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PRINTED IN POLAND

W R O C Ł A W S K A   D R U K A R N I A   N A U K O W A

ACTA ARITHMETICA
XI (1966)

A uniform result on almost primes

by

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In an earlier paper [8] I have shown that if $F(x)$ is a polynomial with integral coefficients then there are an infinite number of integers m such that $F(m)$ has a bounded number of prime factors. My goal here is to prove that there is a positive integer m, which is bounded above by a specific function of the coefficients of $F(x)$, for which $F(m)$ is an almost prime. To be exact, I shall prove

THEOREM 1. *Let $F(x)=f_1(x)\dots f_k(x)$ where, for $1\leqslant i\leqslant k$, $f_i(x)$ is an irreducible polynomial with integral coefficients which is of degree n_i. Suppose that no irreducible factor of $F(x)$ is a constant multiple of any other and that $F(x)$ has no fixed prime divisors. Let $A(f_i)$ denote the maximum of the set of the absolute values of the coefficients of $f_i(x)$ and $B(F)=A(f_1)\dots A(f_k)$. Let n be the maximum of $n_1,n_2,\dots,n_k$ and*

$$L(k)=k\sum_{j=1}^{k}(1/j).$$

Then for any positive δ there is a constant $c(k,\delta,n)$ which depends on k,δ, and n and a positive integer $m\leqslant\exp\big(c(k,\delta,n)(B(F))^{\delta}\big)$ for which $F(m)$ has at most $n_1+n_2+\dots+n_k+L(k)+k\log(2n+1)+1$ prime factors, multiple prime factors being counted multiply.

The proof of this theorem will be based on Selberg's sieve method and one of Fogels' results on Hecke's zeta function.

Under certain circumstances the bound on m given above can be reduced. Using another result of this paper, Theorem 4 of section 2, one can prove

THEOREM 2. *Let $F(x)$ be defined as in Theorem 1. Let, for $1\leqslant i\leqslant k$, θ_i be a zero of $f_i(x)$, K_i be the field obtained by adjoining θ_i to the rationals, and D_i be the absolute value of the discriminant of K_i. Suppose that $D_i\leqslant(\log x)^E$, where E is any positive number. Then there is a constant $c(k,E,n)$ which depends on k,E, and n and a positive integer $m\leqslant c(k,E,n)\big(B(F)\big)^{35n}$ for which $F(m)$ has at most $n_1+n_2+\dots+n_k+L(k)+$*

$k\log(2n+1)+1$ *prime factors. Furthermore, let* λ *be any number such that* $\lambda \geqslant 2(n'+1)/n'$ *and let* $g(\lambda) = 2[(2R-1)n'\lambda]^{-1}$ *where* $R = (2n+1)(6n^2)[(2n+3)(6n^2+1)]^{-1}$ *and* n' *is the minimum of* $n_1, n_2, \ldots,$ *and* n_k. *Then there is a constant* $c(k, E, n, \lambda)$ *and a positive integer* $m \leqslant c(k, E, n, \lambda)(B(F))^{g(\lambda)}$ *for which* $F(m)$ *has at most* $[n_1\lambda]+\ldots+[n_k\lambda]-k+L(k)+k\log(2n+1)+1$ *prime factors, multiple prime factors being counted multiply.* ($[x]$ denotes the integral part of x here.)

Further reductions on the bound for m can be made if $F(x)$ has a simple form. If, for example, the irreducible factors of $F(x)$ are linear it is possible to prove that there is an integer $m \leqslant c(k)(B(F))^{6+\frac{2}{3}}$ for which $F(m)$ has at most $(1.98)k+L(k)$ prime factors. Thus if $F(x) = ax+b$ where $(a, b) = 1$ then there is an $m \leqslant c[\max(|a|, |b|)]^{6+\frac{2}{3}}$ for which $F(m)$ has at most 2 prime factors.

Fluch ([3]) has shown that if $F(x) = ax+b$, $(a, b) = 1$, and $a > b \geqslant 1$ then there is an integer $m \leqslant a^{6+\frac{9}{11}+\varepsilon}$ for which $F(m)$ has at most 2 prime factors, provided that $a > c(\varepsilon)$. Pan ([10]) proved the existence of an integer $m \leqslant a^c$, where $c \leqslant 5448$, for which $F(m)$ has one prime factor. One of Schinzel's conjectures ([12], [1]) can be interpreted to say that if $g(x)$ is a given polynomial of degree n which has integral coefficients and whose leading coefficient a_0 is positive then, if N is any sufficiently large positive integer such that $N-g(x)$ is an irreducible polynomial, there is a positive integer $m \leqslant (N/a_0)^{1/n}$ for which $N-g(m)$ is an almost-prime. If the discriminant of the field associated with $N-g(x)$ satisfies the assumptions of Theorem 2 its proof can be modified to yield this assertion. If not we can, using Brun's method, prove

THEOREM 3. *Let* $F(x)$ *be a polynomial which is of degree* n, *has integral coefficients, and has no fixed prime divisors. Let* $A(F)$ *denote the maximum of the set of the absolute values of the coefficients of* $F(x)$. *Let* λ *be any number such that* $\lambda > \frac{27}{28}$ *and* $h(\lambda) = [n(\frac{28}{27}\lambda-1)]^{-1}$.

Then there is a constant $c(n)$ *and a positive integer* $m \leqslant c(n)(A(F))^{h(\lambda)}$ *for which* $F(m)$ *has at most* $[7n^2\lambda]$ *prime factors, multiple prime factors being counted multiply.*

1. Preliminaries.

We begin with a uniform estimate on the number of ideals in a field.

LEMMA 1. *Let* K *be an algebraic number field of degree* n, D *be the absolute value of its discriminant,* $\zeta_K(s)$ *be the zeta function of* K, r *be the residue of this function at the point* $s = 1$, *and* $G(m)$ *be the number of ideals in* K *whose norm is equal to* m. *Then*

$$\sum_{1\leqslant m\leqslant x} G(m) = rx+O[\lambda(D)x^{t(D)}]$$

where

$$\lambda(D) = (\log(2D))^{n+2}, \quad t(D) = 1-(10n^2\log(2D))^{-1},$$

and the constant implied by the O-term depends only on n.

The next lemma, which is a slightly different form of Theorem 3 of the appendix of [11] will serve as a frame of reference for the proof of Lemma 1.

LEMMA 2. *Let* $s = \sigma+it$ *and let*

$$f(s) = \sum_{m=1}^{\infty} a_m m^{-s}.$$

Suppose that the series is absolutely convergent for $\sigma > 1$,

(1) $$|a_m| \leqslant c_1(f)\beta(m),$$

and

(2) $$\sum_{m=1}^{\infty} |a_m| m^{-\sigma} \leqslant c_2(f)(\sigma-1)^{-a}$$

as $\sigma \to 1+0$, *where* $c_1(f)$ *and* $c_2(f)$ *are constants that depend on the function* $f(s)$, $\beta(x)$ *is a positive eventually increasing function of* x, *and* a *is a positive number. Let* $1 < b \leqslant 2$, $T > 0$, *and* $x \geqslant 1$. *Then*

(3) $$\sum_{m<x} a_m = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{f(s)x^s}{s}\,ds+E(x, T)$$

where

$$E(x, T) = O\left[\frac{c_2(f)x^b}{T(b-1)^a}+\frac{c_1(f)\beta(2x)x\log(2x)}{T}+c_1(f)\beta(2x)\right],$$

and the constant implied by the O-term is independent of the function $f(s)$ *and the numbers* x, T, b, *and* a.

If we let $f(s) = \zeta_K(s)$ it is not difficult to determine a function $\beta(x)$ and numbers $c_1(\zeta_K)$, $c_2(\zeta_K)$, and a which satisfy (1) and (2). We have $a_m = G(m)$, the number of ideals in K whose norm is equal to m. Since there are at most n prime ideals which contain the rational prime p in a field of degree n, since the norm of any one of these prime ideals is divisible by p, and since integers and ideals factor uniquely it follows that $G(m) \leqslant (d(m))^n$, where $d(m)$ is the number of divisors of m. Since (I-5.2, [11])

$$d(m) \leqslant c_3(\varepsilon)\exp[(1+\varepsilon)\log 2(\log m)(\log\log m)^{-1}]$$

for any fixed $\varepsilon > 0$ and any $m \geqslant 2$ it follows that if we set $\varepsilon = 1$ we can let

(4) $$\beta(x) = \exp[n\log 4(\log x)(\log\log 3x)^{-1}]$$

for $x \geqslant 1$; moreover, $c_1(\zeta_K)$ will be a constant that depends only on n. Since, for $\sigma > 1$,

$$|\zeta_K(\sigma+it)| \leqslant \zeta_K(\sigma) \leqslant \big(\zeta(\sigma)\big)^n$$

where $\zeta(s)$ is the zeta function of the rational number field, $c_2(\zeta_K)$ will be a constant that depends only on n and we can let $a = n$. Thus we have

$$\sum_{m<x} G(m) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta_K(s)x^s}{s}\,ds + E_1(x, T) \tag{5}$$

where

$$E_1(x, T) = O\left[\frac{x^b}{T(b-1)^n} + \frac{\beta(x)\log(2x)}{T} + \beta(2x)\right],$$

the constant implied by the O-term depends only on n, and $\beta(x)$ is defined as in (4).

From this point on the symbols $c_1, c_2, \ldots$ will represent constants that depend only on n.

The integral that appears in (5) will be evaluated with the aid of the following result of Fogels: If $n \geqslant 2$ and $0 < \delta < \big((\log 2)/3\big)(\log D)^{-1}$ then there is a constant c_1 such that the inequality

$$\left|\frac{s-1}{s-2}\zeta_K(s)\right| \leqslant c_1\delta^{-n}D^{\frac{1}{2}(1-\sigma)}(1+|t|)^{\frac{1}{2}(1+\delta-\sigma)n} \tag{6}$$

holds uniformly for $-\delta \leqslant \sigma \leqslant 1+\delta$.

This inequality appears in the proof of Lemma 4 of [4]. Fogels' assumption that $0 < \delta < (\log D)^{-1} < \frac{1}{2}$ can be replaced by the one above since his proof only requires that $0 < \delta < 2^{-1}$ and that D^δ be bounded.

If we let $\delta = (n^4\log D)^{-1}$ and apply (6) to $\zeta_K(s)$ on the rectangle with the vertices $1-a\pm iT$ and $b\pm iT$, where $a = (n^2\log D)^{-1}$, $b = 1+(n^4\log D)^{-1}$, and $T \geqslant 1$ we have

$$|\zeta_K(s)| \leqslant c_2(\log D)^{n+1}(1+|t|)^{m(D)}, \tag{7}$$

where $m(D) = (n^{-1}+n^{-3})(2\log D)^{-1}$, on the line $s = 1-a+it$. On the lines $s = \sigma\pm iT$, $1-a \leqslant \sigma \leqslant b$, we have

$$|\zeta_K(s)| \leqslant c_3(\log D)^n(1+T)^{m(D)}. \tag{8}$$

The residue theorem, (7) and (8) imply that

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta_K(s)x^s}{s}\,ds = rx + E_2(x, T) \tag{9}$$

where

$$E_2(x, T) = O\left[(\log D)^{n+2}x^{1-a}T^{m(D)} + (\log D)^n x^b T^{m(D)-1}\right].$$

If we set $T = \exp(n^{-2}\log x)$ equations (5) and (9) give us

$$\sum_{m<x} G(m) = rx + O\left[(\log D)^{n+2}\big(x^\alpha + x^\beta + x^\gamma\beta(2x)\log 2x\big)\right], \tag{10}$$

where $\alpha = 1-a+m(D)n^{-2}$, $\beta = b+\big(m(D)-1\big)n^{-2}$, and $\gamma = b-n^{-2}$. Since α, β and γ are each less than $1-(10n^2\log D)^{-1} = t_1(D)$ and since $\beta(x)$, as well as $\log 2x$, is bounded above by a constant multiple of x^λ where $\lambda = (8n^2)^{-1}$ it follows that

$$\sum_{m<x} G(m) = rx + O\left[(\log D)^{n+2}x^{t_1(D)}\right]. \tag{11}$$

Equality can be allowed in the index of summation in (11), for

$$\sum_{m\leqslant x} G(m) - \sum_{m<x} G(m) \leqslant G([x]) = O\big(\beta(x)\big) = O\big((\log D)^{n+2}x^{t_1(D)}\big). \tag{12}$$

The conclusion of Lemma 1 follows from (11) and (12).

LEMMA 3. *Let K, n, D, r and $G(m)$ be defined as in Lemma 1 and suppose that*

$$\sum_{m\leqslant x} G(m) = rx + O[\lambda(D)x^t]$$

where $\lambda(D)$ and t are functions of D and n but are independent of x, $0 < t < 1$, and the constant implied by the O-term depends only on n. Let P denote a prime ideal in K and $N(P)$ be its norm. Then for $x \geqslant 2$

$$\sum_{N(P)\leqslant x} \frac{1}{N(P)} = \log\log x + \log r + v(K) + O\left[\frac{1}{(1-t)}\left(1+\frac{\lambda(D)}{r}\right)\frac{1}{\log x}\right]$$

where

$$v(K) = v_0 - \sum_{m=2}^{\infty}\frac{1}{m}\sum_P \frac{1}{(N(P))^m}$$

and v_0 is Euler's constant.

COROLLARY. *Let $\Omega(p)$ denote the number of prime ideals of the first degree in K which divide the rational prime p. Then for $x \geqslant 2$*

$$\sum_{p\leqslant x} \frac{\Omega(p)}{p} = \log\log x + \log r + v_1(K) + O\left[\frac{1}{(1-t)}\left(1+\frac{\lambda(D)}{r}\right)\frac{1}{\log x}\right]$$

where $v_1(K)$ is a constant that depends on K but whose absolute value is bounded above by a number that depends only on n.

Lemma 3 can be proved by an argument that is similar to the one given on pages 114-115 and 149-151 of [5].

We now turn to the problem of finding the summatory function of a Dirichlet series that is essentially the product of k zeta functions.

LEMMA 4. *Let the polynomial $F(x) = f_1(x)\ldots f_K(x)$ be defined as in Theorem 1; K_i be the field generated by a zero of $f_i(x)$, $\zeta_i(s)$, D_i, r_i, and $G_i(m)$ be defined in a manner analogous to that of Lemma 1; $\Omega_i(p)$ be the number of prime ideals of the first degree in K_i which divide the rational prime p and $\Omega(p) = \Omega_1(p)+\ldots+\Omega_k(p)$. Let $\omega(d)$ be the number of solutions of the congruence $F(x)\equiv 0 \bmod d$; $1/f(d) = \omega(d)/d$, and*

$$\frac{1}{f'(d)} = \frac{1}{f(d)}\prod_{p|d}\left(1-\frac{1}{f(p)}\right)^{-1}.$$

Set $H(s) = J_1(s)J_2(s)$ where

$$J_1(s) = \prod_p\left(1+\frac{\omega(p)}{p^{s-1}(p-\omega(p))}\right)\left(1-\frac{1}{p^s}\right)^{\Omega(p)},$$

$$J_2(s) = \prod_{i=1}^{k}\prod_{P'\epsilon K_i}\left(1-\frac{1}{N(P')^s}\right),$$

and P' denotes a prime ideal of the second or higher degree. Let $a_m m^{-s}$ be the m-th term of the Dirichlet series that represents $H(s)$ and $H_1(s)$ be the series whose m-th term is $|a_m|m^{-s}$. Suppose that

$$\sum_{m\leqslant x} G_i(m) = r_i x+\theta_i(x)\beta(D_i)x^{t(i)}$$

where $|\theta_i(x)|\leqslant 1$ for $x\geqslant 1$ and $\frac{2}{3}\leqslant t(i) < 1$. Let

$$\eta(k, D) = \frac{r_1+\beta(D_1)}{1-t(1)}\cdots\frac{r_k+\beta(D_k)}{1-t(k)},$$

$$E_k = r_1\ldots r_k(k!)^{-1}H(1),$$

$$H_2\big(1, w, t(1)\big) = H_1\big(t(1)\big),$$

and, for $k\geqslant 2$,

$$H_2\big(k, w, t(1)\big) = H_1(w)(1-w)^{-1}$$

where w is any number such that $\frac{1}{2} < w < 1$. Let $\mu(m)$ denote the Möbius function. Then, for $k\geqslant 1$, we have

$$\sum_{m\leqslant x}\frac{\mu^2(m)}{f'(m)} = E_k(\log x)^k+O\big[c_1^k\eta(k, D)H_2\big(k, w, t(1)\big)(\log x)^{k-1}\big] \tag{13}$$

where the constant c_1, as well as the one implied by the O-term, depends only on n, the maximum of the degrees of $f_1(x), f_2(x), \ldots,$ and $f_k(x)$.

Proof. We begin by noting that

$$\sum_{m=1}^{\infty}\frac{\mu^2(m)}{f'(m)m^{s-1}} = H(s)\zeta_1(s)\ldots\zeta_K(s)$$

and that $H(s)$ can be represented as a Dirichlet series that is absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$.

The next step consists of showing, by induction, that if

$$\zeta_1(s)\ldots\zeta_k(s) = \sum_{m=1}^{\infty}c_m m^{-s}$$

then

$$\sum_{m\leqslant x}c_m = \frac{r_1\ldots r_k}{(k-1)!}x(\log x)^{k-1}+\theta(x)\eta'(k, D)\tau(k, x)x \tag{14}$$

where $|\theta(x)|\leqslant 1$ for $x\geqslant 1$, $\eta'(1, D) = \big(1-t(1)\big)\eta(1, D)$, $\eta'(k, D) = 2\cdot 3^{k-2}\eta(k, D)$ for $k\geqslant 2$, $\tau(1, x) = x^{-1+t(1)}$, and $\tau(k, x) = (\log x)^{k-2}$ for $k\geqslant 2$.

We can also show, as in Lemma 3.1 of [8], that if

$$U(s) = H(s)Z(s)$$

where $H(s)$ is a series that is absolutely convergent for the $\mathrm{Re}(s) > \frac{1}{2}$ and $Z(s)$ is a product of k zeta functions whose summatory function is given by (14) then the summatory function of $U(s)$, i.e. the sum

$$\sum_{m\leqslant x}\frac{\mu^2(m)}{f'(m)}m,$$

is equal to

$$r_1H(1)x+O\big[\eta'(1, D)H_1\big(t(1)\big)x^{t(1)}\big]$$

if $k = 1$, or

$$\frac{r_1\ldots r_k}{(k-1)!}H(1)x(\log x)^{k-1}+O\left[2^k\eta'(k, D)\frac{H_1(w)}{1-w}x(\log x)^{k-2}\right]$$

if $k\geqslant 2$. The conclusion of the lemma follows from these results.

LEMMA 5. *Let $B(F)$ be defined as in Theorem 1. Then we have*

$$\sum_{m\leqslant x}\mu^2(m)/f'(m) = E_k(\log x)^k+O\,[\psi(c_1, k, D)(\log x)^{k-1}]$$

where

$$\psi(c_1, k, D) = \big[\exp\big(c_1^k\log_3 e^3B(F)\big)\big]\,[(\log 2D_1)\ldots(\log 2D_k)]^{c_2}$$

and $\log_3 x = \log\log\log x$.

The proof of Lemma 5 has two parts. The first consists of showing that

$$H_2\big(k, w, t(1)\big)\leqslant(\exp c_1^k)\exp\big[c_2k\log_2\big(d(F)+3\big)\big]$$

where $H_2(k, w, t(1))$ is the quantity defined in Lemma 4 and $d(F)$ is a number that depends on the polynomial $F(x)$. The second consists of proving that

$$d(F) \leqslant c_2^k \log eB(F).$$

Several definitions are in order. Let

$$a(p) = \omega(f_1, p) + \ldots + \omega(f_k, p) - \omega(p),$$

$$b(p) = \omega(g_1, p) - \omega(f_1, p) + \ldots + \omega(g_k, p) - \omega(f_k, p)$$

and

$$c(p) = \Omega_1(p) - \omega(g_1, p) + \ldots + \Omega_k(p) - \omega(g_k, p),$$

where $g(y) = a^{n-1} f(x)$, $y = ax$, a is the leading coefficient of $f(x)$, n is the degree of $f(x)$, and $\omega(f, p)$ is the number of solutions of the congruence $f(x) \equiv 0 \bmod p$. Note that

$$\Omega(p) = \Omega_1(p) + \ldots + \Omega_k(p) = \omega(p) + a(p) + b(p) + c(p)$$

and that we always have $a(p) \geqslant 0$, $b(p) \geqslant -M$, and $c(p) \geqslant -M$ where $M = n_1 + n_2 + \ldots + n_k$ and n_i is the degree of $f_i(x)$. Let

$$Q = \{p \colon a^2(p) + b^2(p) + c^2(p) > 0\}$$

and let $d(F)$ denote the number of primes contained in the set Q.

The quantity $H_2(k, w, t(1))$ is equal to $H_1(w)(1-w)^{-1}$ if $k \geqslant 2$; $H_1(w)$ is the series we get if we replace a_m by $|a_m|$ in the equation

$$H(w) = \sum_{m=1}^{\infty} a_m m^{-w}.$$

In order to find a bound for $H_1(w)$ we begin by recalling that (see Lemma 4)

$$H(w) = \prod_p \left(1 + \frac{\omega(p)}{p^{w-1}(p - \omega(p))}\right)\left(1 - \frac{1}{p}\right)^{\Omega(p)} J_2(w).$$

Since $\Omega(p) = \omega(p) + a(p) + b(p) + c(p)$ we can say that

$$H(w) = \prod_p \left(1 + A'(p)\right) \prod_{p \in Q} \left(1 - \frac{1}{p^w}\right)^{a(p)+b(p)+c(p)} J_2(w),$$

where

$$A'(p) = \frac{-\omega^2(p)}{p^w(p - \omega(p))} + \sum_{j=2}^{\omega(p)+1} (-1)^j \left(\binom{\omega(p)}{j} - \frac{p\,\omega(p)}{p - \omega(p)}\binom{\omega(p)}{j-1}\right)\frac{1}{p^{jw}}.$$

Hence, since $a(p) + b(p) + c(p) \geqslant 2M$, it follows that

$$H_1(w) \leqslant \prod_p \left(1 + A(p)\right) \prod_{p \in Q} \left(1 - \frac{1}{p}\right)^{-2M} \prod_{i=1}^{k} \prod_{P' \in K_i} \left(1 + \frac{1}{N(P')^w}\right)$$

$$\leqslant \prod_p \left(1 + A(p)\right) \prod_{p \in Q} \left(1 + \frac{1}{p^w}\right)^{2M} \left(\zeta(2w)\right)^{2M} \left(\frac{\zeta(2w)}{\zeta(4w)}\right)^M$$

where

$$A(p) = \frac{\omega^2(p)}{(p - \omega(p))p^w} \sum_{j=2}^{\omega(p)+1} \left|\binom{\omega(p)}{j} - \frac{p\,\omega(p)}{p - \omega(p)}\binom{\omega(p)}{j-1}\right| \frac{1}{p^{jw}}$$

and $\zeta(s)$ is the zeta function of the rational number field. Since

$$\binom{\omega(p)}{j} \leqslant 2^{\omega(p)}, \quad \left(1 - \frac{1}{p^c}\right)^{-1} \leqslant 4, \quad \omega(p)\left(1 - \frac{\omega(p)}{p}\right)^{-1} \leqslant M(M+1),$$

and $p^{-w-1} < p^{-2w}$ for $w < 1$ it follows that

$$A(p) \leqslant \frac{M^2(M+1)}{p^{2w}} + \frac{2^{\omega(p)+2}}{p^{2w}}\left[M(M+1)+1\right] \leqslant \frac{2^{M+2}}{p^{2w}}(M+1)^3.$$

Thus

$$\prod_p \left(1 + A(p)\right) \leqslant \left(\zeta(2w)\right)^R$$

where $R = 2^{M+2}(M+1)^3$.

To continue, let q be the $(d(F))$ th prime, suppose that $q > e^3$, and let $w = 1 - (\log q)^{-1}$. Then

$$\prod_{p \in Q} \left(1 + \frac{1}{p^w}\right)^{2M} \leqslant \exp\left(2Me \sum_{p \leqslant q} \frac{1}{p}\right) \leqslant \exp\left[c_1 M \log\log\left(d(F) + 3\right)\right].$$

This bound can also be used if $q \leqslant e^3$; we need only set $w = \frac{2}{3}$ and adjust the constant in the exponential term.

At this point we have

$$\text{(15)} \qquad H_1(w) \leqslant \left(\zeta(\tfrac{4}{3})\right)^S \left(\zeta(\tfrac{8}{3})\right)^{-M} \exp\left[c_1 M \log\log\left(d(F) + 3\right)\right]$$

where $S = 2^{M+2}(M+1)^3 + 2M + M$ and w is the maximum of $\frac{2}{3}$ and $1 - (\log q)^{-1}$. If we use the inequalities $M \leqslant c_2 k$ and $(1-w)^{-1} = \log q \leqslant c_3 \log\left(d(F) + 3\right)$ we can conclude that

$$\text{(16)} \qquad H_1(w)(1-w)^{-1} \leqslant (\exp c_4^k)\left(\exp c_5 k \log\log\left(d(F) + 3\right)\right).$$

If $k=1$ the number $H_2(1, k, t(1)) = H_1(t(1))$ can be dealt with in the following way: If $t(1) \leqslant w$, where w is defined as in the previous paragraph, the exponent of x in the error term of the corresponding ideal theorem can be increased to w; we then get $H_1(w)(1-w)^{-1}$ in the error term of (13) and this can be treated as before. If $w < t(1)$ then, since $H_1(\sigma)$ increases as σ decreases $H_1(t(1))$ can be replaced by $H_1(w)$ and the previous estimate of $H_1(w)$ can be employed.

We now turn to the problem of finding a bound for $d(F)$, which was defined to be the number of primes p such that $a(p) \neq 0$ or $b(p) \neq 0$ or $c(p) \neq 0$. Suppose that $f(x)$ and $g(x)$ are polynomials with integral coefficients which are of degree m and n respectively. Let $R(f, g)$ denote their resultant and $S(f, g)$ denote the maximum of $A(f)$ and $A(g)$; $A(g)$ is the maximum of the set of absolute values of the coefficients of $g(x)$. Then it is known that:

$$|R(f, g)| \leqslant (n+m)!\,(S(f, g))^{n+m};$$

there are polynomials $c(x)$ and $d(x)$ with integral coefficients such that

$$c(x)f(x)+d(x)g(x) = R(f, g),$$

and if $f(x)$ and $g(x)$ are relatively prime polynomials then $R(f, g) \neq 0$. Furthermore if $g(x)$ is an irreducible polynomial whose leading coefficient is 1 and D is the absolute value of the discriminant of the field, generated by a zero of $g(x)$ then $|R(g, g')| = I(g)D$ where $I(g)$ is an integer.

Let us apply them. The number

$$a(p) = \omega(f_1, p)+\ldots+\omega(f_k, p)-\omega(p)$$

will be positive when there is a prime p, a pair of indices i and j with $1 \leqslant i < j \leqslant k$, and an integer u, where $0 \leqslant u < p$, such that

$$f_i(u) \equiv f_j(u) \equiv 0 \bmod p. \tag{17}$$

But if (17) holds then p divides $R(f_i, f_j)$. Thus if we let $v(d)$ denote the number of distinct prime divisors of d and $a(1)$ be the number of primes p such that $a(p) \neq 0$ it follows that

$$a(1) \leqslant \sum_{1\leqslant i<j\leqslant k} v[R(f_i, f_j)] \leqslant c_1^k \log eB(F).$$

As for the numbers $b(p)$, we have

$$b(p) = \omega(g_1, p)-\omega(f_1, p)+\ldots+\omega(g_k, p)-\omega(f_k, p)$$

where $g_i(y) = a_i f_i(x)$, $y = a_i x$, and a_i is the leading coefficient of $f_i(x)$. Since $\omega(f_i, p) = \omega(g_i, p)$ if $(a_i, p) = 1$, we have

$$b(1) \leqslant v(a_1)+\ldots+v(a_k) \leqslant (\log 2)^{-1}\log eB(F),$$

where $b(1)$ is the number of primes such that $b(p) \neq 0$.

Finally, since

$$c(p) = \Omega_1(p)-\omega(g_1, p)+\ldots+\Omega_k(p)-\omega(g_k, p)$$

and since $\Omega_i(p) = \omega_i(p)$ provided that $(p, I(g_i)) = 1$ ([7], p. 63) we have

$$c(1) \leqslant \sum_{i=1}^{k} v[R(f_i, f_i)] \leqslant c_2 k^3 \log eB(F),$$

where $c(1)$ is the number of primes p such that $c(p) \neq 0$.

If we combine these results with (16) we have for $k \geqslant 1$,

$$H_2(k, w, t(1)) \leqslant \exp[c_5^k \log_3 e^3 B(F)].$$

Now, returning to the error term of (13), we have

$$\eta(k, D) = \frac{r_1+\beta(D_1)}{1-t(1)} \cdots \frac{r_k+\beta(D_k)}{1-t(k)}.$$

According to Lemma 1,

$$(1-t(i))^{-1} = 10 n_i^2 \log 2D_i \quad \text{and} \quad \beta(D_i) \leqslant c_1(\log 2D_i)^{n_i+2};$$

Landau ([6], Lemma 1) has shown that $r_i \leqslant c_2(\log 2D_i)^{n_i-1}$. Thus it is possible to conclude that the quantity $c_1^k \eta(k, D) H_2(k, w, t(1))$ of formula (13) is bounded above by

$$\psi(c_1, k, D) = [\exp(c_1^k \log_3 e^3 B(F))][(\log 2D_1)\ldots(\log 2D_k)]^{c_2}.$$

This completes the proof of Lemma 5.

Lemma 6. *Let $e(p) = a(p)+b(p)+c(p)$. Then for $x \geqslant \frac{3}{2}$.*

$$\sum_{p\leqslant x} \frac{\omega(p)}{p} = k\log\log x + \log r_1 \ldots r_k - \sum_{p\leqslant x} \frac{e(p)}{p} + v(k) + \Delta(x)$$

where $v(k) \leqslant c_1 k$ and

$$\Delta(x) = O\left(\frac{\psi(c_1, k, D)}{r_1 \ldots r_k} \cdot \frac{1}{\log x}\right).$$

This follows directly from the corollary to Lemma 3 since $\omega(p) = \Omega_1(p)+\ldots+\Omega_k(p)-e(p)$. We assume, when computing the bound on $\Delta(x)$, that $r_i^{-1} \geqslant 2$, and use Landau's bound on r_i for $\frac{3}{2} \leqslant x \leqslant 2$.

LEMMA 7. *Let $H(1)$ be defined as in Lemma 4. Then we have*

$$\exp[-c_1k^2\log_3 e^3 B(F)] \leqslant H(1) \leqslant \exp[c_1k^2\log_3 e^3 B(F)].$$

We know, by definition, that $H(1)$ is equal to

$$\prod_p \left(1-\frac{\omega(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{\omega(p)} \prod_p \left(1-\frac{1}{p}\right)^{r(p)} \prod_{i=1}^{k} \prod_{P'\epsilon K_i} \left(1-\frac{1}{N(P')}\right)$$

where $r(p) = a(p)+b(p)+c(p)$. The first product of this expression is equal to

$$\exp\left[\sum_p \left(\sum_{m=2}^{\infty} \frac{1}{m}\left(\frac{\omega(p)}{p}\right)^m - \omega(p)\sum_{m=2}^{\infty}\frac{1}{mp^m}\right)\right].$$

But this quantity is bounded from below by 1 and from above by

$$\exp\left(\sum_{p\leqslant 2M} \frac{\omega^2(p)}{2p(p-\omega(p))} + M^2 \sum_{p>2M}\frac{1}{p^2}\right) \leqslant \exp(c_2M^2),$$

since

$$\frac{\omega^2(p)}{p(p-\omega(p))} = \omega(p)\frac{\omega(p)}{p(p-\omega(p))} \leqslant M.$$

An argument similar to the one used in the paragraph preceeding formula (15) can be employed to show that, since

$$-2M \leqslant r(p) = a(p)+b(p)+c(p) \leqslant 3M,$$

we have

$$\exp[-c_3k^2\log_3 e^3 B(F)] \leqslant \prod_p \left(1-\frac{1}{p}\right)^{r(p)} \leqslant \exp[c_3k^2\log_3 e^3 B(F)].$$

Finally, since P' is an ideal of the second or higher degree and since K_i contains at most n_i prime ideals whose norm is divisible by the prime p we have

$$\left(\frac{1}{\zeta(2)}\right)^M \leqslant \prod_{i=1}^{k}\prod_{P'\epsilon K_i}\left(1-\frac{1}{N(P')}\right) \leqslant \left(\frac{\zeta(2)}{\zeta(4)}\right)^M.$$

These bounds yield the conclusion of the lemma.

2. We are now in a position to apply Selberg's method. As in [8], we begin by defining a set of numbers $\{\varrho_d\}$ in terms of two other sets of numbers, $\{\gamma_a\}$ and $\{\lambda_d\}$, by means of the equation

$$\varrho_d = \sum_{[a,b,c]=d} \gamma_a\lambda_b\lambda_c,$$

where $[a, b, c]$ is the least common multiple of the integers a, b, and c. Now, for any integer m,

$$\sum_{d|m} \varrho_d = \Big(\sum_{a|m}\gamma_a\Big)\Big(\sum_{b|m}\lambda_b\Big)^2.$$

Thus if we set $\gamma_1 = T$, $\gamma_p = -T$ if $p \leqslant z^\varepsilon$, $\gamma_p = -1$ if $z^\varepsilon < p \leqslant z$, $\gamma_a = 0$ if $a > z$ or if a is composite integer, and $\lambda_d = 0$ if $d > z^\alpha$, where T, z, α, and ε are positive parameters that will be chosen later, we shall have:

(I) $$\varrho_d = 0 \quad \text{if} \quad d > z^{1+2\alpha},$$

(II) $$0 \leqslant \sum_{d|m}\varrho_d \leqslant \max_m \Big(\sum_{d|m}\gamma_a\Big)\Big(\sum_{b|m}\lambda_b\Big)^2 = M_1,$$

if m has no prime factors less than or equal to z^ε and no more than T prime factors greater than z^ε and less than or equal to z, and

(III) $$\sum_{d|m}\varrho_d \leqslant 0$$

in all other cases.

Let $\Phi(F, x)$ be the number of positive integers $m \leqslant x$ for which the polynomial value $F(m)$ has no prime factors less than or equal to z^ε and at most T prime factors greater than z^ε and less than or equal to z. Arguing as in [8] we have

(18) $$M_1\Phi(F, x) \geqslant x\sum_d \frac{\varrho_d}{f(d)} + O\Big[\sum_d |\varrho_d|\,\omega(d)\Big]$$

where $1/f(d) = \omega(d)/d$, and

(19) $$\sum_d \frac{\varrho_d}{f(d)} = \sum_{a\leqslant z}\frac{\gamma_a}{f(a)} \sideset{}{'}\sum_{\substack{r\leqslant z^\alpha \\ (r,a)=1}} f'(r)\Big(\sideset{}{'}\sum_{\sigma|a} f'(\sigma)y_{\sigma r}\Big)^2$$

where

$$\frac{1}{f'(r)} = \frac{1}{f(r)}\prod_{p|r}\left(1-\frac{1}{f(p)}\right)^{-1}, \quad y_d = \sum_{d|a}\frac{\lambda_a}{f(a)}, \quad \frac{\lambda_d}{f(d)} = \sum_r \mu(r)y_{rd},$$

and the prime on the summation symbol indicates that the summation is restricted to those integers b ($b = r$ or $b = \sigma$) for which $\omega(d) > 0$.

Set $y_d = 0$ if $d > z^\alpha$ and

$$y_d = \mu(d)[E_kf'(d)(\log z^\alpha)^k]^{-1}$$

if $1 \leqslant d \leqslant z^a$. Then applying Lemmas 5, 6, and 7 we have

(20)
$$\sum_d \frac{\varrho_d}{f(d)} \geqslant \frac{1}{E_k(\log z^a)^k}\left[\varphi(k, T, a)+E(z, a)+O\left(\frac{T\psi(c_1, k, D)}{r_1\ldots r_k}\cdot\frac{\log\log z}{\log z^a}\right)\right]$$

where $\varepsilon = \beta a$, a is subject to the condition $\log(1/a) < T$,

$$\varphi(k, T, a) = T-L(k, 1)-(T-1)L(k, \beta)-k\log(1/a),$$

$$L(k, \beta) = k\sum_{j=1}^{k}[1-(1-\beta)^j]/j \quad \text{for} \quad 0 < \beta \leqslant 1,$$

$$E(z, a) = -\sum_{p\leqslant z^a}\frac{\gamma_p e(p)}{p}\cdot\frac{(\log z^a)^k-(\log z^a/p)^k}{(\log z^a)^k}+\sum_{z^a<p\leqslant z}\frac{e(p)}{p};$$

it is assumed that z is large enough so that the quantity under the O-term does not exceed 1. Set

$$T = k\sum_{j=1}^{k}1/j+k\log(1/a)+\varphi$$

where $0 < \varphi \leqslant 1$. Then if we set $\beta = \varphi[100k^3\log(ke/a)]^{-1}$ we have $(T-1)L(k, \beta) \leqslant \varphi 10^{-2}$. Thus given φ and a it is possible to choose β so that $\varphi(k, T, a) \geqslant 99\varphi 10^{-2}$.

The error term of (18) can be estimated by a slight refinement of the argument given in section 5 of [8]. Doing so we get

(21)
$$\sum_d |\varrho_d|\,\omega(d) \leqslant c_1 B(k, \eta)z^\tau[1+\psi(c, k, D)(r_1\ldots r_k\log z^a)^{-1}]$$

where

$$\tau = (1+2a)(1+\eta),$$

$$B(k, \eta) = \log(ek/a)\exp[\exp c_2\eta^{-1}\log(c_3 k)],$$

and η is any positive number that is less than $\frac{1}{2}$.

Now, let $G(x)$ be the set of integers which are counted by $\Phi(F, x)$. As in [8], we wish to discard those values of m for which $F(m)$ is divisible by the square of a prime. To this end, let $G_0(x)$ be the set of integers m in $G(x)$ for which there is a prime p such that $z^\varepsilon < p \leqslant z$ and such that $p^2 | F(m)$. Let $G_1(x) = G(x)-G_0(x)$ and let $\Phi_1(F, x)$ and $\Phi_0(F, x)$ be, respectively, the number of elements in $G_1(x)$ and $G_0(x)$. We have $\Phi_1(F, x) = \Phi(F, x)-\Phi_0(F, x)$ and

$$\Phi_0(F, x) \leqslant x\sum_{z^\varepsilon<p\leqslant z}\frac{\omega(p^2)}{p^2}+\sum_{z^\varepsilon<p\leqslant z}\omega(p^2) \leqslant c_1 k\big(1+d'(F)\big)(z+xz^{-\varepsilon})$$

where $d'(F)$ is the number of primes p such that $\omega(p^2) \neq \omega(p)$.

If this last result is combined with (18), (20), and (21) we have

(22)
$$\Phi_1(F, x) \geqslant \frac{x}{M_1E_k(\log z^a)^k}\left[\frac{99}{100}\varphi+E(z, a)+O\left(\frac{T\psi(c_1, k, D)}{r_1\ldots r_k}\cdot\frac{\log\log z}{\log z^a}\right)\right]+$$
$$+O\left[\frac{B(k, \eta)}{M_1}z^\tau\left(1+\frac{\psi(c_1, k, D)}{r_1\ldots r_k\log z^a}\right)\right]+O\big[k\big(1+d'(F)\big)(z+xz^{-\varepsilon})\big].$$

We can now prove

THEOREM 4. *If δ is any given positive number and*

$$z \geqslant \max\left(\exp(D_1\ldots D_k)^\delta, \exp\left[\left(\frac{\log(e/a)}{\varphi a}\right)^4\exp[c^k(\delta, n)\log_3 e^3 B(F)]\right]\right)$$

then

(23)
$$\Phi_1(F, x)$$
$$\geqslant \frac{a\varphi}{r_1\ldots r_k H(1)}\cdot\frac{x}{(\log z)^k}+\theta\left[\left(\exp\exp\left(\frac{1}{\eta}\log c_1 k\right)\right)(z^\tau+z^{1+\gamma}+xz^{-\gamma})\right],$$

where a is an absolute constant, $c(\delta, n)$ is a constant that depends only on δ and n, and θ is a number such that $|\theta| \leqslant 1$. The numbers a, φ, γ, η, and τ are parameters that satisfy the conditions: $0 < a \leqslant 1$, $0 < \varphi \leqslant 1$, $\gamma = (\beta a)/2$, $\beta = \varphi[100\,k^3\log(ke/a)]^{-1}$, $0 < \eta \leqslant \frac{1}{2}$, and $\tau = (1+2a)(1+\eta)$.

We shall first prove that

(24)
$$\Phi_1(F, x) \geqslant a(F)\left(\frac{99}{100}\varphi+\theta_1 L(F, T)\right)\frac{x}{(\log z^a)^k}+\theta_2\left[\frac{c_1}{T}B(k, \eta)z^\tau\right]+$$
$$+\theta_3\big[c_1k^3\big(\log eB(F)\big)(z+xz^{-\varepsilon})\big]$$

where

$$a(F) = k![2T(\tfrac{5}{2})^T r_1\ldots r_k H(1)]^{-1},$$

$$T = L(k)+k\log(1/a)+\varphi,$$

$$L(F, T) = \frac{T}{a}(D_1\ldots D_k)^{\delta/2}\big[\exp\big(c^k(\delta, n)\log_3 e^3 B(F)\big)\big]\frac{\log_2 z}{\log z},$$

$$B(k, \eta) = \big(\log(ek/a)\big)\exp\exp(c_2\eta^{-1}\log c_3 k),$$

and θ_i is a number such that $|\theta_i| \leqslant 1$ for $i = 1, 2, 3$. Inequality (23) will then follow from (24).

The definitions following (20) and relation (22) furnish us a starting point. We have

$$E(z, a) = -\sum_{p\leqslant z^a}\gamma_p\frac{e(p)}{p}\cdot\frac{[(\log z^a)^k-(\log z^a/p)^k]}{(\log z^a)^k}+\sum_{z^a<p\leqslant z}\frac{e(p)}{p},$$

where $e(p) = a(p)+b(p)+c(p)$, $d(F)$ is the number of primes p for which at least one of the numbers $a(p)$, $b(p)$ or $c(p)$ is not zero and $d(F) \leqslant c_1^k \log eB(F)$. Thus the second sum on the right hand side of the above equation does not exceed $3Md(F)z^{-a}$. The first sum can be estimated by expanding $(\log z^a - \log p)^k$, reversing the order of summation of the resulting double sum, and then comparing the jth inner sum with a similar sum whose index of summation is the first $e^j + d(F)$ primes. If we do this we find that we are dealing with a quantity that is bounded above by

$$T(c_2 k)^k \big(\log_2 e^3 B(F)\big)^k (\log z^a)^{-1}.$$

Thus we have

$$|E(z, a)| \leqslant T(c_2 k)^k \frac{\big(\log_2 e^3 B(F)\big)^k}{\log z^a} + c_3 k c_1^k \frac{\big(\log e^3 B(F)\big)}{z^a}.$$

Since $z^a \geqslant \log e^3 B(F)$ it follows that

$$|E(z, a)| \leqslant T(c_4 k)^k \frac{\big(\log_2 e^3 B(F)\big)^k}{\log z^a}.$$

Let us go on to the term

$$T \frac{\psi(c_1, k, D)}{r_1 \ldots r_k} \cdot \frac{\log_2 z}{\log z^a}$$

of (22). According to a well-known result of Brauer's, [2], for any positive δ there is a constant $c(\delta, n)$ which depends only on δ and n such that

$$r_i^{-1} \leqslant c(\delta, n) D_i^{\delta/4}.$$

Consequently, by Lemma 5,

$$\left| \frac{\psi(c_1, k, D)}{r_1 \ldots r_k} b_1 \right| \leqslant \exp c_1^k(\delta, n) (D_1 \ldots D_k)^{\delta/2} \exp[c_1^k \log_3 e^3 B(F)]$$

where b_1 is the constant implied by the first O-term in (22). If this result is combined with the one in the previous paragraph we obtain the quantity $L(F, T)$ of (24).

As for M_1, we have

$$M_1 = \max_m \Big(\sum_{a|m} \gamma_a\Big) \Big(\sum_{d|m} \lambda_d\Big)^2 \leqslant \max_{m'} T \Big(\sum_{d|m'} |\lambda_d|\Big)^2,$$

where m' runs through the integers having at most T prime factors greater than z^ε and less than or equal to z and none less than z^ε. Since

$$\lambda_d = f(d) \sum_r \frac{\mu(r)\mu(rd)}{f'(rd)} \cdot \frac{1}{E_k (\log z^a)^k}$$

we have

$$|\lambda_d| \leqslant \frac{f(d)}{f'(d)} \left[1 + O\left(\frac{\psi(c_1, k, D)}{E_k \log z^a}\right)\right].$$

If $z^\varepsilon > 10M$ then, for any d that divides m',

$$\frac{f(d)}{f'(d)} = \prod_{p|d} \left(1 - \frac{1}{f(p)}\right)^{-1} \leqslant \left(\frac{5}{4}\right)^{T/2}$$

since we are only interested in those $d \leqslant z^a$ which have prime factors greater than z^ε. Since any m' has at most 2^T divisors it follows that

$$M_1 \leqslant T(\tfrac{5}{2})^T \left[1 + O\left(\frac{\psi(c_1, k, D)}{E_k \log z^a}\right)\right]^2. \tag{25}$$

Now, $E_k = r_1 \ldots r_k (k!)^{-1} H(1)$. Thus, by Lemma 7 and the results of the last paragraph,

$$\left| \frac{\psi(c_1, k, D) b_2}{E_k \log z^a} \right| \leqslant \left| \frac{\psi(c_2, k, D)}{r_1 \ldots r_k \log z^a} \right| \leqslant L(F, T),$$

where b_2 is the constant implied by the O-term in (25). If z is taken large enough so that $L(F, T) \leqslant \frac{1}{4}$ we then have

$$M_1 \leqslant 2T(\tfrac{5}{2})^T.$$

This bound gives us the number $a(F)$ of (24).

A lower bound on M_1 is needed in order to estimate the second error term of (22). To get one set $m = 1$ in the equation that defines M_1. Then $M_1 \geqslant T\lambda_1^2$ and, arguing as before, we have the second error term of (24).

The number $d'(F)$, the number of primes p such that $\omega(p^2) \neq \omega(p)$, does not exceed the number of primes p for which the system

$$F(x) \equiv 0 \bmod p,$$
$$F'(x) \equiv 0 \bmod p$$

is solvable. Since the irreducible factors of $F(x)$ are not constant multiples of each other $F(x)$ and $F'(x)$ are relatively prime; consequently if this system is solvable for p then p divides $R(F, F')$, the resultant of these two polynomials. Since the coefficient of $F(x)$ which has the largest absolute value does not exceed

$$(k-1)!\, n^{k-1} A(f_1) \ldots A(f_k)$$

we have

$$R(F, F') \leqslant (2M-1)! [M(k-1)!\, n^{k-1} A(f_1) \ldots A(f_k)]^{2M-1},$$

which implies that

$$d'(F) \leqslant c_1 k^3 \log eB(F).$$

This completes the proof of (24).

The lower bound for z given in Theorem 4 is a consequence of demanding that $L(F, T) \leqslant \varphi/100$. This will be the case if

$$\frac{T}{\alpha}\,[\exp c^k(\delta, n)\log_3 e^3 B(F)]\left(\frac{4}{e}\right)\frac{1}{(\log z)^{1/4}} \leqslant \frac{\varphi}{100}$$

and

$$(D_1 \ldots D_k)^{\delta/2}/(\log z)^{1/2} \leqslant 1.$$

The assumption of Theorem 4 then follows, for $T \leqslant 2k\log(ke/\alpha)$; the functions of k and the absolute constants that occur can be absorbed in the term, $\exp c^k(\delta, n)$.

As for the constant a, we have

$$\frac{a(F)}{a^k} = \frac{k!}{2\alpha^k T(\frac{5}{2})^T r_1 \ldots r_k H(1)}.$$

Since

$$T = k\sum_{j=1}^{k}(1/j) + k\log(1/\alpha) + \varphi$$

it is not difficult to show that the function

$$f(\alpha, k) = k!\,[2\alpha^k T(\tfrac{5}{2})^T]^{-1}$$

has a minimum for $0 < \alpha \leqslant 1$ and $k \geqslant 1$. Set α equal to this minimum.

We also have

$$\frac{c_1}{T}B(k, \eta) = \frac{c_1}{T}\log(ek/\alpha)\exp\left[\exp\frac{c_2}{\eta}\log c_3 k\right] \leqslant \exp\exp\left[\frac{c_4}{\eta}\log c_3 k\right],$$

since $\log(ek/\alpha)T^{-1} \leqslant 2$.

The only other condition that must be met if (23) is to hold is the inequality

$$c_1 k^3 \log eB(F) z^{-\gamma} \leqslant 1,$$

where $\gamma = \varepsilon/2$, and this is the case if $c(\delta, n)$ is sufficiently large. This completes the proof of Theorem 4.

To obtain Theorem 1 from Theorem 4 set $\alpha = (2n+1)^{-1}$, $\eta = (12n^2)^{-1}$, $\varphi = 1$, and $z = x^R$ where

$$R = \left(1 - \frac{2}{2n+3}\right)\left(1 - \frac{1}{6n^2+1}\right)$$

and n is the maximum of $n_1, n_2, \ldots$, and n_k. Since

$$D_i \leqslant |R(f_i, f_i')| \leqslant (2n_i-1)!\,|n_i A(f_i)|^{2n_i-1}$$

we have

$$D_1 \ldots D_k \leqslant c_1^k\big(A(f_1) \ldots A(f_k)\big)^{2n}.$$

Thus, if we replace δ by $\delta/2n$, the first part of the assumption of Theorem 4 will take the form

(26) $$x \geqslant \exp[c_1^k(\delta, n)B(F)^\delta].$$

The second will hold if

(27) $$x \geqslant \exp\exp[c_2^k(\delta, n)\log_3 e^3 B(F)].$$

As for inequality (23), we have $z^\tau = x^S$, $z^{1+\gamma} \leqslant x^U$, and $xz^{-\gamma} \leqslant x^V$ where $S = 1-(12n^2+2)^{-1}$, $U = (2n+2)(6n^2)[(2n+3)(6n^2+1)]^{-1}$, $V = 1 - c_1 k^{-4}$, and c_1 is a positive number that is less than one-half. Since

$$r_i \leqslant c_2(\log 2D_i)^{n_i-1} \leqslant c_3\big(\log eA(f_i)\big)^n$$

and

$$H(1) \leqslant \exp[c_4 k^2 \log_3 e^3 B(F)],$$

$\Phi_1(F, x)$ will be positive if (26) and (27) hold and if

(28) $$x > \big[\big(\log eA(f_1)\big) \ldots \big(\log eA(f_k)\big)\big]^a\big[\exp\big(c_7 k^6 \log_3 e^3 B(F)\big)\big](\exp\exp c_5 k)$$

where $a = c_6 k^4$. Inequalities (26), (27), and (28) can be replaced by the single inequality

(29) $$x > \exp\big[c_3(k, \delta, n)\big(B(F)\big)^\delta\big].$$

If we assume, as in Theorem 2, that $D_i \leqslant (\log x)^E$ and set $\delta = (2Ek)^{-1}$ the analogues to (26), (27), and (28) can be replaced by the single inequality

(30) $$x > \exp\exp[c_1(k, E, n)\log_3 e^3 B(F)]$$

where $c_1(k, E, n)$ is a constant that depends on k, E, and n.

Suppose that (29) holds. Let $f_i(x)$ be any one of the irreducible factors of $F(x)$, let m be one of the integers counted by $\Phi_1(F, x)$, and suppose that $f_i(m)$ has n_i+1 or more prime factors greater than or equal to $z = x^R$. We then have

$$x^a \leqslant |f_i(m)| \leqslant n_i A(f_i) x^b$$

where $a = R(n_i+1)$ and $b = n_i$. Since $a - b = R(n_i+1) - n_i \geqslant (35n)^{-1}$ we have a contradiction as soon as $x > \big(n_i A(f_i)\big)^{35n}$. This completes the proof of Theorem 1, for if $c_1(\delta, k, n)$ is a sufficiently large constant then

$$\exp\big[c_1(\delta, k, n)\big(B(F)\big)^\delta\big] > \big(nB(F)\big)^{35n} \geqslant \big(n_i A(f_i)\big)^{35n}.$$

If we assume that (30) holds, a similar argument will yield Theorem 2.

The results for the linear case can be obtained by setting $\alpha = 1/2.71$, $\eta = \frac{1}{2}\cdot\frac{49}{65831}$, $\varphi = 10^{-2}z = x^R$ and $R = (1+2\alpha)^{-1}(1+2\eta)^{-1}$, since we then have $z = x^{161/280}$ and $\log 1/\alpha \leqslant \cdot 97$. A weak form of the Goldbach conjecture can also be obtained from Theorem 4. For if $F(y) = y(2N-y)$ then, by (30), $\Phi_1(F, N)$ is positive if N is sufficiently large. If m is one of the integers counted by $\Phi_1(F, N)$ and either m or $2N-m$ have two or more prime factors greater than or equal to $N^{161/280}$ we have $N^{332/280} \leqslant m \leqslant N$ or $N^{332/280} \leqslant 2N-m \leqslant 2N$ both of which are impossible for $N \geqslant 2^{6+\frac{2}{3}}$. Thus there is an integer a such that $2N = a+m$ and such that am has at most six prime factors.

Theorem 3 is a consequence of

THEOREM 5. *Let $F(x)$ be a polynomial with integral coefficients and suppose that $\omega(p) < p$ for every prime p, where $\omega(p)$ is the number of solutions of the congruence $F(x) \equiv 0 \bmod p$. Let $\Phi_2(F, x)$ denote the number of positive integers $m \leqslant x$ such that all the prime factors of $F(m)$ are greater than or equal to z, where $z \geqslant 2$. Then for $x \geqslant 2$*

$$\Phi_2(F, x) \geqslant b_1 x \prod_{p \leqslant z}\left(1-\frac{\omega(p)}{p}\right) + \theta[c(n)z^a],$$

where b_1 is an absolute constant, $|\theta| \leqslant 1$, $a = (26n)/4$, and n is the degree of $F(x)$.

This can be proved with the aid of several of the results in [9], the pertinent parts of that paper being Lemmas 1.2, 1.4, 1.5, and 1.6. Lemma 1.4 has to be modified slightly, the k that appears there must be changed to n. If this is done then the conclusion of the lemma will follow if one employs the estimate $\omega(p) \leqslant n$ and assumes that $y > e^{bn}$, where b is an absolute constant. As for the parameters h and h_0 of [9], set $h = \exp(2.5n)^{-1}$ and $h_0 = \exp(2.49n)^{-1}$.

To obtain Theorem 3 make use of the fact that

$$\prod_{p \leqslant z}\left(1-\frac{\omega(p)}{p}\right) > \frac{e^{-bn^2}}{(\log z)^n}$$

where b is an absolute constant, set $z = x^c$ where $c = 4(27n\lambda)^{-1}$, and then proceed as before.

References

[1] P. T. Bateman and R. A. Horn, *Primes represented by polynomials in one variable*, Procedings of Symposia in Pure Mathematics, 8 (1965), pp. 119-132, Amer. Math. Soc., Providence, R. I., 1965.

[2] R. Brauer, *On the zeta-functions of algebraic number fields*, Amer. J. Math. 69 (1947), pp. 243-250.

[3] W. Fluch, *Verwendung der Zeta-Funktion beim Sieb von Selberg*, Acta Arith. 5 (1959), pp. 381-405.

[4] E. Fogels, *On the zeros of Hecke's L-functions*, Acta Arith. 7 (1962), pp. 87-106.

[5] E. Landau, *Über die zu einem algebraischen Zahlkörper gehörige Zeta Function*, J. Reine Angew. Math. 125 (1903), pp. 64-188.

[6] — *Verallgemeinerung eines Polyaschen Satzes auf algebraische Zahlkörper*, Gött. Nachr. (1918), pp. 478-488.

[7] H. B. Mann, *Introduction to algebraic number theory*, Columbus, Ohio, 1955.

[8] R. J. Miech, *Almost primes generated by a polynomial*, Acta Arith. 10 (1964), pp. 10-30.

[9] — *Primes, polynomials, and almost primes*, Acta Arith. 11 (1965), pp. 35-56.

[10] Cheng-Tung Pan, Science Abstracts of China no. 1 (1958), pp. 7-8.

[11] K. Prachar, *Primzahlverteilung*, Berlin-Göttingen-Heidelberg 1957.

[12] A. Schinzel, *A remark on a paper of Bateman and Horn*, Math. Comp. 17 (1963), pp. 445-447.

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Reçu par la Rédaction le 22.4. 1965