative and positive-homogeneous functionals vanishing in zero. As usual a pseudonorm may assume the value zero on non-zero element.

Suppose X and Y are subsets of the same set Z and Y is provided with some topology τ. We say that X is *of the second category in* (Y, τ) iff $X \cap Y$ is of the second category in (Y, τ).

Consider two linear topological spaces (X_i, τ_i), $i = 1, 2$. We say that (X_1, τ_1) is *coarser* than (X_2, τ_2) and we write $(X_1, \tau_1) \leqslant (X_2, \tau_2)$ iff X_2 is a subspace of X_1 and the identical injection of (X_2, τ_2) into (X_1, τ_1) is continuous.