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A summation formula and an identity for a class of Dirichlet series

by

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1. Introduction. The sum $\sum\limits_{x<n\leqslant y} a(n)f(n)$, where $a(n)$ is a sequence of complex numbers and f is a complex-valued function defined on $(0,\infty)$, can sometimes be expressed by a formula involving the series $\sum\limits_{n=1}^{\infty} a(n)f(n)$. The Poisson and Voronoï summation formulae are of this type, and many other such formulae have been obtained since Voronoï raised the problem in 1904 ([4]). In 1956 A. Sklar ([2]) derived a general formula from which many important examples can be obtained, including the two mentioned above, the Hardy-Landau formula for $\sum\limits_{x<n\leqslant y} r_2(n)f(n)$, and the formula, due to Sklar, for $\sum\limits_{x<n\leqslant y} \tau(n)f(n)$ (where $\tau(n)$ is Ramanujan's τ-function). In this paper we generalize the work of Sklar ([2], [3]) and present a simple and straightforward proof of a summation formula applicable to a large class of generalized Dirichlet series. We then show that this leads directly to an identity for the functions which are associated with the series.

2. Preliminaries. The letter s will denote a complex variable with real part σ and imaginary part t. A summation sign with the index of summation omitted will always precede an expression containing n. Accordingly, a bare summation sign will indicate summation over all positive integral values of n, while the symbol $\sum^{y}$, for $y>0$, will denote summation over all n such that $\lambda_n \epsilon (0, y\rangle$, $\sum_y$ will denote summation over all n such that $\lambda_n \epsilon (y,\infty)$, and $\sum_x^y$ will mean summation over all n for which $\lambda_n \epsilon (x, y\rangle$. We will use $\int\limits_{(a)}$, for real a, to denote the integral $\int\limits_{a-i\infty}^{a+i\infty}$.

The series $\sum a(n)\lambda_n^{-s}$ which we will consider are those with the following properties:

(i) the coefficients $a(n)$ are complex numbers;

(ii) the sequence λ_n is positive, strictly increasing, and unbounded;

(iii) the series $\sum a(n)\lambda_n^{-s}$ has a non-negative abscissa a of absolute convergence;

(iv) the analytic function $\varphi(s)$ defined by the series for $\sigma > a$ can be continued outside this half-plane and is meromorphic in the entire s-plane. Further, $\varphi(s)$ is of finite order in every vertical strip of finite width;

(v) there exists a Dirichlet series $\sum b(n)\mu_n^{-s}$ which is absolutely convergent for $\sigma > b \geqslant 0$ and thus defines an analytic function $\varphi^*(s)$ in the half-plane $\sigma > b$;

(vi) there exists a real number k such that the function

$$H(s) = \varphi(k-s)/\sum b(n)\mu_n^{-s}$$

can be continued outside the half-plane $\sigma > b$ and is meromorphic in the entire s-plane;

(vii) there exists a number $\alpha > 0$ such that $H(s) = O(|t|^{\alpha(2\sigma-k)})$ for $|t|$ sufficiently large, in any vertical strip of finite width.

Note that $\varphi(s)$ has no representation in the vertical strip $k-b \leqslant \sigma \leqslant a$, called the *critical strip* of $\varphi(s)$. However, the range of validity of the identity which we will derive in section 3 can be extended to cover this region.

For q complex and $v > 0$, let $R_q(v)$ be the sum of the residues of

$$\frac{v^{s+q-1}\Gamma(s)}{\Gamma(s+q)}\varphi(s)$$

in the critical strip of $\varphi(s)$. Then, for fixed v, $R_q(v)$ is an entire function of q (and, for fixed q, an analytic function of v).

For q complex, $\mathrm{Re}\,q$ sufficiently large, and $v > 0$, let

$$L_q(v) = \frac{1}{2\pi i}\int_{(b+\delta)} v^{k-1-s+q}\frac{\Gamma(k-s)}{\Gamma(k-s+q)}H(s)\,ds.$$

The number δ is chosen so that no singularities of the integrand lie in the strip $b < \sigma \leqslant b+\delta$. For v in a finite interval and $\mathrm{Re}\,q > \alpha(2b-k)+1$, the integral is absolutely and uniformly convergent with respect to v. $L_q(v)$ can be extended over the whole q-plane by analytic continuation; therefore it is an entire function of q for fixed v (and an analytic function of v for fixed q).

An identity relating these functions is given in the following lemma:

LEMMA. *For $v > 0$, $\mathrm{Re}\,q$ sufficiently large,*

$$\frac{1}{\Gamma(q+1)}\sum^{v} a(n)(v-\lambda_n)^q = R_{q+1}(v) + \sum b(n)\mu_n^{-q-k}L_{q+1}(\mu_n v). \tag{1}$$

In particular, if $\mathrm{Re}\,q > \alpha(2b-k)$ the series on the right is absolutely and uniformly convergent with respect to v, for v in a finite interval.

Proof. By the theorem of residues,

$$R_{q+1}(v) = \frac{1}{2\pi i}\left(\int_{(a+\varepsilon)} - \int_{(k-b-\delta)}\right) v^{s+q}\frac{\Gamma(s)}{\Gamma(s+q+1)}\varphi(s)\,ds$$

where $\varepsilon > 0$, and $\delta > 0$ is chosen so that no singularities of the integrand lie in the strip $k-b-\delta \leqslant \sigma \leqslant k-b$. For v in a finite interval and $\mathrm{Re}\,q > \alpha(2b-k)$, the integrals are absolutely and uniformly convergent with respect to v. Thus

$$R_{q+1}(v) + \frac{1}{2\pi i}\int_{(k-b-\delta)} v^{s+q}\frac{\Gamma(s)}{\Gamma(s+q+1)}\varphi(s)\,ds = \frac{1}{2\pi i}\int_{(a+\varepsilon)} v^{s+q}\frac{\Gamma(s)}{\Gamma(s+q+1)}\left(\sum a(n)\lambda_n^{-s}\right)ds.$$

We can interchange the order of integration and summation on the right and, using a well-known formula (see Erdélyi et al., *Tables of Integral Transforms*, vol. 1, formula 7.3.20) we obtain

$$\frac{1}{\Gamma(q+1)}\sum^{v} a(n)(v-\lambda_n)^q$$

on the right hand side. The integral on the left can be written as

$$\frac{1}{2\pi i}\int_{(k-b-\delta)} v^{s+q}\frac{\Gamma(s)}{\Gamma(s+q+1)}\left(\sum b(n)\mu_n^{s-k}\right)H(k-s)\,ds.$$

Absolute convergence again permits an interchange of integration and summation; and replacing s by $k-s$ in the integral above we get

$$\sum b(n)\mu_n^{-k-q}\left(\frac{1}{2\pi i}\int_{(b+\delta)} (\mu_n v)^{k-s+q}\frac{\Gamma(k-s)}{\Gamma(k-s+q+1)}H(s)\,ds\right)$$

which is

$$\sum b(n)\mu_n^{-k-q}L_{q+1}(\mu_n v).$$

This establishes (1) for $\mathrm{Re}\,q > \alpha(2b-k)$, and it can be continued into any domain which overlaps this half plane and within which the series on the right is uniformly convergent.

3. The summation formula. We now proceed to derive our principal results.

THEOREM. *Let* (1) *hold for* $q = m > a(2b-k)$, *where* m *is a positive integer. Then for* f *such that* $f^{(m)}$ *is absolutely continuous on* $\langle x, y\rangle$,

$$\sum_x^y a(n)f(\lambda_n) \tag{2}$$

$$= \sum_{j=0}^{m} \frac{(-1)^j}{j!}\Big[f^{(j)}(y)\sum^y a(n)(y-\lambda_n)^j - f^{(j)}(x)\sum^x a(n)(x-\lambda_n)^j\Big] +$$

$$+(-1)^{m+1}\int_x^y R_{m+1}(v)f^{(m+1)}(v)\,dv +$$

$$+(-1)^{m+1}\int_x^y \Big[\sum b(n)\mu_n^{-m-k}L_{m+1}(\mu_n v)\Big] f^{(m+1)}(v)\,dv.$$

Proof. Let $x \leqslant u \leqslant y$. Expanding $f(u)$ in a Taylor series about y with integral remainder for the $(m+1)^{st}$ term, we obtain two expressions for the remainder $f_{my}(u)$:

$$f_{my}(u) = \frac{(-1)^{m+1}}{m!}\int_u^y f^{(m+1)}(v)(v-u)^m\,dv; \tag{a}$$

$$f_{my}(u) = f(u) - \sum_{r=0}^{m}\frac{(-1)^r}{r!}f^{(r)}(y)(y-u)^r. \tag{b}$$

Then, for $x \leqslant \lambda_n \leqslant y$,

$$\sum_x^y a(n)f_{my}(\lambda_n) = \frac{(-1)^m}{m!}\sum^x a(n)\int_x^y (v-\lambda_n)^m f^{(m+1)}(v)\,dv + \tag{a$'$}$$

$$+(-1)^{m+1}\int_x^y \Big\{\frac{1}{m!}\sum^v a(n)(v-\lambda_n)^m\Big\} f^{(m+1)}(v)\,dv = I_1 + I_2;$$

$$\sum_x^y a(n)f_{my}(\lambda_n) = \sum_x^y a(n)f(\lambda_n) - \tag{b$'$}$$

$$-\sum_{r=0}^{m}\frac{(-1)^r}{r!}f^{(r)}(y)\sum^y a(n)(y-\lambda_n)^r + \sum_{r=0}^{m}\frac{(-1)^r}{r!}f^{(r)}(y)\sum^x a(n)(y-\lambda_n)^r.$$

To find I_1, we integrate by parts and get

$$\sum_{j=0}^{m}\frac{(-1)^j}{j!}\Big\{f^{(j)}(y)\sum^x a(n)(y-\lambda_n)^j - f^{(j)}(x)\sum^x a(n)(x-\lambda_n)^j\Big\}.$$

From (a$'$), (b$'$) and the above expression for I_1 we get

$$\sum_x^y a(n)f(\lambda_n)$$

$$= I_2 + \sum_{j=0}^{m}\frac{(-1)^j}{j!}\Big[f^{(j)}(y)\sum^y a(n)(y-\lambda_n)^j - f^{(j)}(x)\sum^x a(n)(x-\lambda_n)^j\Big].$$

By Lemma 1,

$$I_2 = (-1)^{m+1}\int_x^y \Big[R_{m+1}(v) + \sum b(n)\mu_n^{-m-k}L_{m+1}(\mu_n v)\Big] f^{(m+1)}(v)\,dv,$$

and this completes the proof of the theorem.

4. The identity. In order to obtain an identity for $\varphi(s)$, we first let $f(u) = u^{-s}$ in equation (2). Then

$$f^{(r)}(u) = (-1)^r\frac{\Gamma(s+r)}{\Gamma(s)}u^{-s-r},$$

and since (1) holds for $q = m > a(2b-k)$, whence summation and integration may be interchanged, we have

$$\sum_x^y a(n)\lambda_n^{-s} \tag{3}$$

$$= \sum_{j=0}^{m}\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\Big[y^{-s-j}\sum^y a(n)(y-\lambda_n)^j - x^{-s-j}\sum^x a(n)(x-\lambda_n)^j\Big] +$$

$$+\frac{\Gamma(s+m+1)}{\Gamma(s)}\int_x^y R_{m+1}(v)v^{-s-m-1}\,dv +$$

$$+\frac{\Gamma(s+m+1)}{\Gamma(s)}\sum b(n)\mu_n^{-m-k}\int_x^y L_{m+1}(\mu_n v)v^{-s-m-1}\,dv.$$

Next, we add $\sum^x a(n)\lambda_n^{-s}$ to both sides in (3) and then let y increase to $+\infty$. For $\sigma > a+\varepsilon$, $\varepsilon > 0$,

$$\lim_{y\to\infty}\sum_{j=0}^{m}\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\Big[y^{-s-j}\sum^y a(n)(y-\lambda_n)^j\Big]$$

$$= \sum_{j=0}^{m}\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\lim_{y\to\infty}y^{-\varepsilon}\sum^y a(n)\lambda_n^{-s+\varepsilon}(\lambda_n/y)^{-s-\varepsilon}(1-\lambda_n/y)^j,$$

which equals 0 since the last two factors are bounded, the series converges, and $y^{-\varepsilon} \to 0$ as $y \to \infty$. Since $R_{m+1}(v)$ is a finite sum of residues with the greatest power of v occuring at $\sigma = a$, $R_{m+1}(v) = O(v^{a+m})$. Thus

$$\int_x^\infty R_{m+1}(v) v^{-s-m-1} dv$$

converges for $\sigma > a$, since it is bounded by

$$\int_x^\infty v^{-\sigma-m-1} v^{a+m} dv.$$

Finally,

$$L_{m+1}(\mu_n v) = O(\mu_n^{k-(b+\delta)+m} v^{k-(b+\delta)+m},$$

so that $\int_x^\infty L_{m+1}(\mu_n v) v^{-s-m-1} dv$ is bounded by $\int_x^\infty v^{-\sigma+k-(b+\delta)-1} dv$ and hence converges for $\sigma > k-(b+\delta)$. Thus we have proved the following identity.

THEOREM. *For $\sigma > a$, $m > a(2b-k)$, $v > 0$, $0 < x < y$,*

$$\varphi(s) = \sum^x a(n)\lambda_n^{-s} - \tag{4}$$

$$- \sum_{j=0}^{m} \frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)} x^{-s-j} \sum^x a(n)(x-\lambda_n)^j +$$

$$+ \frac{\Gamma(s+m+1)}{\Gamma(s)} \int_x^\infty R_{m+1}(v) v^{-s-m-1} dv +$$

$$+ \frac{\Gamma(s+m+1)}{\Gamma(s)} \sum b(n)\mu_n^{-m-k} \int_x^\infty v^{-s-m-1} L_{m+1}(\mu_n v) dv.$$

By taking $m > a(2b-k)+a+b-k+\varepsilon$, where $\varepsilon > 0$, the range of validity of (4) is extended to include the critical strip $k-b \leqslant \sigma \leqslant a$ of $\varphi(s)$.

We remark that, by estimating the terms in an identity which is equivalent to (4), Chandrasekharan and Narasimhan ([1]) obtained an approximate functional equation for a similar class of functions.

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