J. WRZESIEN (Wrocław)

*HORNECKER'S ALGORITHM
FOR RATIONAL APPROXIMATION(*)*

1. Introduction. The values of a function of one variable are usually calculated by means of a polynomial which approximates the function sufficiently well. Replacing the polynomial by a rational function often shortens the calculations and limits the number of necessary constants. Hornecker has given in [2] a method for the calculation of a rational function ω which well approximates in the interval $\langle -1, 1 \rangle$ a continuous function given in series expansion

$$(1) \quad \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j \tau_j(t),$$

where $\tau_j(t) = \cos(j \arccos t)$ is the j th Chebyshev polynomial. In [3] Paszkowski has presented a simplified version of this method, which, for given integers m ($m \geq -1$) and n ($n \geq 1$), yields the function ω in the form of a quotient

$$(2) \quad \frac{\lambda}{\mu}$$

or sum

$$(3) \quad \kappa + \frac{\lambda^*}{\mu},$$

where

$$(4) \quad \mu(t) = e_0 + e_1 t + \dots + e_n t^n \quad (e_n \neq 0),$$

$$(5) \quad \lambda(t) = f_0 + f_1 t + \dots + f_{m+n} t^{m+n},$$

$$(6) \quad \kappa(t) = b_0 + b_1 t + \dots + b_m t^m,$$

$$(7) \quad \lambda^*(t) = f_0 + f_1 t + \dots + f_{n-1} t^{n-1}$$

(*) This paper is part of a diploma work prepared at the Department of Numerical Methods, Wrocław University, and has been presented at the Department seminar on 10. 12. 1964.

(the coefficients of polynomial λ^* are denoted similarly to those of polynomial λ but naturally have another meaning). For $m = -1$ one assumes $\kappa \equiv 0$ and $\lambda \equiv \lambda^*$.

The method presented in [3] is expressed in this paper in ALGOL as the *Hornecker* procedure (§ 2). In § 3 are given examples of rational functions which approximate certain elementary functions. These examples were calculated on the Elliott 803 computer by using the library program YU2 based on the *Hornecker* procedure. The program uses double-length floating point numbers having a 68-bit mantissa.

2. Description of the *Hornecker* procedure.

procedure *Hornecker* (*a*, *p*, *e*, *n*, *f*, *b*, *m*, *eps*, *P*, *Q*, *exit*);

value *p*, *n*, *m*;

integer *p*, *n*, *m*;

real *eps*;

Boolean *P*, *Q*;

array *a*, *e*, *f*, *b*;

label *exit*;

comment The *Hornecker* procedure uses as data the integral numbers $m \geq -1$, $n \geq 1$, $p > m + 2 \times n$, the array $a[0:p]$ of coefficients of the series expansion of the approximated function (1), the label *exit* and the Boolean variables *P*, *Q*. The procedure which realizes the algorithm from [3] approximates function (1) by means of rational functions (2) or (3) giving the array $e[0:n]$ of coefficients of polynomial (4) and either (if $P = \text{true}$) the array $f[0:m+n]$ of coefficients of polynomial (5), or (if $P = \text{false}$) the arrays $b[0:m]$ and $f[0:n-1]$ of coefficients of polynomial (6) and (7). The procedure gives also *eps*, the approximate estimation from above of the maximum error of this approximation in the interval $[-1, 1]$. The greater the difference $p - (m + 2 \times n)$ and the faster the series (1) converges, the better is the estimation. For the first application of the procedure for fixed *p* and *a* must be $Q = \text{true}$. If for a given *n* Westerfield estimation [5] indicates that the denominator (4) may have a zero in the interval $[-1, 1]$ then control is transferred from the procedure body to the exit labelled statement. For calculations with a new and greater *n* must be $Q = \text{false}$ and the *Hornecker* procedure statement has to be used once more. In the body of this procedure the procedure CP from [4] is used, which calculates the coefficients of a polynomial which is a linear combination of Chebyshev polynomials with given coefficients. In the *Hornecker* procedure body comments constitute among others things the Roman numbers of calculation steps similar to [3];

```

begin
  integer  $i, j, k, l$ ;
  own integer  $np$ ;
  real  $r, s, W$ ;
  own real  $U$ ;
  array  $a1[0:p]$ ;
  own array  $d[-1:n], d1[0:n]$ ;
  procedure  $coeff(A, N)$ ;
    value  $N$ ;
    integer  $N$ ;
    array  $A$ ;
    comment VII or VIII asterisk;
    begin
       $r := .5 \times A[0] \times e[0]$ ;
      for  $k := 1$  step 1 until  $N$  do
        begin
           $r := r + A[l] \times e[l]$ ;
           $s := A[k] \times e[0]$ ;
          for  $l := 1$  step 1 until  $n$  do
             $s := s + (A[abs(k-l)] + A[k+l]) \times e[l]$ ;
           $f[k] := .5 \times s$ 
        end  $k$ ;
       $f[0] := 5 \times r$ ;
    end  $coeff$ ;
  procedure  $ni(K)$ ;
    value  $K$ ;
    integer  $K$ ;
    comment I, III;
    begin
      for  $k := K$  step 1 until  $n$  do
        begin
           $r := W := 0$ ;
          for  $i := 2 \times (k-1)$  step  $-1$  until 0 do
            begin
               $s := 0$ ;
               $l := i - k + 1$ ;
              if  $l < 0$ 
                then  $l := 0$ ;
              for  $j := i - l$  step  $-1$  until  $l$  do
                 $s := s + d[j] \times d[i-j]$ ;
            end
          end
        end
      end
    end
  end

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     $r := r + s \times a[m + i + 2];$ 
     $W := W + s \times a[m + i + 1]$ 

    end  $i$ ;
     $r := r / W;$ 
     $s := W / U;$ 
     $U := W;$ 
     $d[k] := d1[k] := d1[k - 1] := 0;$ 
    for  $j := k$  step  $-1$  until  $0$  do
        begin
             $W := d[j];$ 
             $d[j] := d[j - 1] - r \times W - s \times d1[j];$ 
             $d1[j] := W$ 
        end  $j$ 
    end  $k$ ;
     $r := s := 0;$ 
    for  $i := n - 1$  step  $-1$  until  $0$  do
        begin
             $W := \text{abs}(d[i]) \uparrow (1 / (n - i));$ 
            if  $W > r$ 
            then begin
                 $s := r;$ 
                 $r := W$ 
            end  $W > r$ 
            else if  $W > s$ 
            then  $s := W$ 
            end  $i$ ;
             $r := r + s;$ 
            if  $r \geq 1$ 
            then go to exit
        end  $ni$ ;
    comment  $I, III$ ;
    if  $Q$ 
    then begin
         $d[-1] := 0;$ 
         $d[0] := -a[m + 2] / a[m + 1];$ 
         $d[1] := d1[0] := 1;$ 
         $U := a[m + 1];$ 
         $np := n;$ 
         $ni(2)$ 
    end

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    end Q
  else  $ni(np+1)$ ;
comment IV, V;
 $eps := 0$ ;
 $l := m+2 \times n$ ;
for  $j := m+1$  step 1 until  $l$  do
   $a1[j] := a[j]$ ;
for  $j := l+1$  step 1 until  $p$  do
  begin
     $k := j - n$ ;
     $s := 0$ ;
    for  $i := n-1$  step  $-1$  until 0 do
       $s := s - d[i] \times a1[i+k]$ ;
     $a1[j] := s$ ;
     $eps := eps + abs(a[j] - s)$ 
  end j;
comment VI;
for  $k := 0$  step 1 until  $n$  do
  begin
     $s := 0$ ;
    for  $i := k$  step 1 until  $n$  do
       $s := s + d[i-k] \times d[k]$ ;
     $e[k] := s$ 
  end k;
if P
  then begin
    comment VII, VIII;
     $coeff(a, m+n)$ ;
     $CP(f, m+n)$ 
  end P
else begin
  comment VII asterisk;
  for  $j := m$  step  $-1$  until 0 do
    begin
       $s := 0$ ;
      for  $i := 1$  step 1 until  $n$  do
         $s := s - d[i] \times a1[i+j]$ ;
       $a1[j] := s/d[0]$ ;

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      b[j] := a[j] - a1[j]
    end j;
  comment VIII asterisk, IX asterisk;
  b[0] := .5 × b[0];
  CP(b, m);
  coeff(a1, n-1);
  CP(f, n-1)
end ¬P;
e[0] := .5 × e[0];
CP(e, n)
end Hornecker

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It is worth while to supplement the description of the *Hornecker* procedure by two remarks. First, if function (1) is even ($a_1 = a_3 = \dots = 0$) or odd ($a_0 = a_2 = \dots = 0$), m should be chosen odd or even, respectively. Second, if for $P = \mathbf{false}$ during calculation an overflow occurs, sometimes it is possible to obtain even then good results by repeating the calculations with $P = \mathbf{true}$.

3. Examples of rational approximation. By means of a program using the *Hornecker* procedure nine elementary functions have been approximated whose series expansion coefficients for the Chebyshev polynomial have been taken from [1]. Tables 1-9 present the estimations *eps* of the error of approximation with rational functions calculated for various m and n . The numbers *eps* are given in floating point form: the mantissa before / followed by the exponent. In each column the sum $m+2n$ is constant or, in other words, the errors printed in this column correspond to rational functions (with $n > 0$) or polynomials (with $n = 0$, for comparison purposes) with the same number of coefficients. From tables 1-9 it follows that for each of these 9 elementary functions and for any value $m+2n$ it is possible to find a rational function which approximates the given elementary function better than a polynomial with the same number of coefficients. The error of approximation by means of rational functions in comparison with polynomial approximation diminishes slightly for some functions, e.g. 4 times for $\frac{1}{t} \sin \pi t$ and $m+2n = 15$, and very much for others, e.g. 300 times for e^t and $m+2n = 11$, and nearly 10^9 times for $\frac{1}{t} \tan \frac{\pi}{4} t$ and $m+2n = 13$.

Tables 10-18 contain the coefficients of rational functions, calculated by Hornecker's method, which for a fixed value of $m+2n$ yield the best approximation of the given elementary functions with error of approximation 10^{-11} .

TABLE 1

e^t											
9			10			11			12		
$m+2n=$	m	$m+n$	m	$m+n$	eps	m	$m+n$	eps	m	$m+n$	eps
0	9	9	10	10	.260/-10	11	11	.108/-11	12	12	.413/-13
1	7	8	8	9	.284/-11	9	10	.106/-12	10	11	.371/-14
2	5	7	6	8	.691/-12	7	9	.231/-13	8	10	.729/-15
3	3	6	4	7	.283/-12	5	8	.836/-14	6	9	.235/-15
4	1	5	2	6	.176/-12	3	7	.453/-14	4	8	.112/-15
5	-1	4	0	5	.160/-12	1	6	.351/-14	2	7	.760/-16
6						-1	5	.380/-14	0	6	.702/-16

TABLE 2
(1/t) sin πt

15			17			19		
$m+2n=$	m	$m+n$	m	$m+n$	eps	m	$m+n$	eps
0	15	15	17	17	.156/-12	19	19	.928/-15
2	11	13	13	15	.415/-13	15	17	.216/-15
4	7	11	9	13	.282/-13	11	15	.126/-15
6	3	9	5	11	.395/-13	7	13	.143/-15
8	-1	7	1	9	.112/-12	3	11	.305/-15
10						-1	9	.128/-14

TABLE 3

15			17			19		
$m+2n=$	m	$m+n$	m	$m+n$	eps	m	$m+n$	eps
0	15	15	17	17	.935/-12	19	19	.614/-14
2	11	13	13	15	.267/-12	15	17	.216/-15
4	7	11	9	13	.199/-12	11	15	.126/-15
6	3	9	5	11	.316/-12	7	13	.143/-15
8	-1	7	1	9	.107/-11	3	11	.305/-15
10						-1	9	.128/-14

TABLE 7 $\cotan(\pi t/2)$

$m+2n=$	11			13			15		
n	m	$m+n$	eps	m	$m+n$	eps	m	$m+n$	eps
0	11	11	.431/-6	13	13	.311/-7	15	15	.223/-8
2	7	9	.648/-9	9	11	.107/-10	11	13	.171/-12
4	3	7	.925/-10	5	9	.510/-12	7	11	.314/-14
6	-1	5	.174/-9	1	7	.295/-12	3	9	.718/-15
8							-1	7	.116/-14

TABLE 8 $(1/t) \sinh^{-1} t$

$m+2n=$	19			21			23		
n	m	$m+n$	eps	m	$m+n$	eps	m	$m+n$	eps
0	19	19	.502/-9	21	21	.754/-10	23	23	.114/-11
2	15	17	.108/-10	17	19	.134/-11	19	21	.171/-12
4	11	15	.916/-12	13	17	.922/-13	15	19	.974/-14
10	-1	9	.109/-11						

TABLE 9 $(1/t) \tan^{-1} t$

$m+2n=$	15			17			19		
n	m	$m+n$	eps	m	$m+n$	eps	m	$m+n$	eps
0	15	15	.102/-6	17	17	.157/-7	19	19	.244/-8
2	11	13	.233/-8	13	15	.283/-9	15	17	.357/-10
4	7	11	.263/-10	9	13	.244/-10	11	15	.243/-11
6	3	9	.694/-10						

TABLE 10 e^t

$\lambda(t) =$	.50348 60152 82553 62017	$\mu(t) =$	.50348 60152 82530 34546
	.25206 18131 43127 51823 t		-.25142 42021 38934 08981 t
	.05608 85196 08864 47593 t^2		.05576 97141 07949 25428 t^2
	.00702 01094 46870 08430 t^3		-.00695 18394 81325 10574 t^3
	.00050 18983 28716 15338 t^4		.00049 43304 61472 29590 t^4
	.00001 67235 93625 90644 t^5		-.00001 63477 99084 24575 t^5

TABLE 11 $(1/t) \sin \pi t$

$\lambda(t) =$	1.50960 96642 80939 7131	$\mu(t) =$	.48052 36804 19220 68920
	-.236199 42606 76079 4802 t^2		.03858 36475 65589 90636 t^2
	1.02921 23046 15099 9944 t^4		.00101 45501 14329 58101 t^4
	-.19480 90695 96966 4251 t^6		
	.01893 84983 59450 2767 t^8		
	-.00098 01530 91948 0681 t^{10}		
	.00002 30161 09517 5640 t^{12}		

TABLE 12

 $\cos \pi t$

$\lambda(t) =$	.47777 18266 33727 2357	$\mu(t) =$	.47777 18266 33822 56117
	—2.31375 54646 28371 5811 t^2		.04395 39967 86868 73644 t^2
	1.72358 79002 29480 5223 t^4		.00135 38742 97359 16056 t^4
	— .46623 54098 45252 3143 t^6		
	.05923 91729 44999 7428 t^8		
	— .00379 37948 74111 3155 t^{10}		
	.00010 60718 21572 6298 t^{12}		

TABLE 13

 $(1/t) \sin(\pi t/2)$

$\lambda(t) =$	.77167 98479 11469 00212	$\mu(t) =$	.49126 66491 18057 70182
	— .29003 93163 29420 50230 t^2		.01738 05482 20043 80742 t^2
	.02826 57166 86477 72195 t^4		.00021 81042 77151 32072 t^4
	— .00105 57755 37531 68159 t^6		
	.00001 48288 84271 68637 t^8		

TABLE 14

 $\cos(\pi t/2)$

$\kappa(t) =$	—10090 .25155 78839 65808		
	190 .84706 04432 25403 t^2		
	—3 .40246 95277 34440 t^4		
	.04872 88722 30906 t^6		
	— .00040 53794 39017 t^8		
$\lambda^*(t) =$	4998 .16935 58162 46641 5	$\mu(t) =$	.49529 72708 23789 91439
			.00942 76786 33524 42446 t^2

TABLE 15

 $(1/t) \tan(\pi t/4)$

$\lambda(t) =$	.45948 47424 94420 83704	$\mu(t) =$	.58503 41443 46349 84687
	— .03714 45327 20217 72080 t^2		— .16758 67122 18197 58606 t^2
	.00054 63703 42669 30841 t^4		.00547 32321 35874 24465 t^4
	— .00000 12116 63022 24077 t^6		— .00003 52970 25816 93380 t^6
	— .00000 00012 15622 61205 t^8		

TABLE 16

 $t \cotan(\pi t/2)$

$\lambda(t) =$	.37839 55370 93315 04766	$\mu(t) =$	.59438 23197 41952 17680 t
	— .43120 68221 38512 24230 t^2		— .18847 82290 83954 50804 t^2
	.05401 74263 74198 00036 t^4		.01024 74695 56945 36914 t^4
	— .00120 61413 28877 94251 t^6		— .00006 91085 11905 25160 t^6

TABLE 17

 $(1/t) \sinh^{-1} t$

$\kappa(t) =$	.81475 72508 28640 78767		
	— .05897 75784 18273 00001 t^2		
	.00885 84914 47930 25923 t^4		
	— .00136 07799 05330 34149 t^6		
	.00016 66600 94867 23976 t^8		
	— .00001 15158 90092 68960 t^{10}		
$\lambda^*(t) =$	.05345 60883 46057 58792	$\mu(t) =$	.28857 31753 89835 41688
	.04192 99236 07314 80996 t^2		.39411 04632 90871 17221 t^2
			.12607 63713 96493 63600 t^4

TABLE 18

 $(1/t) \tan^{-1}t$

$\lambda(t) =$	.23420 54069 50762 16967	$\mu(t) =$	.23420 54069 50867 35061
	.36222 33898 22510 06919 t^2		.44029 18587 80980 56298 t^2
	.15431 72000 00940 10867 t^4		.25424 00725 40717 46738 t^4
	.01352 17958 31377 17494 t^6		.04366 80143 16215 17929 t^6
	-.00058 41372 34410 17607 t^8		
	.00004 41909 82082 85862 t^{10}		
	-.00000 24683 52272 52939 t^{12}		

References

- [1] C. W. Clenshaw, *Chebyshev series for mathematical functions*, London 1962.
 [2] G. Hornecker, *Approximations rationnelles voisines de la meilleure approximation au sens de Tchebycheff*, Compt. Rend. Hebd. Acad. Sci. 249 (1959), pp. 939-941.
 [3] S. Paszkowski, *Approximation uniforme des fonctions continues par les fonctions rationnelles*, Zastosow. Matem. 6 (1963), pp. 441-458.
 [4] С. Пашковский, *Алгоритм вычисления коэффициентов приближающего многочлена*, Zastosow. Matem. 8 (1965), pp. 75-84.
 [5] E. C. Westerfield, *A new bound for the zeros of polynomials*, Amer. Math. Monthly 40 (1933), pp. 18-23.

DEPARTMENT OF NUMERICAL METHODS, WROCLAW UNIVERSITY

Received on 15. 3. 1965

J. WRZESIEN (Wrocław)

ALGORYTM APROKSYMACJI WYMIERNEJ METODĄ HORNECKERA

STRESZCZENIE

§ 2 pracy zawiera wyrażony w ALGOLu opis procedury *Hornecker*. Służy ona do obliczania zmodyfikowaną metodą Horneckera ([3]) funkcji wymiernej dobrze aproksymującej w przedziale $\langle -1, 1 \rangle$ funkcję ciągłą o danym rozwinięciu w szereg względem wielomianów Czebyszewa. W § 3 znajdują się przykłady funkcji wymiernych obliczonych za pomocą procedury *Hornecker* dla 9 funkcji elementarnych. Tablice 1-9 zawierają oszacowania z góry błędów aproksymacji za pomocą funkcji wymiernych o liczniku stopnia $\leq m+n$ i mianowniku stopnia n , a tablice 10-18 — współczynniki funkcji wymiernych aproksymujących funkcje elementarne z błędem rzędu 10^{-11} , mniejszym niż błąd aproksymacji za pomocą wielomianów z tą samą liczbą współczynników.

Я. ВЖЕСЕНЬ (Вроцлав)

**АЛГОРИТМ РАЦИОНАЛЬНОЙ АППРОКСИМАЦИИ МЕТОДОМ
ГОРНЕКЕРА****РЕЗЮМЕ**

§ 2 настоящей работы содержит выраженное на языке АЛГОЛ описание процедуры *Hornesker*. Она применяется при вычислениях модифицированным методом Горнекера [3] рациональной функции хорошо аппроксимирующей на отрезке $\langle -1, 1 \rangle$ непрерывную функцию с заданным разложением в ряд по многочленам Чебышева. В § 3 имеются примеры рациональных функций, вычисленных с помощью процедуры *Hornesker* для 9 элементарных функций. Таблицы 1-9 содержат оценку сверху погрешности аппроксимации с помощью рациональных функций с числителем степени $\leq m+n$ и знаменателем степени n , а таблицы 10-18 содержат коэффициенты рациональных функций аппроксимирующих элементарные функции с погрешностью порядка 10^{-11} , меньшей чем погрешность аппроксимации с помощью полиномов с тем же числом коэффициентов.