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*IMBEDDED MARKOV CHAINS IN TWO EXAMPLES
OF QUEUEING MODELS*

Introduction. In the note two models from the queueing theory are considered. The first model is specified by the assumptions:

- (a) Poisson input with constant arrival rate λ ;
- (b) no restriction on the length of the queue;
- (c) m parallel channels with constant service-time T .

The second model is similarly determined by:

- (a) Poisson input with arrival rate depending on the length of the queue;
- (b) no restriction on the length of the queue;
- (c) single service channel with arbitrary service-time distribution.

The method of imbedded Markov chains, defined by Kendall in [1] will be applied to both models. In both cases the stochastic process $n(t)$ (the number of units in the system at the moment t) is not a Markov process. It may be shown, however, that for any of those models there exists a sequence of moments t_r ($r = 1, 2, \dots$) such that the discrete parameter process N_r defined as $N_r = n(t_r)$ is a Markov chain. Namely, the process N_r is the so called *imbedded Markov chain*. Having constructed the imbedded Markov chains, we will then examine limit probabilities of the states of the system.

Model I. To the assumptions specifying the first model we shall now add the special service discipline. We will assume that the first channel has a priority to serve any item arriving at the system when this channel is empty. If all the channels are busy, arriving items have to wait in queue, the item at the head of the queue being served in the first available channel. Items arriving at the moment when the first channel is busy while some channels are empty choose the channel at random. Without loss of generality we can assume that $T = 1$ (the constant service-time being accepted as a unit of time).

Let us denote by $n(t)$ the state of the system at the moment t , where $n(t)$ denotes the number of items in the system. It can easily be shown that $n(t)$ is not a Markov process. But the following lemma allows us to imbed a Markov chain.

LEMMA I. *The random sequence $N_r = n(t_r)$, where t_r ($r = 1, 2, \dots$) is the moment of termination of the r -th service in the first channel (the departing unit not included in N_r), is a Markov chain.*

Proof. Let us denote by t_r, t_{r+1} two successive moments of termination of service in the first channel. Two cases should be distinguished. If at the moment t_r , $N_r = 0$, then the state of the system N_{r+1} depends only upon the number of arrivals in the interval of length 1. If at the moment t_r , $N_r > 0$, then the state of the system N_{r+1} depends upon both N_r and the number of arrivals in the interval of length 1. Thus in both cases the N_{r+1} depends upon the state N_r and upon the number of arrivals in the unit interval and does not depend upon the course $(N_{r-1}, N_{r-2}, \dots)$ of the process at the preceding moments. Thus the proof is completed.

Now we shall pass to the study of asymptotic probabilities of the states of chain N_r . Notice that, owing to the specified service discipline, for any $r = 1, 2, \dots$ we know that:

1° if $N_r \geq m$ then at the moment t_r all the channels are busy and there are $N_r - m$ units waiting in the queue;

2° if $N_r < m$ then at the moment t_r the first channel is empty with probability one.

Let us denote by $P_r(n) = P(N_r = n)$ the probability of state n at the moment t_r and by $P_{r,r+1}(i, j) = P(N_{r+1} = j | N_r = i)$ the probability of transition from the state i at the moment t_r to the state j at the moment t_{r+1} . We easily get the probabilities of transition:

$$P_{r,r+1}(i, j) = P(\xi = j) \quad \text{for} \quad 0 \leq i \leq m,$$

$$P_{r,r+1}(i, j) = P(\xi = j - i + m) \quad \text{for} \quad m < i \leq j + m,$$

$$P_{r,r+1}(i, j) = 0 \quad \text{for} \quad i > j + m,$$

where $P(\xi = k) = \frac{\lambda^k}{k!} e^{-\lambda}$.

The system of Chapman-Kolmogorov equations in this model is of the form

$$\begin{aligned} P_{r+1}(0) &= [P_r(0) + P_r(1) + \dots + P_r(m)] e^{-\lambda}, \\ (1) \quad P_{r+1}(1) &= [P_r(0) + P_r(1) + \dots + P_r(m)] \frac{\lambda}{1!} e^{-\lambda} + P_r(m+1) e^{-\lambda}, \\ P_{r+1}(2) &= [P_r(0) + P_r(1) + \dots + P_r(m)] \frac{\lambda^2}{2!} e^{-\lambda} + P_r(m+1) \frac{\lambda}{1!} e^{-\lambda} + \\ &\quad + P_r(m+2) e^{-\lambda}, \\ &\dots \end{aligned}$$

$$P_{r+1}(n) = [P_r(0) + P_r(1) + \dots + P_r(m)] \frac{\lambda^n}{n!} e^{-\lambda} + P_r(m+1) \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} + \dots + P_r(m+n) e^{-\lambda},$$

On the ground of the results obtained by Ryll-Nardzewski ([2]) we are aware of the existence of non-negative limits $\lim_{r \rightarrow \infty} P_r(n) = p_n$ which for $\lambda \leq m$ fulfil the equation $\sum_{n=0}^{\infty} p_n = 1$.

Passing with r to infinity in the whole system (1) we get

$$\begin{aligned} e^{\lambda} p_0 &= p_0 + p_1 + p_2 + \dots + p_m, \\ e^{\lambda} p_1 &= \frac{\lambda}{1!} (p_0 + p_1 + p_2 + \dots + p_m) + p_{m+1}, \\ (2) \quad e^{\lambda} p_2 &= \frac{\lambda^2}{2!} (p_0 + p_1 + p_2 + \dots + p_m) + \frac{\lambda}{1!} p_{m+1} + p_{m+2}, \\ &\dots \\ e^{\lambda} p_n &= \frac{\lambda^n}{n!} (p_0 + p_1 + p_2 + \dots + p_m) + \frac{\lambda^{n-1}}{(n-1)!} p_{m+1} + \\ &\quad + \frac{\lambda^{n-2}}{(n-2)!} p_{m+2} + \dots + p_{m+n}, \\ &\dots \end{aligned}$$

System (2) is homogeneous and has to be solved under the conditions

$$(3) \quad \sum_{k=0}^{\infty} p_k = 1,$$

$$(4) \quad p_k \geq 0.$$

Systems (2) and (3) without inequality (4) have an infinite number of solutions, because to solve them we can accept some $p_0, p_1, \dots, p_{m-1}$ and solve the systems recurrently. Adding the inequality (4) we get a system with a unique solution ([2]). In some cases the systems may be effectively solved by inserting m additional equations, which may be obtained from the generating function of the probabilities p_k . Let us denote by $\Phi_r(s)$ the generating function

$$(5) \quad \Phi_r(s) = \sum_{n=0}^{\infty} s^n P_r(n).$$

Multiplying the k -th equation of system (1) by s^k ($k = 0, 1, \dots$) and adding them all, we get

$$(6) \quad \Phi_{r+1}(s) = \frac{e^{\lambda(s-1)}}{s^m} \{P_r(0)(s^m-1) + P_r(1)(s^m-s) + \dots + \\ + P_r(m-1)(s^m-s^{m-1}) + \Phi_r(s)\}.$$

When $r \rightarrow \infty$ we have $\lim_{r \rightarrow \infty} \Phi_r(s) = \Phi(s) = \sum_{n=0}^{\infty} s^n p_n$ and equation (6) becomes:

$$(7) \quad \Phi(s) = \frac{e^{\lambda(s-1)}}{s^m} \{p_0(s^m-1) + p_1(s^m-s) + \dots + (s^m-s) + \Phi(s)\}.$$

We may obtain additional equations by taking the first m derivatives of formula (7) and putting $s = 1$. The first two of those relations are of the form:

$$(8) \quad p_0 m + p_1(m-1) + \dots + p_{m-1} = m - \lambda,$$

$$(9) \quad m(m-1)(p_0 + \dots + p_{m-1}) - [2 \cdot 1 p_2 + \dots + (m-2)(m-3)p_{m-2}] \\ = m(m-1) + 2L(m-\lambda) + \lambda(\lambda-2m),$$

where $L = \sum_{k=0}^{\infty} k p_k$.

Model II. In this model we assume that the process of arrivals is a Poisson process with arrival rate $\lambda_n = \lambda[n(t)]$ depending on the state $n(t)$ of the system at any moment of time t . A special case of this kind of feedback may be observed in a model of service facility with impatient customers. The process of arrival is a Poisson process with constant intensity λ but arriving units decide to join the queue with a certain probability r_n dependent on the state of system n . Units joining the queue form a Poisson process with intensity $\lambda_n = \lambda r_n$.

Let $G(x)$ be the distribution function of the service time in a single channel. The number of units in the system $n(t)$ (t a continuous parameter) is not a Markov process. But the following lemma shows the imbedded Markov chain in this system.

LEMMA II. *The random sequence $N_r = n(t_r)$, where t_r ($r = 1, 2, \dots$) is the moment of departure of the r -th unit from the service channel, is a Markov chain.*

Proof. For any $r = 1, 2, \dots$ we have

$$(10) \quad N_{r+1} = N_r + \xi(t_{r+1} - t_r) - 1,$$

where $\xi(\tau)$ denotes the number of arrivals during the interval of length τ . Thus random variable N_{r+1} depends on random variable N_r and the number of arrivals during one service, but does not depend on the behaviour of the process before moment t_r . Thus it is a Markov process.

We denote by $P_r(n) = P(N_r = n)$ the probability of state $N_r = n$ at the moment t_r and by $P(\xi(\tau) = k | n)$ the probability of k arrivals during the time interval of length τ if at the beginning of this interval there are n units in the system.

The latter probability may be expressed for $k = 0$ as

$$P(\xi(\tau) = 0 | n) = e^{-\lambda_n \tau}$$

and for $k \geq 1$ as

$$(11) \quad P(\xi(\tau) = k | n) = \lambda_n \dots \lambda_{n+k-1} \int \dots \int_{\mathcal{O}} e^{-(\lambda_n x_0 + \dots + \lambda_{n+k} x_k)} dx_0 \dots dx_{k-1},$$

where $\Omega = \{(x_0, \dots, x_k): x_i \geq 0, x_0 + \dots + x_k = \tau\}$.

Hence the distribution of the number of arrivals during the service of a unit if n units are waiting at the beginning of the service is

$$(12) \quad P(\xi = k, n) = \int_0^\infty P(\xi(x) = k | n) dG(x).$$

We denote by $P_{r,r+1}(i, j)$ the probability of transition from state $N_r = i$ to state $N_{r+1} = j$. We can express them as

$$P_{r,r+1}(0, j) = P_{r,r+1}(1, j) = P(\xi = j, 1),$$

$$P_{r,r+1}(i,j) = \begin{cases} P(\xi = j-i+1) & \text{for } i-1 \leq j, \\ 0 & \text{for } i-1 > j. \end{cases}$$

The Chapman-Kolmogorov system of equations has in this model the form

[illegible]

Differentiating formula (18) twice and inserting $s = 1$, we obtain

$$(19) \quad p_0 = 1 - \lambda_1 T,$$

$$(20) \quad L = \frac{\lambda_1 T (\lambda_1 T + 2)}{2},$$

where $L = \sum_{k=1}^{\infty} k p_k$ is the mean number of items in the system.

References

- [1] D. G. Kendall, *Stochastic processes occurring in the theory of queues and their analysis by the imbedded Markov chain*, Ann. Math. Stat. 24 (1953), p. 338-354.
- [2] C. Ryll-Nardzewski (in preparation).

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WŁOŻONE ŁAŃCUCHY MARKOWA W DWÓCH MODELACH KOLEJKOWYCH

STRESZCZENIE

W pierwszym modelu kolejkowym rozważanym w tej pracy zakładamy: (a) poissonowski strumień zgłoszeń; (b) nieograniczoną kolejkę; (c) m równoległych kanałów obsługowych ze stałym czasem obsługi. Zakładamy, że jednostka czekająca w kolejce wchodzi do obsługi natychmiast po zakończeniu poprzedniej obsługi. Ponadto zakładamy, że pierwszy kanał ma pierwszeństwo obsługi jednostki zgłaszającej się kiedy jest on wolny, w przeciwnym razie wybór kanału jest dowolny.

W drugim modelu kolejkowym zakładamy: (a) poissonowski strumień zgłoszeń z intensywnością zależną od długości kolejki; (b) nieograniczoną kolejkę; (c) pojedynczy kanał obsługowy z dowolnym rozkładem czasu obsługi.

W tych modelach stan systemu $n(t)$ określony jako ilość jednostek przebywających w systemie w chwili t nie jest procesem Markowa. Jednakże można w tych modelach wyróżnić ciąg momentów $t_1, t_2, \dots, t_r, \dots$ taki, że $N_r = n(t_r)$ jest procesem Markowa. Proces N_r nazywamy za Kendallem *włożonym łańcuchem Markowa*.

W pierwszym modelu ciąg $\{t_r\}$ definiujemy jako momenty zakończenia obsługi w pierwszym kanale, natomiast w drugim modelu ciąg $\{t_r\}$ stanowią momenty zakończenia obsługi.

W pracy rozpatrujemy graniczne własności prawdopodobieństw stanów procesu N_r w tych modelach. W drugim modelu rozważamy również szczególny przypadek stałego czasu obsługi i szczególnego sprzężenia pomiędzy procesem zgłoszeń a długością kolejki.

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ВЛОЖЕННЫЕ ЦЕПИ МАРКОВА В ДВУХ СИСТЕМАХ МАССОВОГО ОБСЛУЖИВАНИЯ**РЕЗЮМЕ**

В первой из рассматриваемых в работе систем предполагается (а) пуассоновский поток заявок, (в) неограниченную очередь; (с) m параллельных каналов с постоянной длиной времени обслуживания. Кроме этого предполагаем, что поступающие заявки обязаны занимать первый канал если он свободный. Если первый канал занят, то поступающие заявки выбирают случайно один из свободных каналов.

В другой системе предполагается: (а) пуассоновский поток заявок с интенсивностью которая зависит от состояния системы; (в) неограниченную очередь; (с) один канал обслуживания с произвольным распределением времени обслуживания. В обоих случаях состояние системы $n(t)$ определенное количеством заявок находящихся в системе в момент времени t не является процессом Маркова. Однако можно выбрать такую последовательность моментов $t_1, t_2, \dots, t_r, \dots$ что $N_r = n(t_r)$ является цепью Маркова. По Кендаллу эту цепь называется *вложенной цепью Маркова*. В первой из этих систем последовательность $\{t_r\}$ определяется моментами окончания обслуживания в первом канале. В другой — это моменты окончания обслуживания всех клиентов.

В статье рассмотрены предельные свойства вероятностей состояний цепи N_r . Во второй модели рассматривается тоже частный случай постоянного времени обслуживания и особого сопряжения между входящим потоком и состоянием системы.