A remark on a theorem of Banach

by

R. E. WILLIAMSON (Hanover, N. H.)

In [3] Steinhaus describes Banach's construction of an orthogonal system complete in L^2 but not in L^1 (see [1]). It is worth noticing both that the theorem holds more generally and that the proof, while mimicking Banach's original construction, can be simplified as follows. (A method similar to that used in the sequel was considered in paper [2] in the special case $V = L^1$).

THEOREM. *Let V and W be vector spaces with $V \not\subseteq W$. Let $\{\varphi_a\}_{a\in A}$ be a total family in V^*. Let Φ be the subspace of V^* spanned by $\{\varphi_a\}$. Then there is a set $\{\psi_a\}_{a\in A}$ in Φ such that $\{\psi_a\}$ is total on W, but not on V.*

Proof. Let $v \in V \setminus W$ be such that $\varphi_{a_0}(v) = 1$ for some a_0. Define $\psi_a = \varphi_a - \varphi_a(v)\varphi_{a_0}$. Then $\psi_a(v) = \varphi_a(v) - \varphi_a(v)\varphi_{a_0}(v) = 0$. But $v \neq 0$, so $\{\psi_a\}$ is not total on V. To show that $\{\psi_a\}$ is total on W let $w \in W$ be such that $\psi_a(w) = 0$ for all $a \in A$. By the definition of ψ_a

$$\psi_a(w) = \varphi_a(w) - \varphi_a(v)\varphi_{a_0}(w) = \varphi_a\big(w - \varphi_{a_0}(w)v\big) = 0, \qquad a \in A.$$

Then $w = \varphi_{a_0}(w)v$ since $\{\varphi_a\}$ is total, and either $w = 0$ or else $\varphi_{a_0}(w) \neq 0$ and $v \in W$. Thus $v \in V \setminus W$ implies $\{\psi_a\}$ is total on W.

To obtain Banach's result let the φ's be the trigonometric system and orthonormalize the ψ's.

References

[1] S. Banach, *An example of an orthogonal development*, Proc. London Math. Soc. 21 (1922), p. 95.

[2] W. Orlicz, *Beiträge zur Theorie der Orthogonalreihenentwicklungen*, Studia Math. 1 (1929), p. 5.

[3] H. Steinhaus, *Stefan Banach*, ibidem, Seria specjalna 1 (1963), p. 11.

Reçu par la Rédaction le 24. 8. 1964