# $m$-Semigroups, semigroups, and function representations

by

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An *$m$-semigroup* is an algebraic structure $\mathfrak{A} = \langle A, (\ )\rangle$ such that $A$ is a nonempty set, $(\ )$ is an $m$-ary operation on $A$, and the following associative law holds:

$$(1)\qquad \big((x_0 \ldots x_{m-1})\, x_m \ldots x_{2m-2}\big) = \big((x_0 \ldots x_i (x_{i+1} \ldots x_{i+m})\, x_{i+m+1} \ldots x_{2m-2}\big)\,,$$

for all $i < m-1$ and all $x_0, \ldots, x_{2m-2} \in A$. Thus a 2-semigroup is just an ordinary semigroup. In the articles Sioson [5], [6], [7] and Gluskin [2] various results known for semigroups were generalized to the theory of $m$-semigroups. In this paper we are concerned with the relationship of $m$-semigroups with semigroups, and with the problem of representing an $m$-semigroup as an $m$-semigroup of functions. The first question has previously been considered by Banach, Łoś [3], and Gluskin [2].

For $n \in \{0, 1, 2, \ldots\}$ an $n$-termed sequence is denoted by $a = \langle a_0 \ldots a_{n-1}\rangle$; we write $l(a) = n$ for the *length* of $a$. The empty set, 0, is also the empty sequence, the unique 0-termed sequence. As usual, we identify $a$ with $\langle a\rangle$. The *concatenation* or *juxtaposition* of two sequences $a = \langle a_0 \ldots a_{n-1}\rangle$ and $b = \langle b_0 \ldots b_{p-1}\rangle$ is the sequence

$$a^\frown b = ab = \langle a_0 \ldots a_{n-1} b_0 \ldots b_{p-1}\rangle\,.$$

We will assume throughout that $m > 2$. In any $m$-semigroup $\mathfrak{A} = \langle A, (\ )\rangle$ the operation $(\ )$ has a natural extension, still denoted by $(\ )$, to the set of all sequences of any length $k(m-1)+1$, with $k > 0$. Namely, by recursion with $k > 1$,

$$(2)\qquad (x_0 \ldots x_{k(m-1)}) = \big((x_0 \ldots x_{m-1})\, x_m \ldots x_{k(m-1)}\big)\,,$$

for all $x_0, \ldots, x_{k(m-1)} \in A$. The following general associative law then holds (for the proof, see Bruck [1], p. 38):

THEOREM A. *If $\mathfrak{A}$ is $m$-semigroup, $a, b, c$ are finite sequences of elements of $A$, $l(abc) = k(m-1)+1$ and $l(b) = h(m-1)+1$ for some $k, h > 0$, then $(ab) = \big(a(b)c\big)$.*

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**1. Embedding an $m$-semigroup in a semigroup.** Given a semigroup $\mathfrak{A} = \langle A, \cdot \rangle$ we introduce an $m$-ary operation ( ) on $A$ by defining

$$(x_0, \ldots, x_{m-1}) = x_0 \cdot \ldots \cdot x_{m-1}, \tag{3}$$

for all $x_0, \ldots, x_{m-1} \in A$. Clearly $\mathfrak{B} = \langle A, (\ ) \rangle$ is then an $m$-semigroup; we say that $\mathfrak{B}$ is the *$m$-semigroup reduct* of $\mathfrak{A}$. In the next section we will give an example of an $m$-semigroup which is not the reduct of any semigroup (for each $m > 2$); such an example was first given by Banach for $m = 3$ (unpublished; see Łoś [1]); the general case is implicit in Post [4], p. 230. If $\mathfrak{C}$ is an $m$-semigroup which is a subalgebra of $\mathfrak{B}$, then we say that $\mathfrak{C}$ is a *subreduct* of $\mathfrak{A}$. Łoś [3] proved that every 3-semigroup is a subreduct of a semigroup, and Gluskin [2] stated (without proof) that every $m$-semigroup is a subreduct of a semigroup, for any $m > 2$ [1]. In this section we will prove two theorems which constitute improvements of Gluskin's theorem. The method of proof applied will be useful in section 2.

THEOREM 1. *Any $m$-semigroup $\mathfrak{A}$ is a subreduct of a semigroup $\mathfrak{B}$ such that $A$ generates $\mathfrak{B}$ and if $\mathfrak{A}$ is a subreduct of a semigroup $\mathfrak{C}$ then there is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$ which is the identity on $A$.*

Proof. Let $B$ be the set of all finite sequences of elements of $A$; under concatenation, $B$ forms a semigroup $\mathfrak{B} = \langle B, \hat{\ } \rangle$ (the free semigroup with identity on $A$). Let $R$ be the relation between elements of $B$ such that $aRb$ iff there exist $c, d, e \in B$ such that $l(d) = m$, $a = cde$, and $b = c(d)e$, for all $a, b \in B$. Let $S$ be the smallest equivalence relation with field $B$ which includes $R$. Thus

(4) *$aSb$ iff there is a finite sequence $a = c_0, \ldots, c_{p-1} = b$ of elements of $B$ such that $c_i R c_{i+1}$ or $c_{i+1} R c_i$ for each $i < p-1$,*

for all $a, b \in R$ (we may have $p = 1$, so that $a = b$). If $a, b, c \in B$ and $aRb$ it is clear that $[ac]R[bc]$ and $[ca]R[cb]$. Hence by (4) we infer that $S$ is a congruence relation on $\mathfrak{B}$. The semigroup $(\mathfrak{B}/S)^-$ will essentially play the role of the semigroup $\mathfrak{B}$ of the statement of the theorem, where $(\mathfrak{B}/S)^-$ is $\mathfrak{B}/S$ with the identity $0/S$ removed. A similar notation $\mathfrak{E}^-$ is used for any semigroup $\mathfrak{E}$ with an external (i.e. prime) identity.

Note that if $a, b \in B$ and $aRb$, then $l(a) = l(b) + m - 1$. Hence from (4),

(5) *if $a, b \in B$ and $aSb$ then $l(a) \equiv l(b) \pmod{m-1}$.*

The following statement will be found useful in the proof of the theorem:

(6) *if $x \in A$, $b \in B$, and $xSb$, then either $b = x$, or else $l(b)$ has the form $k(m-1)+1$ for some $k > 0$, and $(b) = x$.*

[1] This result was obtained independently, but definitely later, by the present authors.



To prove (6), by (4) let $x = c_0, \ldots, c_{p-1} = b$ be a finite sequence of elements of $B$ such that $c_i R c_{i+1}$ or $c_{i+1} R c_i$ for each $i < p-1$. By induction on $i$ we prove the conclusion of (6), with $b$ replaced by $c_i$, for each $i \leqslant p-1$. The case $i = 0$ is trivial. We assume, inductively, that $0 < i \leqslant p-1$. By the inductive assumption (on $i-1$), we have two possibilities. First, we may have $c_{i-1} = x$. Then clearly $c_i R c_{i-1} = x$, and hence $l(c_i) = m$ and $(c_i) = x$, as desired. Second, we may have $c_{i-1}$ of length $k(m-1)+1$ for some $k > 0$, and $(c_{i-1}) = x$. We know that $c_{i-1} R c_i$ or $c_i R c_{i-1}$. Both cases are treated similarly, so we treat only the case $c_{i-1} R c_i$. Thus there are elements $d, e, f \in B$ such that $l(e) = m$, $c_{i-1} = def$, and $c_i = d(e)f$. Hence, using Theorem $A$,

$$(c_i) = \big(d(e)f\big) = (def) = (c_{i-1}) = x,$$

as desired. Thus (6) holds.

Now for each $x \in A$ let $f(x) = x/S$, the $S$-equivalence class of $x$. From (6) it follows that $f$ is 1-1. If $x_0, \ldots, x_{m-1} \in A$, then

$$\begin{aligned} f\big((x_0 \ldots x_{m-1})\big) = (x_0 \ldots x_{m-1})/S &= \langle x_0 \ldots x_{m-1} \rangle/S \\ &= x_0/S \cdot \ldots \cdot x_{m-1}/S = f(x_0) \cdot \ldots \cdot f(x_{m-1}). \end{aligned}$$

Thus $\mathfrak{A}$ is isomorphic under $f$ to a subreduct $\mathfrak{C}$ of $\mathfrak{B}/S$; it is also clear that $\mathfrak{C}$ generates $(\mathfrak{B}/S)^-$. Now suppose that $\mathfrak{C}$ is a subreduct of a semigroup $\mathfrak{D}$. Define $g$ from $\mathfrak{B}^-$ into $\mathfrak{D}$ by:

$$g(\langle x_0 \ldots x_{n-1} \rangle) = f(x_0) \cdot \ldots \cdot f(x_{n-1}),$$

for all $x_0, \ldots, x_{n-1} \in A$, $n > 0$. Clearly $g$ is a homomorphism of $\mathfrak{B}^-$ into $\mathfrak{D}$. Now suppose that $aRb$; choose, accordingly, $c, d, e \in B$ such that $l(d) = m$, $a = cde$, and $b = c(d)e$. Then

$$\begin{aligned} g(a) &= g(c) \cdot g(d) \cdot g(e) \\ &= g(c) \cdot f(d_0) \cdot \ldots \cdot f(d_{m-1}) \cdot g(e) \\ &= g(c) \cdot f\big((d_0 \ldots d_{m-1})\big) \cdot g(e) \\ &= g(c) \cdot f\big((d)\big) \cdot g(e) \\ &= g(b). \end{aligned}$$

It follows that if $aSb$ then $g(a) = g(b)$, for all $a, b \in B^-$. Hence there is a homomorphism $h$ of $(\mathfrak{B}/S)^-$ into $\mathfrak{D}$ such that $h(a/S) = g(a)$ for all $a \in B^-$. If $x \in A$, then $h\big(f(x)\big) = h(x/S) = g(x) = f(x)$. The theorem now follows by the well-known replacement principle.

For any $a$ and any $n \in \{0, 1, 2, \ldots\}$ we let $a^n = aa \ldots a$ ($n$ times); thus $a^0 = 0$. An element $e$ of an $m$-semigroup $\mathfrak{A}$ is called an *identity* of $\mathfrak{A}$ if $(e^i a e^j) = a$ whenever $a \in A$ and $i + j = m - 1$. It was shown in Sioson [7] that an $m$-semigroup with identity is a reduct of a semigroup.

In semigroups an identity can always be adjoined externally; only one new element, the identity, needs to be adjoined. This is no longer possible for $m$-semigroups, as an example in the next section shows. Nevertheless, an obvious consequence of Theorem 1 is the following

COROLLARY. *Any $m$-semigroup can be extended to an $m$-semigroup with identity.*

An $m$-semigroup $\mathfrak{A}$ is *commutative* if

$$(a_0 \dots a_{m-1}) = (a_{f(0)} \dots a_{f(m-1)})$$

for all $a_0, \dots, a_{m-1} \in A$ and for every permutation $f$ of $\{0, 1, \dots, m-1\}$. Analogously to Theorem 1 we have

THEOREM 2. *Every commutative $m$-semigroup $\mathfrak{A}$ is a subreduct of a commutative semigroup $\mathfrak{B}$ such that $A$ generates $\mathfrak{B}$ and if $\mathfrak{A}$ is a subreduct of a commutative semigroup $\mathfrak{C}$ then there is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$ which is the identity on $A$.*

Proof. We modify the proof of Theorem 1 by specifying that $aRb$ also if there exist $c, d, e \in B$ such that $l(d) = 2$, $a = cde$, and $b = cd_1d_0e$. The remainder of the proof of Theorem 1 goes through with the obvious changes; $\mathfrak{B}/S$ is a commutative semigroup, and $\mathfrak{D}$ must be assumed to be a commutative semigroup.

COROLLARY. *Any commutative $m$-semigroup can be extended to a commutative $m$-semigroup with identity.*

**2. Representation as an $m$-semigroup of functions.** From Theorem 1 we know that every $m$-semigroup is isomorphic to an $m$-semigroup of functions, the $m$-ary operation being defined as $(m-1)$-fold composition of functions, since the corresponding result holds for semigroups. However, there is a more natural notion of $m$-semigroup of functions, which we now describe.

If $A_0, \dots, A_{m-2}$ are non-empty sets we denote by $\mathcal{S}(A_0 \dots A_{m-2})$ the set of all functions with domain $\bigcup_{i<m-1} A_i$ which map $A_i$ into $A_{i+1}$ for $i < m-2$ and which map $A_{m-2}$ into $A_0$. An element of $\mathcal{S}(A_0 \dots A_{m-2})$ is called an *$(A_0 \dots A_{m-2})$-function*. For any functions $f, g$, $f \circ g$ is the *composition* of $f$ and $g$: $(f \circ g)(x) = f(g(x))$ if $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$. If $f_0, \dots, f_{m-1} \in \mathcal{S}(A_0 \dots A_{m-2})$, we let

$$(f_0 \dots f_{m-1}) = f_0 \circ \dots \circ f_{m-1} . \tag{7}$$

Clearly then $(f_0 \dots f_{m-1}) \in \mathcal{S}(A_0 \dots A_{m-2})$. A non-empty subset of $\mathcal{S}(A_0 \dots A_{m-2})$ closed under ( ) is called an *$m$-semigroup of $(A_0 \dots A_{m-2})$-functions*, or just an *$m$-semigroup of functions*. Note that the natural representation yielded by Theorem 1 and the theory of semigroups is a representation as an $m$-semigroup of $(A_0 \dots A_{m-2})$-functions, with $A_0 = \dots = A_{m-2}$.



If the sets $A_0, \dots, A_{m-2}$ are pairwise disjoint, then $f \circ g \notin \mathcal{S}(A_0 \dots A_{m-2})$ if $f, g \in \mathcal{S}(A_0 \dots A_{m-2})$. In this case an $m$-semigroup of $(A_0 \dots A_{m-2})$-functions is called a *disjoint $m$-semigroup of functions*.

EXAMPLE. The following example illustrates the notion just introduced and clears up two natural questions which arose in section 1. Let $p$ be the least positive prime divisor of $m-1$. Let $A_0, \dots, A_{m-2}$ be pairwise disjoint sets, each with exactly $p$ elements, say $A_i = \{a_{i0}, \dots, a_{i,p-1}\}$ for each $i < m-1$. Let $f$ be the $(A_0 \dots A_{m-2})$-function such that $f(a_{ij}) = a_{i+1,j}$ for $i < m-2$, $j < p$, $f(a_{m-2,j}) = a_{0,j+1}$ for $j < p-1$, and $f(a_{m-2,p-1}) = a_{00}$. Thus $f$ is a permutation of $\bigcup_{i<m-1} A_i$; it is easily checked that the order of $f$ (in the group of all permutations of $\bigcup_{i<m-1} A_i$) is $p(m-1)$. For each $i < p$ let $g_i = f^{1+i(m-1)}$ (i.e., $f$ composed with itself $1+i(m-1)$ times). Then the elements $g_0, \dots, g_{p-1}$ are all distinct, and $\{g_0, \dots, g_{p-1}\}$ forms an $m$-semigroup $\mathfrak{B}$ under the natural operation (7). For $i < p$ we have

$$(g_i^m) = f^{m(1+i(m-1))} = f^{1+(mi+1)(m-1)};$$

since $mi+1 \equiv i \pmod{p}$, it follows that $(g_i^m) \neq g_i$. Thus $\mathfrak{B}$ does not have an identity.

It is impossible to adjoint just one new element $e$ to $\mathfrak{B}$ to form a new $m$-semigroup in which $e$ is an identity. To prove this, we suppose on the contrary that this can be done. We derive a contradiction in each of the cases $m$ odd and $m$ even.

For $m$ odd we have $p = 2$, and $\mathfrak{B}$ has just two elements $g_0$ and $g_1$. The operation ( ) is determined as follows: for any $a_0, \dots, a_{m-1} \in B$,

$$(a_0 \dots a_{m-1}) = \begin{cases} g_1 & \text{if there is an even number of } i < m \\ & \text{such that } a_i = g_1 , \\ g_0 & \text{otherwise} . \end{cases}$$

Now $(e^{m-2}g_0^2) \in \{g_0, g_1, e\}$. If $(e^{m-2}g_0^2) = e$, then

$$g_0 = (e^{m-1}g_0) = (e^{m-2}e^{m-2}g_0^2g_0) = (e^{m-3}g_0^3) = \dots = (g_0^m) = g_1 ,$$

a contradiction. If $(e^{m-2}g_0^2) = g_0$, then

$$(e^{m-2}g_0g_1) = (e^{m-2}e^{m-2}g_0^2g_1) = (e^{m-3}g_0^2g_1) = \dots = (g_0^{m-1}g_1) = g_0 ,$$

and hence

$$g_0 = (e^{m-2}g_0g_1) = (e^{m-2}e^{m-2}g_0g_1^2) = (e^{m-3}g_0g_1^2) = \dots = (g_0g_1^{m-1}) = g_1 ,$$

a contradiction. Finally, if $(e^{m-2}g_0^2) = g_1$, then

$$\begin{aligned} g_0 = (g_1^m) &= (e^{m-2}g_0^2g_1^{m-1}) = (e^{m-2}g_0e^{m-1}g_0g_1^{m-1}) = (g_0e^{m-2}g_0e^{m-1}g_1^{m-1}) \\ &= (g_0^2e^{m-2}g_1^{m-1}) = \dots = (g_0^{2(m-1)}g_1) = (g_0^m g_0^{m-2}g_1) = (g_1g_0^{m-2}g_1) = g_1 , \end{aligned}$$

again a contradiction. Thus a contradiction follows from the assumption that $m$ is odd.

Now assume that $m$ is even. Then $(e^2 g_0^{m-2}) \in \{g_0, \ldots, g_{p-1}, e\}$. If $(e^2 g_0^{m-2}) = e$, then

$$e = (e^2 g_0^{m-2}) = (ee^2 g_0^{m-2} g_0^{m-2}) = (e^3 g_0^{2(m-2)}) = \ldots = (g_0^{(m-2)(m-2)}) \neq e\,,$$

which is impossible. Now assume that $(e^2 g_0^{m-2}) = g_i$, where $i < p$. Let $j = p-i$, $s = j(m-1)+m-2$, and $t = \frac{m}{2}s - \left(\frac{m}{2} - 1\right)$. Then

$$\begin{aligned} g_0 &= (g_0^{1+p(m-1)}) = (g_0^{1+i(m-1)} g_0^{j(m-1)}) = (g_i g_0^{j(m-1)}) = (e^2 g_0^s) \\ &= (e^2 g_0 g_0^{s-1}) = (e^4 g_0^s g_0^{s-1}) = (e^4 g^{2s-1}) = \ldots = (e^m g^{\frac{m}{2}s - \left(\frac{m}{2}-1\right)}) \\ &= (e g_0^t) = (e g_0 g_0^{t-1}) = (e^2 g_0^{2t-1}) = g_0^{(m-1)t-(m-2)}\,. \end{aligned}$$

Now it is easily seen that

$$(m-1)t-(m-2) = \frac{m}{2}(s-1)(m-1)+1\,,$$

and that

$$\frac{m}{2}(s-1) = \left(\frac{m}{2}(j+1)-1\right)(m-1)-1\,.$$

Hence $p$ does not divide $\frac{m}{2}(s-1)$, and consequently $g_0^{(m-1)t-(m-2)} \neq g_0$, which contradicts the above calculation.

Hence for $m$ odd or $m$ even it is impossible to adjoin just one new element to $\mathfrak{B}$ to form an $m$-semigroup with identity. In particular it follows that $\mathfrak{B}$ is *not* a reduct of a semigroup, since otherwise this *would* be possible.

THEOREM 3. *Every $m$-semigroup is isomorphic to a disjoint $m$-semigroup of functions.*

Proof. Let $\mathfrak{A} = \langle A, (\ )\rangle$ be any $m$-semigroup. By the corollary to Theorem 1 we may assume that $\mathfrak{A}$ has an identity $e$. We retain the notation from the proof of Theorem 1. For each $i < m-1$ let $E_i = \{M\colon M \in B/S$ and there is an $a \in M$ with $l(a) = i+1\}$. From (5) we see that the sets $E_0, \ldots, E_{m-2}$ are pairwise disjoint. Now, since $S$ is a congruence relation on $\mathfrak{B}$, for each $x \in A$ there is a function $k_x$ such that, for each $a \in B^-$, $k_x(a/S) = xa/S$. Also,

(8) for all $a \in B$ there is a $b \in B$ such that $aSb$ and $l(b) < m$.

For, if $a \in B$ and $n = l(a) > m$, then $aS(a_0 \ldots a_{m-1})a_m \ldots a_{n-1}$, and $l\big((a_0 \ldots a_{m-1})a_m \ldots a_{n-1}\big) = l(a) - m+1$ (cf. (5)); (8) then follows by induction.



From (8) we see that $k_x$ maps $\bigcup_{i<m-1} E_i$ into $\bigcup_{i<m-1} E_i$, for each $x \in A$. If $x \in A$, $a \in B$, and $l(a) = i+1$, then by (5) and (8), $k_x(a/S) \in E_{i+1}$ if $i < m-2$, and $k_x(a/S) \in E_0$ if $i = m-2$. Thus $k_x$ is an $(E_0 \ldots E_{m-2})$-function, for each $x \in A$. If $x_0, \ldots, x_{m-1} \in A$ and $a \in B$, then

$$\begin{aligned} k_{(x_0 \ldots x_{m-1})}(a/S) &= (x_0 \ldots x_{m-1})\,a/S = (x_0 \ldots x_{m-1}\,a/S) \\ &= (k_{x_0} \circ \ldots \circ k_{x_{m-1}})(a/S) = (k_{x_0} \ldots k_{x_{m-1}})(a/S)\,. \end{aligned}$$

Thus $k$ is a homomorphism of $\mathfrak{A}$ into $\langle S(E_0 \ldots E_{m-2}), (\ )\rangle$. If $x, y \in A$ and $k_x = k_y$, then $x/S = (xe^{m-1})/S = xe^{m-1}/S = k_x(e^{m-1}/S) = k_y(e^{m-1}/S) = y/S$, and hence $x = y$ by (6). Thus $k$ is an isomorphism, as desired.

THEOREM 4. *For any $m$-semigroup $\mathfrak{A}$ the conjunction of the following three conditions is necessary and sufficient for $A$ to be isomorphic to a disjoint $m$-semigroup of 1-1 functions.*

(i) *For all $x_0, \ldots, x_{m-2} \in A$, if $y = (yx_0 \ldots x_{m-2})$ for some $y \in A$ then $y = (yx_0 \ldots x_{m-2}) = (x_0 \ldots x_{m-2}y)$ for every $y \in A$.*

(ii) *For all $x_0, \ldots, x_{m-2}, y, z \in A$, the equality $(x_0 \ldots x_{m-2}y) = (x_0 \ldots x_{m-2}z)$ implies that $y = z$.*

(iii) *For all $i < m-2$ and for all $y_0, \ldots, y_i, z_0, \ldots, z \in A$, if $(x_0 \ldots x_{m-i-2}y_0 \ldots y_i) = (x_0 \ldots x_{m-i-2}z_0 \ldots z_i)$ for some $x_0, \ldots, x_{m-1-2} \in A$, then $(uy_0 \ldots y_i v) = (uz_0 \ldots z_i v)$ whenever $u$ and $v$ are sequences of elements of $A$ such that $l(u)+l(v) = m-i-2$.*

Proof. The three conditions clearly hold in any disjoint $m$-semigroup of 1-1 functions. Now assume that (i)-(iii) hold in an $m$-semigroup $\mathfrak{A}$. First we want to see that it can be assumed that $\mathfrak{A}$ has an identity. To this end we modify the proof of Theorem 1 by letting $aRb$ also in the following two cases:

(9) $a = 0$, $l(b) = m-1$, and $y = (yb)$ for some $y \in A$.

(10) $l(a) = f(b) < m$, and there is a $c \in B$ with $l(c) = m - l(a)$ such that $(ca) = (cb)$.

The conditions (5) and (6) still hold (to prove (6) it is necessary to use (ii)). Furthermore, $S$ is still a congruence relation on $\mathfrak{B}$, and $f$ is an isomorphism of $\mathfrak{A}$ onto a subreduct of $\mathfrak{B}/S$ (here it is necessary to use (i) and (iii)). The $m$-semigroup reduct of $\mathfrak{B}/S$ has an identity and satisfies (i)-(iii). To prove this it suffices to show that $\mathfrak{B}/S$ itself is a left-cancellative semigroup. Assume, then, that $x/S \cdot y/S = x/S \cdot z/S$. We may assume that $x, y, z$ have length $< m$, and in fact that $x$ has length $m-1$. If $y = z = 0$, then trivially $y/S = z/S$. Assume that $y = 0$ while $z \neq 0$ (similar to the case $z = 0$ while $y \neq 0$). Then $l(z) = m-1$ by (5), and $xSxzSx_0\dots x_{m-3}(x_{m-2}z)$; hence $x_0x_0x_1\dots x_{m-2}Sx_0x_0x_1\dots x_{m-3}(x_{m-2}z)$, and so $(x_0x_0x_1\dots x_{m-2})S\big(x_0x_0x_1\dots x_{m-3}(x_{m-2}z)\big)$. Thus by (6), $(x_0x_0x_1\dots x_{m-2}) = \big(x_0x_0x_1\dots x_{m-3})x_{m-2}z)\big)$. Then (ii) yields $x_{m-2} = (x_{m-2}z)$, so that $0Sz$, as desired. Finally, suppose that $y, z \neq 0$. Then by (5), $l(z) = l(y)$. Write $x = uv$, where $l(v) = m - l(z)$ ($u = 0$ if $l(z) = 1$). Then $u(vy)SxySxzSu(vz)$. Let $n = m - \big(l(u)+1\big)$. Then $\big(x_0^n u(vy)\big)Sx_0^n u(vy)Sx_0^n u(vz)S\big(x_0^n u(vz)\big)$. Hence by (6), $\big(x_0^n u(vy)\big) = \big(x_0^n u(vz)\big)$. By (ii), $(vy) = (vz)$, and hence $y/S = z/S$.

Therefore we may assume that $\mathfrak{A}$ has an identity. We now apply the same modification as above to the proof of Theorem 1. We find then that the function $k$ in the proof of Theorem 3 is still 1-1. Furthermore, $k_x$ is now 1-1 for each $x \in A$. For, assume that $\langle y_0\dots y_i\rangle/S$ and $\langle z_0\dots z_i\rangle/S$ are members of $E_i$ with $k_x(\langle y_0\dots y_i\rangle/S) = k_x(\langle z_0\dots z_i\rangle/S)$; thus $\langle xy_0\dots y_i\rangle/S = \langle xz_0\dots z_i\rangle/S$. If $j = m-(i+1)$, then $(x^jy_0\dots y_i)Sx^jy_0\dots y_iSx^jz_0\dots z_iS(x^jz_0\dots z_i)$. Hence by (6), $(x^jy_0\dots y_i) = (x^jz_0\dots z_i)$, so $\langle y_0\dots y_i\rangle/S = \langle z_0\dots z_i\rangle/S$.

An $m$-semigroup $\mathfrak{A}$ is an *$m$-group* if for every $i < m$ and for all $x_0, \dots, x_{m-1} \in A$ there is a unique $y \in A$ such that $(x_0\dots x_{i-1}yx_i\dots x_{m-2}) = x_{m-1}$. The theory of $m$-groups was extensively developed in Post [4].

If $A_0, \dots, A_{m-2}$ are non-empty sets and $f \in S(A_0\dots A_{m-2})$, then $f$ is called an *onto* $(A_0\dots A_{m-2})$-function if $f$ maps $A_i$ onto $A_{i+1}$ for each $i < m-2$, and $A_{m-2}$ onto $A_0$.

THEOREM 5. *Every $m$-group is isomorphic to a disjoint $m$-group of* 1-1 *onto functions.*

Proof. An $m$-group is known to satisfy conditions (i)-(iii) of Theorem 4. Hence we may continue the notation of Theorems 1, 3, and 4. The mapping $k$ is one-one even if $\mathfrak{A}$ has no identity, since there is a sequence $y$ of length $m-1$ of elements of $A$ such that $(xy) = x$ for all $x$. It remains to show that $k_x$ is onto for each $x \in A$. If $y \in A$, choose $z_0, \dots, z_{m-2} \in A$ such that $(xz_0\dots z_{m-2}) = y$. Then $k_x(z_0\dots z_{m-2}/S) = y/S$; thus $k_x$ maps $E_{m-2}$ onto $E_0$. Now suppose that $a \in B$ and $1 < l(a) = i < m$. Choose $y_1, \dots, y_{m-1} \in A$ such that $(a_0y_1\dots y_{m-1}) = x$, and choose $z \in A$ such that $(y_1\dots y_{m-1}z) = a_1$. Then

$$[xza_2\dots a_{i-1}]S[a_0y_1\dots y_{m-1}za_2\dots a_{i-1}]S[a_0\dots y\,a_{i-1}],$$

and hence $k_x(za_2\dots a_{i-1}/S) = a/S$, as desired.

COROLLARY (Post's Coset Theorem). *Every $m$-group is a subreduct of a group.*

In Post [4], p. 230, an example is given of an $m$-group which is not a reduct of a group (in fact, as is easily seen, the $m$-group described is not even a reduct of a semigroup; this is another example of the kind described at the beginning of this section).



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