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On the theory of $(\mathcal{F})$-sequences

by

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Introduction. It very often happens that considering an $(\mathcal{F})$-space we are virtually confronted with an inverse sequence of pseudonormed spaces which yields the $(\mathcal{F})$-space as its projective limit.

However, it may happen that together with restricting our attention to the inverse limit only, we are loosing some of important properties of the spaces from the initial sequence.

This paper suggests a method of handling an inverse sequence as a whole. The object introduced for such purposes we call an $(\mathcal{F})$-sequence. The concept of $(\mathcal{F})$-sequence, announced in [5] comes as a consequence of a careful analysis of results and methods of [2]-[4]. Applications to some essential points considered in [2]-[4] will appear separately.

Though the bare notion of $(\mathcal{F})$ and pre-$(\mathcal{F})$-sequences gives scarce intuition as to its most important applications, it is still a very natural thing to consider these notions and their elegant mathematical form should appeal even without any important applications at hand.

Terminology and notation. We denote by $\{x_n\}$ the set of elements of a sequence $x_1, x_2, \ldots$ of elements of some X which justifies writing the inclusion $\{x_n\} \subset X$.

An operation is said to be *linear* iff it is additive and homogeneous and we do not require any kind of continuity. This differs from the standpoint of [1].

Pseudonorms will always be understood as subadditive non-negative and positive-homogeneous functionals vanishing in zero. As usual a pseudonorm may assume the value zero on non-zero element.

Suppose X and Y are subsets of the same set Z and Y is provided with some topology τ. We say that X is *of the second category in* (Y, τ) iff $X \cap Y$ is of the second category in (Y, τ).

Consider two linear topological spaces (X_i, τ_i), $i = 1, 2$. We say that (X_1, τ_1) is *coarser* than (X_2, τ_2) and we write $(X_1, \tau_1) \leqslant (X_2, \tau_2)$ iff X_2 is a subspace of X_1 and the identical injection of (X_2, τ_2) into (X_1, τ_1) is continuous.

A sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$ of linear spaces each provided with a pseudonorm is said to be a *pre-($\mathcal{F}$)-sequence* iff $(V_n, \|\cdot\|_n) \leqslant (V_{n+1}, \|\cdot\|_{n+1})$ for $n = 1, 2, \ldots$ and $x \in \bigcap_{n=1}^{\infty} V_n$ vanishes whenever $\|x\|_n = 0$ for every n.

If the pseudonorms of a pre-($\mathcal{F}$)-sequence $\mathfrak{V}$ are not given directly, we shall write $\|\cdot\|_{\mathfrak{V},n}$ for the pseudonorm of the n-th space of the sequence $\mathfrak{V}$.

With any pre-($\mathcal{F}$)-sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$ we associate the following three notions:

(i) The linear space

$$|\mathfrak{V}| = V_1,$$

(ii) The linear locally convex metrizable space

$$\frown \mathfrak{V} = \Big(\bigcap_{n=1}^{\infty} V_n, \tau\Big),$$

where τ denotes the topology of $\bigcap_{n=1}^{\infty} V_n$ induced by pseudonorms $\{\|\cdot\|_n : n = 1, 2, \ldots\}$,

(iii) The metric-function $\varrho_{\mathfrak{V}}$, defined and translation-invariant on $\mathfrak{V}$ considered as an abelian group with the linear space addition as the group operation, given by fixing the distance $\varrho_{\mathfrak{V}}(x) = \varrho_{\mathfrak{V}}(x-0)$ of $x \in |\mathfrak{V}|$ from 0 as follows:

$$\varrho_{\mathfrak{V}}(x) = \sum_{n=1}^{\infty} 2^{-n} \varrho_{\mathfrak{V},n}(x),$$

where

$$\varrho_{\mathfrak{V},n}(x) = \begin{cases} \|x\|_n/(1+\|x\|_n) & \text{for} \quad x \in V_n, \\ 1 & \text{for} \quad x \in |\mathfrak{V}| - V_n. \end{cases}$$

Since $\varrho_{\mathfrak{V},n}$ are all subadditive, $\varrho_{\mathfrak{V}}$ must be subadditive as well. The metric-function $\varrho_{\mathfrak{V}}$ and the topological group $[\mathfrak{V}]$, where

$$[\mathfrak{V}] \overset{\mathrm{df}}{=} (\mathfrak{V}, \varrho_{\mathfrak{V}}),$$

are said to be *assigned* to $\mathfrak{V}$.

A pre-($\mathcal{F}$)-sequence $\mathfrak{V}$ is said to be an *($\mathcal{F}$)-sequence* iff $[\mathfrak{V}]$ is complete. It is easy to see that $\frown\mathfrak{V}$ is a closed subgroup of $[\mathfrak{V}]$.

Examples. I. Consider the k-dimensional Euclidean space E^k and denote by $\mathcal{D}^m$ the space of all complex valued functions f defined on E^k with continuous derivatives $D^p f$ for every $p = (p_1, \ldots, p_k)$ with $|p| = p_1 + \ldots + p_k \leqslant m$. Let $\|f\| = \sup\{|f(x)| : x \in E^k\}$, $\|f\|_\lambda = \left(\int |f(x)|^\lambda dx\right)^{1/\lambda}$, $\lambda > 1$, $\|f\|_{\mathcal{D}^m} = \sum_{|p| \leqslant m} \|D^p f\|$ and $\|f\|_{\lambda,m} = \sum_{|p| \leqslant m} \|D^p f\|_\lambda$. For any compact $K \subset E^k$ we write $\mathcal{D}^m(K) = \{f \in \mathcal{D}^m : \text{support } f \subset K\}$.

With any descending sequence $\{K_n\}$ of compact subsets of E^k such that each K_n coincides with the closure of its interior, we associate ($\mathcal{F}$)-sequences

$$\mathfrak{D} = \{(\mathcal{D}^{m_n}(K_n), \|\cdot\|_{\mathcal{D}^{m_n}})\}, \qquad \mathfrak{D}_{(\lambda)} = \{(\mathcal{D}^{m_n}(K_n), \|\cdot\|_{\lambda,m_n})\},$$

where $\{m_n\}$ is an increasing sequence of natural numbers.

In the first case the spaces $\left(\mathcal{D}^{m_n}(K_n), \|\cdot\|_{\mathcal{D}^{m_n}}\right)$ are all complete and then $[\mathfrak{D}]$ is automatically complete, and in the second case none of $\left(\mathcal{D}^{m_n}(K_n), \|\cdot\|_{\lambda,m}\right)$ is complete while $[\mathfrak{D}_{(\lambda)}]$ is still complete. Moreover, if $\bigcap_n K_n$ is discrete, then $\frown\mathfrak{D}$ is trivial, i. e. represents ($\mathcal{F}$)-space that consists of only point zero.

II. Consider a k-dimensional complex linear space Z^k. For $z = (z_1, \ldots, z_k) \in Z^k$ we write $|z|^2 = z_1\bar{z}_1 + \ldots + z_k\bar{z}_k$ and $\mathrm{Im}\, z = (\mathrm{Im}\, z_1, \ldots, \mathrm{Im}\, z_k) \in E^k$.

Take any entire function f defined on Z^k. We put

$$\|f\|_m^a = \sup_z (1+|z|)^m |f(z)| e^{a|\mathrm{Im}\, z|}, \qquad a > 0,$$

and

$$\mathcal{I}_a^m = \{f\text{—entire on } Z^k : \|f\|_m^a < \infty\}.$$

For any increasing $\{m_n\}$ and decreasing $\{a_n\}$, $a_n > 0$, tending to some $a \geqslant 0$, we introduce an ($\mathcal{F}$)-sequence

$$\mathfrak{I} = \{(\mathcal{I}_{a_n}^{m_n}, \|\cdot\|_{m_n}^{a_n})\}.$$

It is clear, that for K_n from Example I equal to $\{x \in E^k : |x| \leqslant a_n\}$, the Fourier transform F maps $|\mathfrak{D}|$, where $\mathfrak{D}$ is defined in Example I, into $|\mathfrak{I}|$. It turns out that F is closed in the sense which will be introduced in this paper, every $\mathcal{I}_{a_r}^{m_r}$ contains some $F\mathcal{D}^{m_s}(K_s)$ and, vice versa, every $F\mathcal{D}^{m_r}(K_r)$ contains some $\mathcal{I}_{a_s}^{m_s}$. This can serve as an illuminating illustration to Theorems 3 and 4 yielding in the same time a more general formulation of the Paley-Wiener theorem.

III. Consider a sequence $\mathfrak{Z} = \{(Z_n, |\cdot|_n)\}$ of linear spaces with pseudonorms such that $(Z_n, |\cdot|_n) \geqslant (Z_{n+1}, |\cdot|_{n+1})$ for $n = 1, 2, \ldots$, the linear space $X = \bigcup_{n=1}^{\infty} Z_n$ and a space X' of linear functionals over X, X' closed with respect to the pointwise addition, multiplication by scalars and convergence of sequences of functionals.

We define the polar $\mathfrak{Z}^*$ of $\mathfrak{Z}$ in X' setting $\mathfrak{Z}^* = \{(Z_n^*, |\cdot|_n^*)\}$, where $|x'|_n^* = \sup\{|x'z| : z \in Z_n, |z|_n < 1\}$ and $Z_n^* = \{x' \in X' : |x'|_n^* < \infty\}$.

It is clear that $\mathfrak{Z}^*$ is an $(\mathscr{F})$-sequence and that $\bigcap_{n=1}^{\infty} Z_n^*$ consists of all functionals continuous on the inductive limit of $\mathfrak{Z}$. It is clear, moreover, that the spaces $(Z_n^*, |\cdot|_n^*)$ need not be complete.

IV. Take a topological space R and let $\{R_n\}$ be a sequence of open subsets of R, $R_n \subset R_{n-1}$ for $n = 1, 2, \ldots$, such that $R = \bigcup_{n=1}^{\infty} R_n$. For any $U \subset R$ denote by $C(U)$ the linear space of all continuous scalar-valued functions defined on R and bounded on U. For $f \in C(U)$ let $\|f\|_U = \sup\{|f(t)|: t \in U\}$.

The sequence $\mathfrak{C} = \{(C(R_n), \|\cdot\|_{R_n})\}$ is an $(\mathscr{F})$-sequence. Here $\bigcap_{n=1}^{\infty} C(R_n)$ consists of all continuous functions on R which are bounded on every R_n. The spaces that constitute $\mathfrak{C}$ are generally not complete.

V. Consider two $(\mathscr{F})$-spaces (X_i, τ_i), $i = 1, 2$, where the topologies τ_i are given by pointwise non-decreasing sequences of pseudonorms $\{\|\cdot\|_{i,n}\}$ for $i = 1, 2$ respectively.

Let L be a space of linear mappings of X_2 into X_1 closed with respect to the pointwise addition, multiplication by scalars and convergence. We put

$$|A|_{k,n} = \sup\{\|Ax\|_{1,n} : \|x\|_{2,k} \leqslant 1\}, \qquad L_{k,n} = \{A \in L: |A|_{k,n} < \infty\}.$$

It is easy to see that with any increasing sequence $\{k_n\}$ there is associated an $(\mathscr{F})$-sequence

$$\mathfrak{L} = \{(L_{k_n,n}, |\cdot|_{k_n,n})\}.$$

VI. Consider an $(\mathscr{F})$-space (X, τ), where τ is induced by a pointwise non-decreasing sequence of pseudonorms $\{\|\cdot\|_n\}$ and (Y, ϱ) which is an inductive limit of a sequence $\{(Y_n, |||\cdot|||_n)\}$ of Banach spaces, $(Y_n, |||\cdot|||_n) \geqslant (Y_{n+1}, |||\cdot|||_{n+1})$. Denote by B the linear space of all bilinear functionals defined on $X \times Y$. For any $\varphi \in B$ we define

$$|\varphi|_{k,n} = \sup\{|\varphi(x, y)|: |||y|||_n \leqslant 1, \|x\|_k \leqslant 1\}$$

and further we set

$$B_{k,n} = \{\varphi \in B: |\varphi|_{k,n} < \infty\}.$$

Exactly like in the previous example to any increasing sequence $\{k_n\}$ there corresponds the $(\mathscr{F})$-sequence

$$\mathfrak{B} = \{(B_{k_n,n}, |\cdot|_{k_n,n})\}.$$

Setting $X = X_2$ and $Y =$ the adjoint to (X_1, τ_1), where (X_i, τ_i) are from Example V, we find that assigning to $A \in L$ the functional $\varphi \in B$, where $\varphi(x, x') = x'(Ax)$, we obtain a mapping of $|\mathfrak{L}|$ into $|\mathfrak{B}|$. This, together with Example II, provides the reader with some intuitive background as to what kind of mappings are subject to the discussion presented in this paper.

However, we do not propose to draw any conclusions from Theorems 3 and 4 of this paper as yet, waiting for another paper which together with further development of the theory will bring better opportunities for a careful study of the examples.

PROPOSITION 1. *Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$.*

A. *A sequence $\{x_n\} \subset |\mathfrak{V}|$ tends to zero in $[\mathfrak{V}]$ iff to each p there correspond m_p such that $x_n \in V_p$ for $n \geqslant m_p$ and $\lim_{m_p \leqslant n} \|x_n\|_p = 0$; $\{x_n\}$ tends to x in $[\mathfrak{V}]$ iff $\{x_n - x\}$ tends to zero in $[\mathfrak{V}]$; $\{x_n\}$ satisfies the Cauchy condition in $[\mathfrak{V}]$ iff to every p there correspond m_p such that $x_n - x_m \in V_p$ for $m, n \geqslant m_p$ and $\lim_{m_p \leqslant n,m} \|x_n - x_m\|_p = 0$.*

B. *Denote by $\mathrm{Cl}_{\mathfrak{V}}$ the operation of closure in $[\mathfrak{V}]$. A point $x \in |\mathfrak{V}|$ belongs to the closure $\mathrm{Cl}_{\mathfrak{V}} G$ of some $G \subset |\mathfrak{V}|$ iff for every n the set $G \cap (x + V_n)$ is non-void and*

$$\inf\{\|x - u\|_n : u \in G \cap (x + V_n)\} = 0.$$

C. *If $\{x_n\} \subset |\mathfrak{V}|$ satisfies the Cauchy condition in $[\mathfrak{V}]$, then for every scalar t we have $\{tx_n\}$ satisfying the Cauchy condition in $[\mathfrak{V}]$. If, in particular, the sequence $\{x_n\}$ tends to some $x \in |\mathfrak{V}|$, then $\{tx_n\}$ tends to tx in $[\mathfrak{V}]$.*

D. *Take $x \in |\mathfrak{V}|$ and a sequence of scalars $\{t_n\}$ tending to some scalar t different from countably many t_n. If $\{t_n x\}$ tends to tx in $[\mathfrak{V}]$, then $x \in \bigcap_{n=1}^{\infty} V_n$.*

Proof. Convergence of $\{x_n\}$ to zero in $[\mathfrak{V}]$ is equivalent to $\varrho_{\mathfrak{V},p}(x_n)$ tending to zero for every p separately and then amounts exactly to the condition given in A. The next follows from the fact that $[\mathfrak{V}]$ is a topological group. Similarly, the Cauchy condition for $\{x_n\}$ amounts to $\varrho_{\mathfrak{V},p}(x_n - x_m)$ tending to zero for every p separately which again is equivalent to the condition given in A. An element x belongs to $\mathrm{Cl}_{\mathfrak{V}} G$ iff $\varrho_{\mathfrak{V},n}(x - G) = 0$ for every n. Hence, for every n the set $G \cap (x + V_n)$ must be non-void and $\varrho_{\mathfrak{V},n}(x - G) = \inf\{\|x - u\|_n : u \in G \cap (x + V_n)\} = 0$ which proves B. The last two assertions are simple consequences of the first one.

PROPOSITION 2. *To every pre-$(\mathscr{F})$-sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$ there corresponds an $(\mathscr{F})$-sequence $\mathfrak{V}^- = \{(V_n^-, \|\cdot\|_n^-)\}$ such that $V_n \subset V_n^-$, $V_n^- = \mathrm{Cl}_{\mathfrak{V}^-} V_n$ for $n = 1, 2, \ldots$, where $\mathrm{Cl}_{\mathfrak{V}^-}$ denotes the closure in $[\mathfrak{V}^-]$ and $\|\cdot\|_n$ coincide with $\|\cdot\|_n^-$ restricted to V_n respectively for $n = 1, 2, \ldots$ The $(\mathscr{F})$-sequence $\mathfrak{V}^-$ is called the completion of $\mathfrak{V}$.*

Proof. Let (V^-, ϱ^-) denotes the completion of the abelian topological group $[\mathfrak{V}]$. Then, $|\mathfrak{V}|$ is a dense subgroup of (V^-, ϱ^-) and $\varrho_{\mathfrak{V}}$ is the restriction of ϱ^- to $|\mathfrak{V}|$. From Proposition 1 it follows that the multiplication by scalars in $|\mathfrak{V}|$ can be extended over V^- in such a way that V^- becomes a linear space. Indeed, if $\{x_n\} \subset |\mathfrak{V}|$ tends to some $x \in V^-$ in the sense of ϱ^-, then for any scalar t the sequence $\{tx_n\}$ tends in ϱ^- to some $x_1 \in V^-$. Setting $x_1 = tx$ we obtain a non-ambiguous definition of multiplication by scalars in V^- which makes a linear space out of the group V^-. Since all $\varrho_{\mathfrak{V},n}$ are continuous with respect to $\varrho_{\mathfrak{V}}$ and then ϱ^- as well, we can extend them over V^- to ϱ_n^-. Since ϱ^- is the extension of $\varrho_{\mathfrak{V}}$ over V^*, we shall have

$$\varrho^-(x) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n^-(x)$$

for $x \in V^-$.

Setting $V_n^- = \{x \in V^-: \varrho_n^-(x) < 1\}$ we find that $V^- = V_1^-$, $V_n^- \supset V_n$ and, moreover, for every p the pseudonorm $\|\cdot\|_p$ extends from V_p over V_p^- to $\|\cdot\|_p^-$. Indeed, if $\{x_n\} \subset |\mathfrak{V}|$, $\varrho_p^-(x_n - x) \to 0$ and $\varrho_p^-(x) < 1$, then $\varrho_p^-(x_n) < 1$ for $n \geqslant n_0$ and $\{x_n: n \geqslant n_0\} \subset V_p$. Then $\|x_n - x_m\|_p \to 0$ and setting $\|x\|_p^- = \lim_{n_0 \leqslant n} \|x_n\|_p$ we find that the definition of $\|x\|_p$ does not depend on the choice of $\{x_n\}$ and produces the extension of $\|\cdot\|_p$ to a continuous pseudonorm on V_p^- such that $\varrho_p^-(x) = \|x\|_p^-/(1+\|x\|_p^-)$ for $x \in V_p^-$. It is clear that $\mathfrak{V}^- = \{(V_n^-, \|\cdot\|_n^-)\}$ is an $(\mathscr{F})$-sequence with $[\mathfrak{V}^-] = (V^-, \varrho^-)$, satisfying all the requirements of the Proposition. This finishes the proof of Proposition 2.

Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$. With any natural p we associate the *p-th shift* $p\mathfrak{V}$ of $\mathfrak{V}$ defined as

$$p\mathfrak{V} = \{(V_n, \|\cdot\|_n): n = p+1, p+2, \ldots\},$$

i. e. $p\mathfrak{V}$ is produced from $\mathfrak{V}$ by dropping first p elements of the sequence $\mathfrak{V}$. Using the notion of p-th shift we can express the space that appears in the p-th element of $\mathfrak{V}$ as $|p\mathfrak{V}|$. Hence, we can write now the identity

$$\mathfrak{V} = \{(|n\mathfrak{V}|, \|\cdot\|_{\mathfrak{V},n})\}.$$

Here comes a proposition showing that for every $\mathfrak{V}$ we have the decomposition $|\mathfrak{V}| = \bigcap_{n=1}^{\infty} [|(n+1)\mathfrak{V}| - |n\mathfrak{V}|]$ into disjoint open and closed subsets of $\mathfrak{V}$.

PROPOSITION 3. *Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V}$. For any natural n the identical injection of $[n\mathfrak{V}]$ into $[\mathfrak{V}]$ is bicontinuous and $|n\mathfrak{V}|$ is open and closed in $[\mathfrak{V}]$.*

Proof. The bicontinuity of the identical injection of $[n\mathfrak{V}]$ into $[\mathfrak{V}]$ follows from the fact that the convergence of $\varrho_{\mathfrak{V}}(x_n)$ to zero amounts

to the convergence to zero of $\varrho_{\mathfrak{V},p}(x_n)$ separately for every p taken from any fixed countable set of natural numbers. Hence, for $\{x_n\} \subset |p\mathfrak{V}|$ the condition $\lim_n \varrho_{\mathfrak{V}}(x_n) = 0$ is equivalent to the condition $\lim_n \varrho_{p\mathfrak{V}}(x_n) = 0$.

Every $|p\mathfrak{V}|$ is closed in $[\mathfrak{V}]$. Indeed, if $\{x_n\}$ tends to some $x \in |\mathfrak{V}|$, we have $x - x_n \in |p\mathfrak{V}|$ for sufficiently great n and then $x \in |p\mathfrak{V}|$ as well. Similarly, if $\{x_n\} \subset |\mathfrak{V}| - |p\mathfrak{V}|$ and $\{x_n\}$ tends to some $x \in |\mathfrak{V}|$, we have again $x - x_n \in |p\mathfrak{V}|$ for sufficiently great n and this time, since every x_n does not belong to $|p\mathfrak{V}|$, x does not belong to $|p\mathfrak{V}|$ either. This finishes the proof of Proposition 3.

It is worth noticing that the fact of bicontinuity of the identical imbedding of $[p\mathfrak{V}]$ in $[\mathfrak{V}]$ generalizes to the bicontinuity of such imbedding of any $[\mathfrak{V}_1]$ obtained by dropping of any fixed countable number of elements of $\mathfrak{V}$ in such a way that the remaining is still an infinite sequence. This is which was practically proved in the above given proof.

Consider two pre-$(\mathscr{F})$-sequences $\mathfrak{V}_i$ $(i = 1, 2)$. The sequence $\mathfrak{V}_1$ is said to be *finer* than $\mathfrak{V}_2$, which we write

$$\mathfrak{V}_1 \geqslant \mathfrak{V}_2,$$

iff there exists a number p such that $|p\mathfrak{V}_1|$ is a subspace of $|\mathfrak{V}_2|$ and the identical injection of $[p\mathfrak{V}_1]$ into $[\mathfrak{V}_2]$ is continuous; $\mathfrak{V}_1$ is said to be *equivalent* to $\mathfrak{V}_2$ iff $\mathfrak{V}_1 \geqslant \mathfrak{V}_2$ and $\mathfrak{V}_2 \geqslant \mathfrak{V}_1$. Obviously $\mathfrak{V}_1 \geqslant \mathfrak{V}_2$ implies $\cap \mathfrak{V}_1 \geqslant \cap \mathfrak{V}_2$ and every pre-$(\mathscr{F})$-sequence $\mathfrak{V}$ is equivalent to any subsequence of itself.

For example, the $(\mathscr{F})$-sequences $\mathfrak{D}$ and $\mathfrak{D}_{(2)}$ introduced in Example I are equivalent by virtue of the well-known Sobolev lemma assuring that the uniform convergence of derivatives up to a given order follows from L^2-convergence of derivatives up to some higher order.

Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$ and a linear $[\mathfrak{V}]$-closed subspace L of $[\mathfrak{V}|$. We define the *quotient pre-$(\mathscr{F})$-sequence* (cf. [3])

$$\mathfrak{V}/L = \{(V_n/L, \|\cdot\|_{n/L})\}$$

as follows. We set $|\mathfrak{V}/L| \stackrel{\mathrm{df}}{=} |\mathfrak{V}|/L$ and we denote by V_n/L the subspace of $|\mathfrak{V}/L|$ with representants taken from V_n. We define in the following the pseudonorms

$$\|x/L\|_{n/L} = \inf\{\|x-k\|_n: k \in L \cap (x+V_n)\}$$

for $x/L \in V_n/L$. Notice that $L \cap (x+V_n)$ is non-void for $x/L \in V_n/L$. If $x/L \in \bigcap_{n=1}^{\infty} V_n/L$ and $\|x/L\|_{n/L} = 0$ for $n = 1, 2, \ldots$, then $x \in L$, i. e. $x/L = 0$. Indeed, if $x/L \in V_n/L$ and $\|x/L\|_{n/L} = 0$, then there exists $k_n \in L$ such that $x - k_n \in V_n$ and $M_n\|x-k_n\|_n < 1/n$, where $\{M_n\}$, $M_n > 0$, are chosen in such a way that $M_n\|x\|_n \leqslant M_{n+1}\|x\|_{n+1}$ for $x \epsilon V_n$, $n = 1, 2, \ldots$ Hence $\{k_n\}$ tends to x in $[\mathfrak{V}]$ and, since L is closed in $[\mathfrak{V}]$, we have $x \epsilon L$.

In view of the fact that the relations

$$(V_n/L, \|\cdot\|_{n/L}) \leqslant (V_{n+1}/L, \|\cdot\|_{n+1/L}), \quad n = 1, 2, \ldots,$$

hold as simple consequences of the corresponding relations in $\mathfrak{V}$ itself, we conclude that $\mathfrak{V}/L$ is a pre-$(\mathscr{F})$-sequence.

Looking at $[\mathfrak{V}]$ from the topological group viewpoint we can always construct the factor-group $[\mathfrak{V}]/L$ which is a topological group in the case of closed L and, moreover, the group $[\mathfrak{V}]/L$ is complete whenever the original group $[\mathfrak{V}]$ is complete. In view of the definition of $|\mathfrak{V}/L|$ we know that $|\mathfrak{V}/L|$ and $|\mathfrak{V}|/L$ are identical algebraically.

PROPOSITION 4. *Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V}$ and a linear $[\mathfrak{V}]$-closed subspace $L \subset |\mathfrak{V}|$. The topological groups $[\mathfrak{V}/L]$ and $[\mathfrak{V}]/L$ are topologically the same.*

Proof. Let $\mathfrak{V} = \{(V_n, \|\cdot\|_n)\}$ and take $x_n/L \epsilon |\mathfrak{V}|/L$, where L is a $[\mathfrak{V}]$-closed linear subspace of $|\mathfrak{V}|$. The sequence $\{x_n/L\}$ tends to 0 in $[\mathfrak{V}]/L$ iff there exists $\{k_n\} \subset L$ such that $\varrho_{\mathfrak{V}}(x_n - k_n)$ tends to zero, i. e. $x_n - k_n \epsilon V_p$ for $n \geqslant m_p$ and $\lim\limits_{m_p \leqslant n} \|x_n - k_n\|_p = 0$ for $p = 1, 2, \ldots$, where $\{m_n\}$ is a properly chosen sequence. This can be equivalently expressed by saying that for every p we have $x_n/L \epsilon V_p/L$ for $n \geqslant m_p$ and $\lim\limits_{m_p \leqslant n} \|x_n/L\|_{p/L} = 0$ which means exactly that $\{x_n/L\}$ tends to zero in $[\mathfrak{V}/L]$. This finishes the proof of Proposition 4.

To establish a relation between topologies of the elements of a given pre-$(\mathscr{F})$-sequence $\mathfrak{V}$ and the topology of the assigned group $[\mathfrak{V}]$ we prove the following statement:

PROPOSITION 5. *Let $\mathfrak{V}$ be a pre-$(\mathscr{F})$-sequence and define for any positive r and natural n the following subsets of $\mathfrak{V}$:*

$$K_{\mathfrak{V}}(r) = \{x \epsilon |\mathfrak{V}| : \varrho_{\mathfrak{V}}(x) < r\}, \quad K_{\mathfrak{V},n}(r) = \{x \epsilon |n\mathfrak{V}| : \|x\|_{\mathfrak{V},n} < r\}.$$

A. *To every natural p and $\varepsilon > 0$ there corresponds $\eta > 0$ such that*

$$K_{\mathfrak{V}}(\eta) \subset K_{\mathfrak{V},p}(\varepsilon).$$

B. *To every $\varepsilon > 0$ there correspond a natural p and $\eta > 0$ such that*

$$K_{\mathfrak{V}}(\varepsilon) \supset K_{\mathfrak{V},p}(\eta).$$

Proof. If $x \epsilon V_p \overset{\text{df}}{=} |p\mathfrak{V}|$, then

$$\varrho_n(x) \overset{\text{df}}{=} \varrho_{\mathfrak{V},n}(x) = 1$$

for $n \geqslant p$ and

$$\varrho(x) \overset{\text{df}}{=} \varrho_{\mathfrak{V}}(x) \geqslant \sum_{n=p+1}^{\infty} 2^{-n}\varrho_n(x) = \sum_{n=p+1}^{\infty} 2^{-n} = 2^{-p}.$$

Hence $x \epsilon V_p$ implies $\varrho(x) \geqslant 2^{-p}$ and by the contraposition $\varrho(x) < 2^{-p}$ implies $x \epsilon V_p$. Thus, given natural p we can always find sufficiently small $\eta > 0$ to have $K_{\mathfrak{V}}(\eta) \subset V_p$.

Now, $\eta > \varrho(x) \geqslant 2^{-p}\varrho_p(x)$ implies $\varrho_p(x) < 2^{-p}\eta$ or, for $x \epsilon V_p$ and $2^p\eta < 1$, $\|x\|_{\mathfrak{V},p} < 2^p\eta/(1-2^p\eta)$. Taking η so small that $K_{\mathfrak{V}}(\eta) \subset V_p$ and $0 < 2^p\eta/(1-2^p\eta) < \varepsilon$ we obtain $K_{\mathfrak{V}}(\eta) \subset K_{\mathfrak{V},p}(\varepsilon)$ which proves A.

To prove B take arbitrary but fixed p and assign M_p to it in such a way that $M_p\|x\|_{\mathfrak{V},p} \geqslant \|x\|_{\mathfrak{V},i}$ for $x \epsilon V_i$, $i = 1, 2, \ldots, p$. Then for $x \epsilon K_{\mathfrak{V},p}(r/M_p)$ we find that $\varrho_i(x) < r/(1+r)$ for $i = 1, 2, \ldots, p$ and then $\sum_{i=1}^{p} 2^{-i}\varrho_i(x) < r/(1+r)$. Since $\sum_{i=p+1}^{\infty} 2^{-i}\varrho_i(x) \leqslant 2^{-p}$, we find that $x \epsilon K_{\mathfrak{V},p}(r/M_p)$ implies $\varrho(x) < 2^{-p} + r/(1+r)$. Hence, if we choose p and r in such a way that $2^{-p} + r/(1+r) < \varepsilon$ and if we put $\eta = r/M_p$, then $K_{\mathfrak{V},p}(\eta) \subset K_{\mathfrak{V}}(\varepsilon)$ which finishes the proof of B and then of Proposition 5 as well.

From Proposition 5 we obtain the following equivalent characterizations of the continuity of a pseudonorm defined on $|\mathfrak{V}|$:

PROPOSITION 6. *Consider a pre-$(\mathscr{F})$-sequence $\mathfrak{V}$ and a pseudonorm $\|\cdot\|$ defined on $|\mathfrak{V}|$. The following conditions are equivalent:*

α. *The pseudonorm $\|\cdot\|$ is bounded on at least one $K_{\mathfrak{V}}(r)$.*

β. *The pseudonorm $\|\cdot\|$ is bounded on an open subset of $[\mathfrak{V}]$.*

γ. *The pseudonorm $\|\cdot\|$ is $[\mathfrak{V}]$-continuous in 0.*

δ. *The pseudonorm $\|\cdot\|$ is $[\mathfrak{V}]$-continuous in the whole $|\mathfrak{V}|$.*

ε. *The pseudonorm $\|\cdot\|$ is continuous in at least one $(|n\mathfrak{V}|, \|\cdot\|_{\mathfrak{V},n})$.*

λ. *The pseudonorm is bounded on at least one $K_{\mathfrak{V},n}(r)$.*

Proof. Let us list at the beginning the obvious implications and equivalences. We have the equivalence of ε and λ as the well-known fact out of the elementary properties of pseudonorms. The equivalence of γ and δ follows from the subadditivity of $\|\cdot\|$. The equivalence of α and λ follows directly from Proposition 5. The implication α → β is obvious. To prove the converse we notice that β amounts to the inclusion

$$\{x : \varrho(x - x_0) < r\} \subset \{x : \|x\| \leqslant n_0\}$$

for some $x_0 \epsilon |\mathfrak{V}|$, n_0 and $r > 0$. This implies the inclusion

$$\{y : \|y + x_0\| \leqslant n_0\} \supset K_{\mathfrak{V}}(r),$$

or, using the subadditivity of $\|\cdot\|$ and remembering that $\|x_0\| \leqslant n_0$, the inclusion

$$\{y : \|y\| \leqslant 2n_0\} \supset K_{\mathfrak{V}}(r)$$

which amounts to the boundedness of $\|\cdot\|$ on $K_{\mathfrak{B}}(r)$. Hence α and β are equivalent.

The implication $\varepsilon \to \gamma$ follows directly from Proposition 1 and then to complete the proof it is sufficient to show the implication $\gamma \to \alpha$. We show it proving its contraposition. If $\|\cdot\|$ is not bounded on any $K_{\mathfrak{B}}(r)$, then from A of Proposition 5 it follows that $\|\cdot\|$ is not bounded on any $K_{\mathfrak{B},n}(r)$ and then to each n there correspond $x_n \epsilon K_{\mathfrak{B},n}(1/n)$ such that $\|x_n\| \geqslant 1$. Hence, by virtue of A of Proposition 1, $\{x_n\}$ tends to zero in $[\mathfrak{B}]$ and $\|x_n\| \geqslant 1$ for every n which contradicts γ.

This concludes the proof of Proposition 6.

From now on we shall start using the notion of topological category. Applied in the case of the assigned group this notion admits some extra properties which are explained in the following proposition:

PROPOSITION 7. *Take a pre-($\mathscr{F}$)-sequence $\mathfrak{B}$. If $Y \subset |\mathfrak{B}|$ is of the second category in $[k\mathfrak{B}]$, then Y is of the second category in $[\mathfrak{B}]$ as well.*

Proof. Suppose Y is of the second category in $[k\mathfrak{B}]$ and take a decomposition $Y = \bigcap_{n=1}^{\infty} Y_n$. There must be then a Y_{n_0} which closure contains a ball K_0 taken in $[k\mathfrak{B}]$. Since in view of Proposition 1, $|k\mathfrak{B}|$ is open in $[\mathfrak{B}]$, the ball K_0 is open in $[\mathfrak{B}]$ as well and Y_{n_0} is not nowhere dense in $[\mathfrak{B}]$ which concludes the proof of Proposition 7.

Now comes the well-known result which is often called the Banach and Steinhaus Theorem:

THEOREM 1. *Consider a pre-($\mathscr{F}$)-sequence with the second category assigned group $[\mathfrak{B}]$. Every $[\mathfrak{B}]$-lower semi-continuous pseudonorm defined on $|\mathfrak{B}|$ is $[\mathfrak{B}]$-continuous.*

Proof. In view of the Proposition 6 it is sufficient to show that every $[\mathfrak{B}]$-lower semi-continuous pseudonorm $\|\cdot\|$ is bounded on at least one open subset of $[\mathfrak{B}]$.

We have

$$|\mathfrak{B}| = \bigcup_{n=1}^{\infty} \{x \epsilon |\mathfrak{B}| : \|x\| \leqslant n\}$$

and since $\|\cdot\|$ is lower semi-continuous, every $\{x \epsilon |\mathfrak{B}| : \|x\| \leqslant n\}$ is closed in $[\mathfrak{B}]$. Furthermore, $[\mathfrak{B}]$ is of the second category and then at least one of $\{x \epsilon |\mathfrak{B}| : \|x\| \leqslant n\}$ is not nowhere dense in $[\mathfrak{B}]$. Hence there is n_0 and an open subset U of $|\mathfrak{B}|$ such that $\|x\| \leqslant n_0$ for $x \epsilon U$ and the Theorem follows.

In order to prove some other properties of pre-($\mathscr{F}$)-sequences we shall need certain new notions.

Consider two pre-($\mathscr{F}$)-sequences $\mathfrak{B}_i$ $(i = 1, 2)$ and a linear mapping T of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$.

The mapping T is said to be *nearly-open* iff the following condition holds:

1. To every $\varepsilon > 0$ there correspond $\eta > 0$ such that

$$\mathrm{Cl}_{\mathfrak{B}_1} T K_{\mathfrak{B}_2}(\varepsilon) \supset K_{\mathfrak{B}_1}(\eta),$$

where $\mathrm{Cl}_{\mathfrak{B}_1}$ denotes the operation of closure in $[\mathfrak{B}_1]$ and $K_{\mathfrak{B}_i}(r)$ $(i = 1, 2)$ are defined according to Proposition 5.

The mapping T is said to be *open* iff the following condition holds:

2. To every $\varepsilon > 0$ there correspond $\eta > 0$ such that

$$T K_{\mathfrak{B}_2}(\varepsilon) \supset K_{\mathfrak{B}_1}(\eta),$$

where $K_{\mathfrak{B}_1}(r)$ are defined according to Proposition 5.

The mapping T is said to be *complete-closed* iff the following condition holds:

3. If $\{x_n\} \subset |\mathfrak{B}_2|$ satisfies the Cauchy condition in $[\mathfrak{B}_2]$ and the sequence $\{Tx_n\}$ tends in $[\mathfrak{B}_1]$ to some $y \epsilon |\mathfrak{B}_1|$, then there exists $x \epsilon |\mathfrak{B}_2|$ such that $\{x_n\}$ tends to x and $y = Tx$.

The mapping T is said to be *closed* iff the following condition holds:

4. If $\{x_n\} \subset |\mathfrak{B}_2|$ tends in $[\mathfrak{B}_2]$ to some $x \epsilon |\mathfrak{B}_2|$ and $\{Tx_n\}$ tends in $[\mathfrak{B}_1]$ to some $y \epsilon |\mathfrak{B}_1|$, then $y = Tx$.

The coming proposition will explain the notion of nearly open and open mappings entirely in terms of elements of participating pre-($\mathscr{F}$)-sequences.

PROPOSITION 8. *Consider pre-($\mathscr{F}$)-sequences $\mathfrak{B}_i$ $(i = 1, 2)$ and a linear mapping of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$.*

A. *The mapping T is open if and only if the following condition holds:*

5. *To every n and $\varepsilon > 0$ there correspond m and $\eta > 0$ such that*

$$T K_{\mathfrak{B}_2,n}(\varepsilon) \supset K_{\mathfrak{B}_1,m}(\eta),$$

where $K_{\mathfrak{B}_i,p}(r)$, $i = 1, 2$, are defined according to Proposition 5.

B. *Define*

$$S^r_{n,p}(y) = \{x \epsilon |\mathfrak{B}_2| : Tx \epsilon y + |p\mathfrak{B}_1|\} \cap K_{\mathfrak{B}_2 n}(r).$$

The mapping T is nearly open if and only if the following condition holds:

6. *To every n and every $\varepsilon > 0$ there correspond m and $\eta > 0$ such that for every $y \epsilon K_{\mathfrak{B}_1,m}(\eta)$ and every p we have*

a. *$S^\varepsilon_{n,p}(y)$ is non-void,*

b. $\|TS^\varepsilon_{n,p} - y\|_{\mathfrak{B}_1,p} = \inf\{\|Tx - y\|_{\mathfrak{B}_1,p} : x \epsilon S^\varepsilon_{n,p}(y)\} = 0.$

Proof. Let us put briefly $K_{\mathfrak{B}_i}(r) = K_i(r)$ and $K_{\mathfrak{B}_i,p}(r) = K_{i,p}(r)$ for $i = 1, 2$.

Ad A. Suppose that 2 holds and fix any $\varepsilon > 0$ and natural n. Then, using A of Proposition 5 we find $\varepsilon' > 0$ such that $K_{2,n}(\varepsilon) \supset K_2(\varepsilon')$ and then, applying 2, we find $\eta' > 0$ such that $TK_2(\varepsilon') \supset K_1(\eta')$. Applying again Proposition 5, but now part B, we find $\eta > 0$ and m such that $K_1(\eta') \supset K_{1,m}(\eta)$ and we finally get $TK_{2,n}(\varepsilon) \supset K_{1,m}(\eta)$. This proves the implication $2 \to 5$.

Let conversely 5 hold. Applying B of Proposition 5 we find that to any fixed $\varepsilon > 0$ there correspond $\varepsilon' > 0$ and natural n such that $K_2(\varepsilon) \supset K_{2,n}(\varepsilon')$ and then from 5 we find $\eta' > 0$ and m such that $TK_{2,n}(\varepsilon') \supset K_{1,m}(\eta')$. Now, from A of Proposition 5 it follows that there is $\eta > 0$ such that $K_{1,m}(\eta') \supset K_1(\eta)$ and consequently $TK_2(\varepsilon) \supset K_1(\eta)$ which shows that 5 implies 2 and concludes the proof of the first part of Proposition 8.

Ad B. Write $S_p^r(y) = \{x \epsilon |\mathfrak{V}_2| : Tx \epsilon y + |p\mathfrak{V}_1|\} \cap K_2(r)$.

From Proposition 5 it follows that 6 can be replaced by the following equivalent conditions:

7. *To every $\varepsilon > 0$ there correpond $\eta > 0$ such that for every $y \epsilon K_1(\eta)$ and every p we have*

a. *$S_p^\varepsilon(y)$ is non-void,*

b. $\inf\{\|Tx - y\|_{\mathfrak{V}_1,p} : x \epsilon S_p(y)\} = 0$.

Indeed, suppose that 6 is satisfied and take arbitrary $\varepsilon > 0$. From B of Proposition 5 it follows that there are $\varepsilon' > 0$ and n such that $K_2(\varepsilon) \supset K_{2,n}(\varepsilon')$. From 6 it follows that there are $\eta' > 0$ and m such that for $y \epsilon K_{1,m}(\eta')$ and every p we have $S_{n,p}^{\varepsilon'}(y)$ non-void and $\inf\{\|Tx - y\|_{\mathfrak{V}_1,p} : x \epsilon S_{n,p}^{\varepsilon'}(y)\} = 0$ and then from the previously introduced inclusion we conclude that for $y \epsilon K_{1,m}(\eta')$ the set $S_p^\varepsilon(y)$ is non-void and $\inf\{\|Tx - y\|_{\mathfrak{V}_1,p} : x \epsilon S_p^\varepsilon(y)\} = 0$. Now, from B of Proposition 5 we find $\eta > 0$ such that $K_{1,m}(\eta') \supset K_1(\eta)$ which proves together with the previous facts that a and b of 7 hold for the ε and η. Hence 6 implies 7.

Suppose, conversely, that 7 holds and take arbitrary $\varepsilon > 0$ and natural n. From B of Proposition 5 we find $\varepsilon' > 0$ such that $K_{2,n}(\varepsilon) \supset K_2(\varepsilon')$. From 7 it follows that there is $\eta' > 0$ such that for $y \epsilon K_1(\eta')$ and every p we have $S_p^{\varepsilon'}(y)$ non-void and $\inf\{\|Tx - y\|_{\mathfrak{V}_1,p} : x \epsilon S_p^{\varepsilon'}(y)\} = 0$. Applying the above-given inclusion we find that for $y \epsilon K_1(\eta')$ and every p we have $S_{n,p}^{\varepsilon}(y)$ non-void and $\inf\{\|Tx - y\|_{\mathfrak{V}_1,p} : x \epsilon S_{n,p}^{\varepsilon}(y)\} = 0$. Using B of Proposition 5 we find $\eta > 0$ and natural m such that $K_1(\eta') \supset K_{1,m}(\eta)$ and we conclude that a and b of 6 hold for the ε, n and η, m. This establishes the implication $7 \to 6$ and then also the equivalence of 6 and 7.

To finish the proof of B of Proposition 8 it is sufficient to notice that $TS_p^r(y) = (y + p\mathfrak{V}_1) \cap TK_2(r)$ and then, by virtue of B of Proposition 1, 7 amounts exactly to 1, i. e. states that T is nearly open.

This way Proposition 8 has been proved.

It should be mentioned that Proposition 8 has been introduced only for the explanatory purposes, i. e. to show how great a simplification it is to use the concept of the assigned group instead of translating everything on the language of sequences of pseudonormed spaces. Though we expect to use Proposition 8 in some other ocasions, it will not be used in any proof given in this paper thanks to the application of this efficient tool which is the assigned group.

PROPOSITION 9. *Consider a pre-($\mathscr{F}$)-sequence $\mathfrak{V}$. For every $\varepsilon > 0$ and every natural p we have*

$$|p\mathfrak{V}| \subset \bigcup_{m=1}^{\infty} \{x \epsilon |\mathfrak{V}| : \varrho_{\mathfrak{V}}(x/m) < \varepsilon + 2^{-p}\}.$$

Proof. If $x \epsilon |p\mathfrak{V}|$, then $\lim_n \varrho_{\mathfrak{V},i}(x/m) = 0$ for $i = 1, 2, \ldots, p$ and for any given $\varepsilon > 0$ we can find m such that

$$\sum_{i=1}^{p} 2^{-i} \varrho_{\mathfrak{V},i}(x/m) < \varepsilon.$$

Then

$$\varrho_{\mathfrak{V}}(x/m) < \varepsilon + \sum_{i=p+1}^{\infty} 2^{-i} = \varepsilon + 2^{-p}$$

which proves our assertion.

PROPOSITION 10 (Banach [1]). *Consider pre-($\mathscr{F}$)-sequences $\mathfrak{V}_i$ $(i = 1, 2)$ and a linear mapping T of $|\mathfrak{V}_2|$ into $|\mathfrak{V}_1|$. If to every p there correspond k_p such that $T|p\mathfrak{V}_2|$ is of the second category in $[k_p\mathfrak{V}_1]$, then T is nearly open.*

Proof. From Proposition 7 it follows that we can put all k_p equal to 1. Let $\varrho_i = \varrho_{\mathfrak{V}_i}$ for $i = 1, 2$. Take any $\varepsilon > 0$ and let p be such that $\varepsilon/4 > 2^{-p}$. From Proposition 9 it follows that

$$|p\mathfrak{V}_2| \subset \bigcup_{m=1}^{\infty} \{x \epsilon |\mathfrak{V}_2| : \varrho_2(x/m) < \varepsilon/2\}.$$

Setting $H_m = T\{x \epsilon |\mathfrak{V}_2| : \varrho_2(x/m) < \varepsilon/2\}$ we obtain

$$T|p\mathfrak{V}_2| \subset \bigcup_{m=1}^{\infty} H_m.$$

Since $T|p\mathfrak{V}_2|$ is of the second category in $[\mathfrak{V}_1]$, there exists m_0 such that H_{m_0} is not nowhere dense in $[\mathfrak{V}_1]$. Hence, there are $y_0 \epsilon |\mathfrak{V}_1|$ and $r > 0$ such that $H_{m_0}^- \supset \{y \epsilon |\mathfrak{V}_1| : \varrho_1(y - y_0) < r\}$, where $G^- \overset{\mathrm{df}}{=}$ the closure of G in $[\mathfrak{V}_1]$.

From now on we repeat the proof given in [1]. It follows that $H_1^- \supset \{y \epsilon |\mathfrak{V}_1| : \varrho_1(y - y_0/m_0) < r/m_0\}$. Indeed, $\varrho_1(y - y_0/m_0) < r/m_0$ implies

$$\varrho_1(m_0 y - y_0) = \varrho_1(m_0(y - y_0/m_0)) \leqslant m_0 \varrho_1(y - y_0/m_0) < r$$

and then $m_0 y \in H^-_{m_0}$ or, finally, $y \in H^-_1$. There exists $y_1 \in H_1$ with $\varrho_1(y_0 - m_0 - y_1) < \eta \overset{\mathrm{df}}{=} r/2m_0$. Then

$$\{y \in |\mathfrak{B}_1| : \varrho_1(y - y_0/m_0) < r/m_0\} \supset \{y \in |\mathfrak{B}_1| : \varrho_1(y - y_1) < \eta\}$$

and

$$H^-_1 \supset \{y \in |\mathfrak{B}_1| : \varrho_1(y - y_1) < \eta\}.$$

Take $y \in |\mathfrak{B}_1|$ with $\varrho_1(y) < \eta$. We have $\eta > \varrho_1(y) = \varrho_1((y_1 - y) - y_1)$ and then $y_1 - y \in H^-_1$. Hence $y - y_1 \in H^-_1$, further $y \in y_1 - H^-_1 = (y_1 - H_1)^-$ and we conclude that

$$(*) \qquad \{y \in |\mathfrak{B}_1| : \varrho_1(y) < \eta\} \subset (y_1 - H_1)^-.$$

Moreover, if $y = y_1 - u \in y_1 - H_1$, then there are $x_1, v \in |\mathfrak{B}_2|$ with $\varrho_2(x_1), \varrho_2(v) < \varepsilon/2$ such that $y_1 = Tx_1$ and $u = Tv$. Setting $x = x_1 - v$ we have $y = Tx$, where $\varrho_2(x) \leqslant \varrho_2(x_1) + \varrho_2(v) < \varepsilon$ and we find that

$$(**) \qquad y_1 - H_1 \subset T\{x \in |\mathfrak{B}_2| : \varrho_2(x) < \varepsilon\}.$$

Joining together $(*)$ and $(**)$ we obtain

$$K_1(\eta) = \{y \in |\mathfrak{B}_1| : \varrho_1(y) < \eta\} \subset \mathrm{Cl}_{\mathfrak{B}_1} T\{x \in |\mathfrak{B}_2| : \varrho_2(x) < \varepsilon\} = \mathrm{Cl}_{\mathfrak{B}_1} K_2(\varepsilon)$$

and then the Proposition is proved.

Here comes a statement which explains the connection between complete-closed and closed mappings.

PROPOSITION 11. *Conisder pre-($\mathscr{F}$)-sequences $\mathfrak{B}_i$ $(i = 1, 2)$ and a linear mapping T of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$. If $[\mathfrak{B}_2]$ is complete, then T is complete-closed whenever it is closed.*

Proof. This is a triviality.

The next proposition, which is originally due to Banach, explains the connection between nearly open and open mappings and together with Proposition 10 appears to be the essential part of the theorems known as open-mapping and closed-graph theorems.

PROPOSITION 12 (Banach [1]). *Let $\mathfrak{B}_i$ $(i = 1, 2)$ be two pre-($\mathscr{F}$)-sequences. Every complete-closed nearly open linear mapping of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$ is open.*

Proof. The proof is the almost exact repetition of that given by Banach in [1] but because of some necessary rearrangements of the Banach's proof we should like to repeat it here in full.

Let $\varrho_i \overset{\mathrm{df}}{=} \varrho_{\mathfrak{B}_i}$, $K_i(r) \overset{\mathrm{df}}{=} K_{\mathfrak{B}_i}(r)$ for $i = 1, 2$, take $\varepsilon > 0$ and let $\{\eta_n\}$, $\eta_n > 0$, be such that $\lim_n \eta_n = 0$ and

$$(*) \qquad \mathrm{Cl}_{\mathfrak{B}_1} T\{x \in |\mathfrak{B}_2| : \varrho_2(x) < 2^{-n}\varepsilon\} \supset \{y \in |\mathfrak{B}_1| : \varrho_1(y) < \eta_n\}.$$

Put $\eta \overset{\mathrm{df}}{=} \eta_1$ and let $y \in K_1(\eta)$. We can always find $x_1 \in K_2(2^{-1}\varepsilon)$ such that $\varrho_1(y - Tx_1) < \eta_2$. Suppose we have found $x_1, \ldots, x_{n-1}$ such that

$$(**) \quad \varrho_1\big(y - T(x_1 + \ldots + x_{n-1})\big) < \eta_n \quad \text{and} \quad x_i \in K_2(2^{-i}\varepsilon) \text{ for } i = 1, \ldots, n-1.$$

If such is the case, then applying $(*)$ we find x_n from $K_2(2^{-n}\varepsilon)$ such that $\varrho_1([y - T(x_1 + \ldots + x_{n-1})] - Tx_n) < \eta_{n+1}$ and it follows that we can define by induction a sequence $\{x_n\}$ satysfying $(**)$ for each n. Setting $\bar{x}_n = x_1 + \ldots + x_n$ we find that $\{\bar{x}_n\}$ satisfies the Cauchy condition in $[\mathfrak{B}_2]$ while $\{T\bar{x}_n\}$ tends to y in $[\mathfrak{B}_1]$. Since T is complete-closed, there must be the limit x of $\{\bar{x}_n\}$ in $[\mathfrak{B}_2]$ with $y = Tx$. We have

$$\varrho_2(x) \leqslant \sum_{n=1}^{\infty} \varrho_2(x_n) < \varepsilon \sum_{n=1}^{\infty} 2^{-n} = \varepsilon$$

and then $x \in K_2(\varepsilon)$ which completes the proof of Proposition 12.

We are provided now with all the necessary informations to express the main results, i. e. the open mapping and the closed graph theorems.

THEOREM 2 (The Open Mapping Theorem I). *Consider pre-($\mathscr{F}$)-sequences $\mathfrak{B}_i$ $(i = 1, 2)$ and a complete-closed mapping T of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$. If to every p there correspond k_p such that $T|p\mathfrak{B}_2|$ is of the second category in $|k_p\mathfrak{B}_1|$, then T is open.*

Proof. It follows after succesive application of Propositions 10 and 12.

THEOREM 3 (The Open Mapping Theorem II). *Consider ($\mathscr{F}$)-sequences $\mathfrak{B}_i$ $(i = 1, 2)$ and a closed mapping of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$. If to every p there correspond k_p such that $T|p\mathfrak{B}_2| \supset |k_p\mathfrak{B}_1|$, then T is open.*

Proof. This follows from Theorem 2 after application of Proposition 11 and the Baire theorem on categories.

COROLLARY. *Let $\mathfrak{B}_i$ $(i = 1, 2)$ be two ($\mathscr{F}$)-sequences, $\mathfrak{B}_2 \gg \mathfrak{B}_1$, such that to every p there correspond k_p such that $|p\mathfrak{B}_2| \supset |k_p\mathfrak{B}_1|$. Then $\mathfrak{B}_1 \gg \mathfrak{B}_2$, i. e. $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent.*

Proof. Since $\mathfrak{B}_2$ is finer than $\mathfrak{B}_1$, there exists p_0 such that $|p_0\mathfrak{B}_2| \subset |\mathfrak{B}_1|$ and the identical injection of $[p_0\mathfrak{B}_2]$ into $[\mathfrak{B}_1]$ is continuous. Denote this injection by T. Of course T is closed and $T|p\mathfrak{B}_2| \supset |k_p\mathfrak{B}_1|$ for $p \geqslant p_0$. This means that the assumptions of Theorem 3 are satisfied and T must be open as a mapping of $[p_0\mathfrak{B}_2]$ into $[\mathfrak{B}_1]$. Therefore we have $|q_0\mathfrak{B}_1| \subset |\mathfrak{B}_2|$ for some q_0, the identical injection T^{-1} of $[q_0\mathfrak{B}_1]$ into $[\mathfrak{B}_2]$ is continuous and the Corollary follows.

THEOREM 4 (The Closed Graph Theorem; cf. [2]). *Let $\mathfrak{B}_i$ $(i = 1, 2)$ be two ($\mathscr{F}$)-sequences. Every closed mapping of $|\mathfrak{B}_2|$ into $|\mathfrak{B}_1|$ such that every $|p\mathfrak{B}_1|$ contains some $T|k_p\mathfrak{B}_2|$, is continuous.*

Proof. Let $L = \{x \in |\mathfrak{V}_2| : Tx = 0\}$. The mapping T can be factorized to a one-to-one closed mapping $\tilde{T}$ of $|\mathfrak{V}_2|L|$ into $|\mathfrak{V}_1|$. Here L is closed in $[\mathfrak{V}_2]$ and then $\mathfrak{V}_2/L$ is an $(\mathscr{F})$-sequence. Denote by Y the image of T in $[\mathfrak{V}_1]$ and let $\mathfrak{V} = \{(Y \cap |n\mathfrak{V}_1|, \|\cdot\|_{\mathfrak{V}_1,n})\}$. The mapping $\tilde{T}^{-1}$ is closed, maps $|\mathfrak{V}| = Y$ onto $|\mathfrak{V}_2|$ and, moreover, $\tilde{T}^{-1}|p\mathfrak{V}_1| \supset |k_p(\mathfrak{V}_2/L)|$.

Applying Theorem 3 we find that $\tilde{T}^{-1}$ is open, consequently $\tilde{T}$ is continuous and then T must be continuous as well which finishes the proof of Theorem 4.

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Базисные последовательности, биортогональные системы и нормирующие множества в пространствах Банаха и Фреше

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В монографии Банаха [1], стр. 238, приводится без доказательства результат, что любое бесконечномерное банаховое пространство содержит бесконечномерное замкнутое подпространство с базисом. Начиная с 1958 года появилось в литературе несколько доказательств этого факта ([12], [2], [4] и [9]). Как показано в [2] и [4] результат Банаха остаётся верный также в случае любого бесконечномерного пространства Фреше (= локально-выпуклого, полного, линейно-метрического пространства). Доказательство, данное в [4], основано на неопубликованном доказательстве Мазура для пространств Банаха, изложенном им на одном из заседаний семинара по функциональному анализу в Варшавском Университете в 1955 году. Недавнее доказательство Дэя усиливает результат Банаха, показывая, что в любом бесконечномерном банаховом пространстве X существует биортогональная система (x_n, x_n^*) такая, что $\|x_n\| = \|x_n^*\| = 1$ $(n = 1, 2, \ldots)$ и последовательность (x_n)-базисная, т. е. образующая базис в подпространстве пространства X. Другие конструкции разных видов биортогональных систем и базисных последовательностей можно найти в работах [18], [19], [20], [33] и [7]. В работе [18], см. тоже [43], доказывается существование в любом бесконечномерном пространстве Банаха X условных базисных последовательностей, т. е. образующих не безусловный ([10], стр. 73) базис в подпространстве X. С другой стороны, в [7] показано, что в любом пространстве Фреше, которое не изоморфно никакому банаховому пространству, существует бесконечная безусловная базисная последовательность (натягивающая ядерное пространство). Вопрос [3], [9], [41], существует ли в произвольном бесконечномерном банаховом пространстве бесконечная безусловная базисная последовательность, остаётся открытым, и, повидимому, является трудным. Другие нерешённые вопросы можно найти в реферате [41] и в обзорной статье Зингера [46]. С помощью базисных последовательностей и биортогональных систем можно получить разные характеристики рефлексивности. Так например,