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# On the time optimal control in the case of non-uniqueness

by

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**1. Introduction.** The problem of time optimal control of linear systems has been solved by many authors using various methods ([2], [5]-[10], [12]-[14], [18]). But almost in all those papers the problem has been solved under the assumption that the time optimal control is unique. The only exceptions are papers [12] and [14]. In paper [12] only qualitative results are given, and in [14] one method of finding time optimal control in the case of non-uniqueness is proposed. The purpose of this paper is to give a possibly exhaustive description of the problem of non-unique time optimal control of linear systems. It will be based on the methods given in papers [1], [2], [5]-[10], [13]-[15], and so the basic results of those papers are presented below.

**2. The basic results of application of functional analysis to the time optimal control problem.** There is given a physical system described by the linear differential equation

$$\dot{x}(t) = F(t)x(t)+G(t)u(t), \tag{1}$$

where $F(t)$ and $G(t)$ are matrices $n\times n$-dimensional and $n\times m$-dimensional respectively. Elements of these matrices are measurable and almost everywhere bounded functions of $t$.

An $n$-dimensional space $X$ will be called a *state space* and an $n$-dimensional vector $x(t)\in X$ will be called the *state of system* (1) *at the time* $t$. The $m$-dimensional vector $u(t)$ belonging to the space $E$ will be called the *control of system* (1).

In the space $E$ there is given a set $U(t)$ which will be called the *control region*. We will assume that $U(t)$ is a closed compact and convex polyhedron spanned on $p$ not necessarily different vectors, which are measurable bounded functions of parameter $t$.

Moreover, it will be required that there exist a positive number $a>0$ not depending upon $t$, such that $u(t)\in U(t)$ implies $-au(t)\in U(t)$.

The vector function $u$ defined on some interval $[t_0, T]$ will be called an *admissible control* on that interval if $u(t)\in U(t)$ for almost every $t\in[t_0, T]$.

There is given an initial state of system (1) $x(t_0) = x_0$, and a required terminal state $x_1$.

Our purpose is to find the time optimal control, i.e. such an admissible control function which transfers the trajectory of the system from the state $x_0$ to $x_1$ in minimum time.

The solution of equation (1) satisfying the initial condition $x(t_0) = x_0$ is [3] given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)u(\tau)d\tau, \tag{2}$$

where $\Phi(t, t_0)$ is a fundamental matrix solution of the equation $\dot{x}(t) = F(t)x(t)$ satisfying the initial condition $\Phi(t_0, t_0) = I$.

After simple rearrangement we obtain from equation (2)

$$c(t) = \int_{t_0}^{t} A(\tau)u(\tau)d\tau, \tag{2a}$$

where

$$c(t) = \Phi(t_0, t)x(t) - x_0,$$
$$A(t) = \Phi(t_0, t)G(t). \tag{3}$$

Note that the dependence between $c(t)$ and $x(t)$ is biunique. Let the minimum time of control be $t_1 - t_0$; then system (1) is transferred from the state $x(t_0) = x_0$ to $x(t_1) = x_1$ if and only if it is transferred from $c(t_0) = \theta$ to $c(t_1) = c_1(t_1) = \Phi(t_0, t_1)x_1 - x_0$.

Further we will investigate system (1) in the coordinates $c(t)$ only. To determine the time optimal control first we must find the minimum time of control $t_1 - t_0$. This will be done by the method given by Krasovskiĭ [6], [7].

The control region $U(t)$ can be treated as a unit sphere inducing at the time $t$ the Minkowskian norm $\|\cdot\|$ in the space $E_t$ spanned on $U(t)$ [4], [14], [15].

Now we assume that the time of control is fixed and equals $T - t_0$, and we introduce the function space $B[t_0, T] = L^{\infty}_{E_t}[t_0, T]$ of measurable vector functions $u$ defined on the interval $[t_0, T]$ and such that

$$|u|_T = \operatorname*{ess\,sup}_{t \in [t_0, T]} \|u(t)\| < +\infty. \tag{4}$$

The measurable function $u$, defined on $[t_0, T]$, is an admissible control if and only if $|u|_T \leqslant 1$ (1).

(1) All the results of section 3 are true if instead of the constraint $|u|_T < 1$ we consider constraints of the type

$$\Big[\int_{t_0}^{T} \|u(t)\|^p dt\Big]^{1/p} < 1 \quad (1 < p < +\infty).$$



The further procedure is the following: we solve the so called *minimum norm problem*, i.e. we find the minimum norm (4) of the function $u$ which transfers the system from $\theta$ to $c_1(T) = \Phi(t_0, T)x^1 - x^0$.

In the same way as in [6] one can show that the value of the minimum norm $|u|_T$ is a continuous function of the parameter $T$ and $|u|_T \to \infty$ if $T \to t_0$. The minimum value $t_1$ of the parameter $T$ for which the equality $|u|_T = 1$ holds (if such a value exists) is the minimum time of control. Further we will assume that a finite time of control exists.

To solve the minimum norm problem we put $t = T$ and $c(t) = c_1(T)$ in equation (2a). Rewriting this equation in the form of $n$ scalar equations we obtain

$$c_1^i(T) = \int_{t_0}^{T} \big(a_i(\tau), u(\tau)\big)d\tau = \int_{t_0}^{T} \sum_{j=1}^{m} a_i^j(\tau)u^j(\tau)d\tau \quad (i = 1, 2, \ldots, n), \tag{5}$$

where $c_1^i(T)$ denotes the $i$-th component of the vector $c_1(T)$, the vector $a_i(t)$ is the $i$-th row of the matrix $A(t)$, and $a_i^j(t)$ is the $j$-th component of the vector $a_i(t)$.

It was shown in [15] that the space $B[t_0, T]$ is conjugate of the space $B_*[t_0, T] = L^1_{E_t^*}[t_0, T]$ of $m$-dimensional measurable vector functions $a$ with the norm

$$|a|_T = \int_{t_0}^{T} \|a(t)\|^* dt, \tag{6}$$

where $\|\cdot\|^*$ denotes the norm in $m$-dimensional space $E_t^*$ conjugate of the space $E_t$.

Each linear functional, defined on the space $B_*[t_0, T]$, can be represented in the form of the right-hand side of equation (5) and its norm is given by (4).

We can treat the vector functions $a_i$ as the element of the space $B_*[t_0, T]$. So the minimum norm problem can be formulated as follows: find a minimal norm of a linear functional $u$ which assumes given numbers $c_1^i(T)$ on $n$ given elements $a_i \in B_*[t_0, T]$:

$$u(a_i) = c_1^i(T) \quad (i = 1, 2, \ldots, n). \tag{5a}$$

We assume that the complete controllability condition [8] is fullfiled, i.e.

$$\min_{\lambda^i} \Big|\sum_{i=1}^{n} \lambda^i a_i\Big|_T > 0, \quad \sum_{i=1}^{n} (\lambda^i)^2 > 0 \tag{7}$$

for sufficiently large values of the parameter $T$. Then equation (5a) has the minimum norm solution for each bounded vector $c_1(T)$ [8].

To find the minimum norm of the functional $u$ satisfying equation (5a) we use the method given in [1]. First we introduce a linear functional $\varphi$ defined on the $n$-dimensional subspace $B_*^n[t_0, T] \subset B_*[t_0, T]$ spanned on the elements $a_i$, and assuming the given values $c_1^i(T)$ on the elements $a_i$. This functional is defined uniquely and on the element $a = \sum_{i=1}^{n} \lambda^i a_i$ it assumes the value

$$u(a) = \sum_{i=1}^{n} \lambda^i c_1^i(T). \tag{8}$$

By definition the norm of the functional $\varphi$ equals

$$|\varphi|_T = \max_{\lambda^i} \sum_{i=1}^{n} \lambda^i c_1^i(T), \qquad \Big|\sum_{i=1}^{n} \lambda^i a_i\Big|_T = 1. \tag{9}$$

Let the vector $\lambda_0 = [\lambda_0^1, \lambda_0^2, \ldots, \lambda_0^n]$ maximize the right-hand side of formula (9); then

$$|\varphi|_T = \sum_{i=1}^{n} \lambda_0^i a_i = \lambda_0 A. \tag{9a}$$

In virtue of the Hahn-Banach theorem [4] functional $\varphi$ can be extended with the same norm to the whole space $B_*[t_0, T]$. Thus we obtain the desired functional $u$ with the minimal norm $|u|_T = |\varphi|_T$.

Let $t_1$ be the minimum value of the parameter $T$ for which the equation $|\varphi|_{t_1} = |\varphi| = 1$ holds; then $t_1 - t_0$ is the minimum control time.

Now we turn to the problem of determining the form of the time optimal control.

The extremal element of the functional $\varphi$, i.e. also of the functional $u$, is given by

$$a_0 = \sum_{i=1}^{n} \lambda_0^i a_i = \lambda_0 A. \tag{10}$$

For the element $a_0$ the following equation must be satisfied [1]:

$$u(a_0) = \int_{t_0}^{t_1} \big(a_0(t), u(t)\big)\, dt = |u|\,|a_0| = 1. \tag{11}$$

These results have a simple geometrical interpretation [14].

The functional $\varphi$ determines in the subspace $B_*^n[t_0, t]$ an $(n-1)$-dimensional hyperplane $G^{n-1} = \{a \in B_*^n[t_0, t_1] : \varphi(a) = |\varphi| = 1\}$ supporting the unit sphere $Z_*^n$ of the subspace $B_*^n[t_0, t_1]$. The element $a_0$ is a common point of $Z_*^n$ and $G^{n-1}$. It is shown in Fig. 1 for the case $n=2$.



The functional $u$, which is an extention of the functional $\varphi$ to the whole space $B_*[t_0, t_1]$, determines in this space a hyperplane

$$G = \{a : u(a) = |u| = 1\} \tag{12}$$

such that $G \cap B_*^n[t_0, t_1] = G^{n-1}$.

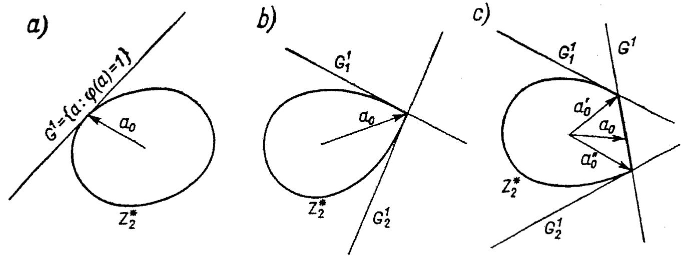


Fig. 1

Note that the reachable region $Z^n$, i.e. the set of all vectors $c(t_1)$ which can be reached from the point $\theta$ in the time $t_1 - t_0$ by using all admissible controls, is identical with the unit sphere $Z^n$ in the space conjugate to the $n$-dimensional subspace $B_*^n[t_0, t_1]$ (see [1]). As a unit sphere this set is closed, convex and bounded and because $B_*^n[t_0, t_1]$ is $n$-dimensional, $Z^n$ is compact and also $n$-dimensional.

The time optimal control is unique if there is only one hyperplane $G$ satisfying (12).

The geometrical interpretation allows us to determine when equation (11) is a sufficient condition for time optimal control, i. e. each function $u$ satisfying (11) must satisfy (5).

If the unit sphere $Z_*^n$ is smooth at the point $a_0$ (see Fig. 1a), then the element $a_0$ is the extremal element of one functional $\varphi$ only and condition (11) is sufficient for time optimal control. But if $Z_*^n$ is not smooth at the point $a_0$ (see Fig. 1b), then $a_0$ is the extremal element of a whole bunch of functionals (each functional belonging to the cone created by the lines $G_1^1$, $G_2^1$ in Fig. 1b). For each of these functionals equation (12) is satisfied and we cannot use it to determine the time optimal control.

Note that more than one vector $\lambda_0$ maximalizing formula (9) can exist. Geometrically it means that there is more than one common point of $G^{n-1}$ and $Z_*^n$. If there are $q$ linearly independent vectors $\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0q}$ satisfying (9), then $G^{n-1}$ includes a $(q-1)$-dimensional flattening $P^{q-1}$ of $Z_*^n$ (the segment $P^1$ in Fig. 1c). All elements $a_0$ belonging to the interior of $P^{q-1}$ are the extremal elements of the same functionals $\varphi$ and $u$. But the elements $a_0$ belonging to the edge of $P^{q-1}$ ($a_0'$ and $a_0''$ in Fig. 1c) can be the extremal elements of some additional functionals (e.g. for $a_0'$ all supporting lines situated between $G^1$ and $G_1^1$).

It is easy to see that the necessary condition for $Z_*^n$ not to be smooth at the point $a_0$ is that there is more than one function $u$ satisfying (11); so we will analise the cases where the solution of (11) is non-unique.

First of all we find the conditions for the time optimal control to be non-unique.

**3. Conditions for the non-uniqueness of the time optimal control.** It was shown in [14] that the function $u$ satisfies (11) if and only if the following equation holds for almost every $t \in [t_0, t_1]$:

$$(a_0(t), u(t)) = \|a_0(t)\|^*. \tag{13}$$

The set $\{u(t): (a_0(t), u(t)) = \|a_0(t)\|^*\}$ defines in the space $E$ an $(m-1)$-dimensional hyperplane $H^{m-1}(t)$ supporting the unit sphere $U(t)$.

The space $E$ is finite-dimensional, and thus the hyperplane $H^{m-1}(t)$ has at least one common point with $U(t)$. Each of those points satisfies equation (13) and is the desired value of the function $u$ at the time $t$.

The vector $u(t)$ is determined non-uniquely if and only if the hyperplane $H^{m-1}(t)$ contains a certain $k$-dimensional face $R^k$ $(k = 1, 2, \ldots, m-1)$ of the unit sphere $U(t)$.

It is obvious that if $R^k(t) \subset H^{m-1}(t)$ there must exist such a one-dimensional edge $R^1(t)$ of $U(t)$ that

$$R^1(t) \subset H^{m-1}(t). \tag{14}$$

Let the edge $R^1(t)$ generate a line $S^1(t)$. Let us assume that dim $[U(t)] = m$. Then the line $S^1(t)$ can be considered as an intersection of $p \geqslant m-1$ $(m-1)$-dimensional hyperplanes $S_1^{m-1}(t), S_2^{m-1}(t), \ldots, S_p^{m-1}(t)$ generated by all $(m-1)$-dimensional faces $R_1^{m-1}(t), R_2^{m-1}(t), \ldots, R_p^{m-1}(t)$ of the polyhedron $U(t)$ which are contiguous to $R^1(t)$. Let $s_j(t)$ denote vectors orthogonal to $S_j^{m-1}(t)$ and directed outside of the polyhedron $U(t)$.

LEMMA 1. *$R^1(t)$ belongs to $H^{m-1}(t)$ if and only if there is a non-negative linear combination*

$$s(t) = \sum_{j=1}^{p} \nu^j s_j(t), \qquad \nu^j \geqslant 0, \tag{15}$$

*of the vectors $s_j(t)$ such that $a_0(t) = s(t)$.*

Proof. Let $K^m(t)$ denote a convex cone generated by the hyperplanes $S_j^{m-1}(t)$ $(j = 1, 2, \ldots, p)$ and containing $U(t)$. $R^1(t)$ is an edge of this cone. Such a cone is shown in Fig. 2 for the case $m = 2$ (a vertex $R^0(t)$ occurs here instead of a one-dimensional edge $R^1(t)$).



$R^1(t)$ belongs to $H^{m-1}(t)$ if and only if $H^{m-1}(t)$ is a hyperplane of support to the cone $K^m(t)$. The hyperplane $H^{m-1}(t)$ is a hyperplane of support to the cone $K^m(t)$ if and only if the intersection of the hyperplane $H_1^{m-1}(t) = \{u(t): (a_0(t), u(t)) = \beta\}$ and the cone $K^m(t)$ is void for sufficiently large values of the parameter $\beta$: $H_1^m(t) \cap K^m(t) = \Lambda$.

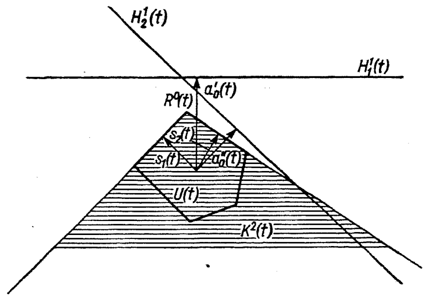


Fig. 2

The cone $K^m(t)$ can be described by the formula

$$K^m(t) = \{u(t): (s_j(t), u(t)) \leqslant \alpha^j,\ j = 1, 2, \ldots, p\}, \tag{16}$$

where $\alpha^j$ are positive numbers, because the zero element of the space $E$ belongs to the interior of the cone $K^m(t)$.

Then $R^1(t) \subset H^{m-1}(t)$ if and only if the system of inequalities

$$\begin{gathered} (-s_j(t), u(t)) \geqslant -\alpha^j \quad (j = 1, 2, \ldots, p), \\ (a_0(t), u(t)) \geqslant \beta \end{gathered} \tag{17}$$

has no solution for $\beta$ sufficiently large.

It was shown in [11] that system (17) has no solution if and only if there exist non-negative numbers $\mu^j \geqslant 0$ $(\sum_{j=0}^{p} \mu^j > 0)$ such that

$$\mu^0 a_0(t) - \sum_{j=1}^{p} \mu^j s^j(t) = \theta, \tag{18a}$$

$$\mu^0 \beta - \sum_{j=1}^{p} \mu^j \alpha^j > 0. \tag{18b}$$

We can put $\mu^0 \neq 0$. It is easy to see that if (18a) holds we can find such $\beta$ that inequality (18b) is satisfied. Then equality (18a) is the necessary and sufficient condition for $R^1(t) \subset H^m(t)$. This equality can be rewritten in the following form:

$$\sum_{j=1}^{p}\frac{\mu^j}{\mu^0}s_j(t) = \sum_{j=1}^{p}\nu^j s_j(t) = s(t) = a_0(t), \qquad \text{q.e.d.}$$

LEMMA 2. *If the vector $a_0(t)$ is a positive linear combination*

$$a_0(t) = \sum_{j=1}^{p}\nu^j s_j(t), \qquad \nu^j > 0, \tag{19}$$

*of $p$ linearly dependent vectors $s_j(t)$ $(j = 1, 2, \ldots, p)$, then there are $q$ linearly independent vectors $s_j(t)$ belonging to the set $\{s_1(t), s_2(t), \ldots, s_p(t)\}$ such that the vector $a_0(t)$ is a positive linear combination of them.*

Proof. The vectors $s_j(t)$ $(j = 1, 2, \ldots, p)$ are linearly dependent, i.e. there exist numbers $\varkappa^j$, not all equal to zero, such that

$$\sum_{j=1}^{p}\varkappa^j s_j(t) = \theta. \tag{20}$$

We can assume that at least one number $\varkappa^j$ is positive. Let us write

$$\tilde{s}_j(t) = \nu^j s_j(t), \qquad \tilde{\varkappa}^j = \frac{\varkappa^j}{\nu^j}. \tag{21}$$

Then equations (19) and (20) can be rewritten in the form

$$a_0(t) = \sum_{j=1}^{p}\tilde{s}_j(t), \tag{19a}$$

$$\theta = \sum_{j=1}^{p}\tilde{\varkappa}^j\tilde{s}_j(t). \tag{20a}$$

Let $\tilde{\varkappa}^k = \max\{\tilde{\varkappa}^j\}$. Dividing both sides of equation (20a) by $-\tilde{\varkappa}^k$ we obtain

$$\theta = -\sum_{\substack{j=1\\ j\neq k}}^{p}\frac{\tilde{\varkappa}^j}{\tilde{\varkappa}^k}\tilde{s}_j(t) - \tilde{s}_k(t), \tag{20b}$$

where $\tilde{\varkappa}^j/\tilde{\varkappa}^k \leqslant 1$.

Adding both sides of equations (19a) and (20b) we obtain in virtue of (21)

$$a_0(t) = \sum_{\substack{j=1\\ j\neq k}}^{p}\left(1 - \frac{\tilde{\varkappa}^j}{\tilde{\varkappa}^k}\right)\nu^j s_j(t). \tag{19b}$$

The coefficients corresponding to $s_j(t)$ are non-negative, and thus the vector $a_0(t)$ is a non-negative linear combination of $p-1$ vectors $s_j(t)$.



The above construction can be repeated successively until we obtain $a_0(t)$ as a non-negative linear combination of $q$ linearly independent vectors $s_j(t)$, q.e.d.

The intersection of hyperplanes $S_j^{m-1}(t)$ $(j = 1, 2, \ldots, p)$ gives a line; so in the set $\{s_j(t)\}$ there are $(m-1)$-linearly independent vectors.

In virtue of Lemmas 1 and 2, $R^1(t) \subset H^{m-1}(t)$ if and only if there are $m-1$ vectors $s_j(t)$ such that $a_0(t)$ is a non-negative linear combination of them, i.e. there are non-negative numbers $\nu^0, \nu^1, \ldots, \nu^{m-1}$ $(\sum_{j=1}^{m-1}\nu^j \neq 0$, $\nu^0 \neq 0)$ which satisfy the homogeneous equation

$$[-a_0(t), s_1(t), s_2(t), \ldots, s_{m-1}(t)]\begin{bmatrix}\nu^0\\ \nu^1\\ \vdots\\ \nu^{m-}\end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} -a_0^1(t), s_1^1(t), \ldots, s_{m-1}^1(t)\\ -a_0^2(t), s_1^2(t), \ldots, s_{m-1}^2(t)\\ \vdots \quad\quad \vdots \quad\quad\quad \vdots\\ -a_0^m(t), s_1^m(t), \ldots, s_{m-1}^m(t)\end{bmatrix}\cdot\begin{bmatrix}\nu^0\\ \nu^1\\ \vdots\\ \nu^{m-1}\end{bmatrix} = \begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix},$$

where $a_0^i(t)$ and $s_j^i(t)$ are components of the vectors $a_0(t)$ and $s_j(t)$ respectively.

The necessary condition for equation (22) to be satisfied is that the determinant of this equation be equal to zero:

$$|-a_0(t), s_1(t), s_2(t), \ldots, s_{m-1}(t)| = 0. \tag{23}$$

Since the vectors $s_j(t)$ $(j = 1, 2, \ldots, m-1)$ are linearly independent, the rank of the matrix $[s_1(t), s_2(t), \ldots, s_{m-1}(t)]$ is equal to $m-1$. Then there is at least one $(m-1)$-dimensional non-zero determinant of this matrix.

Let us assume that we obtain this determinant after removing the last row of the matrix

$$\begin{vmatrix} s_1^1(t), s_2^1(t), \ldots, s_{m-1}^1(t)\\ s_1^2(t), s_2^2(t), \ldots, s_{m-1}^2(t)\\ \vdots \quad\quad \vdots \quad\quad\quad \vdots\\ s_1^{m-1}(t), s_2^{m-1}(t), \ldots, s_{m-1}^{m-1}(t)\end{vmatrix} = |s_1'(t), s_2'(t), \ldots, s_{m-1}'(t)| = M(t) \neq 0. \tag{24}$$

It follows from Kramer's formulas that the coefficients $\nu^j$ are non-negative if and only if the inequalities

$$\frac{M_j(t)}{M(t)} \geqslant 0 \qquad (j = 1, 2, \ldots, m-1) \tag{25}$$

hold, where $M_j(t)$ is the determinant obtained from $M(t)$ by substituting $a_0'(t) = [a_0^1(t), a_0^2(t), \ldots, a_0^{m-1}(t)]$ instead of the $j$-th column of $M(t)$.

The following lemma is a direct consequence of the above considerations:

LEMMA 3. *The necessary condition for $R^1(t)$ to belong to $H^{m-1}(t)$ is that equation* (23) *hold. This condition is also a sufficient one if, in addition, inequalities* (25) *are satisfied.*

Hitherto we assumed that $\dim[U(t)] = m$. If $\dim[U(t)] = k < m$, the above results are also true, but instead of the vector $a_0(t)$ one ought to take its projection on the subspace spanned on $U(t)$. In this case the rank of determinant (29) equals $k-1$.

We can obtain another, more convenient necessary condition. Let $r(t)$ denote the vector parallel to $R^1(t)$. The necessary condition for $R^1(t) \subset H^{m-1}(t)$ is that the vector $r(t)$ be parallel to $H^{m-1}(t)$, i.e. orthogonal to $a_0(t)$. Hence we obtain

LEMMA 4. *The necessary condition for $R^1(t)$ to belong to $H^{m-1}(t)$ is that*

$$(26) \qquad \big(a_0(t), r(t)\big) = 0.$$

COROLLARY 1. *If $U(t)$ is a parallelepiped, condition* (26) *is the necessary and sufficient one.*

Proof. If condition (26) is fulfilled, the vector $a_0(t)$ belongs to the $(m-1)$-dimensional subspace spanned on vectors $s_j(t)$ $(j = 1, 2, \ldots, m-1)$ defined as above. This subspace is identical for each of the mutually parallel edges $R^1(t)$ of the parallelepiped $U(t)$. Then an edge $R_1(t)$ for which all coefficients $\nu^j$ in equation (22) are non-negative must exist, q.e.d.

The solution $u$ of equation (11) is non-unique if and only if there exists such a measurable set $D \subset [t_0, t_1]$, $\operatorname{mes} D > 0$, that for every $t \in D$ the hyperplane $H^{m-1}(t)$ contains some edge $R^1(t)$ of the control region $U(t)$.

In virtue of Lemmas 3 and 4 we can formulate the following

PROPOSITION 1. *The necessary condition for the solution $u$ of equation* (11) *to be non-unique is that there exist a measurable set $D \subset [t_0, t_1]$, $\operatorname{mes} D > 0$, and a measurable vector function $r$ with its values $r(t)$ parallel to some edge $R^1(t)$ of the control region $U(t)$ such that*

$$(27) \qquad \int_D \big(a_0(t), r(t)\big)^2 dt = 0.$$

This condition is a sufficient one if, in addition, there is a positive measure set $D' \subset D$ such that for every $t \in D'$ the inequalities (25) are fulfilled.

Remark 1. Condition (27) is a necessary and sufficient one if the control region $U(t)$ is a parallelepiped.



Hitherto we analysed the cases where equation (11) had more than one solution $u$ for one particular vector function $a_0$. This means that the control function corresponding to the extremal element $a_0$ is non-unique. But it is interesting to find out when non-unique control can occur in the system for any initial and terminal states, i.e. when there is at least one point $a_0 \in Z_*^n$ such that equation (11) has more than one solution.

Each point $a_0$ belonging to the boundary of $Z_*^n$ can be represented in the form

$$a_0 = \sum_{i=1}^{n} \lambda_0^i a_i = \lambda_0 A \qquad (|a_0| = 1),$$

where the vector $\lambda_0$ depends upon initial and terminal states. Note that appropriate initial and terminal states exist for every vector $\lambda_0$, for the unit sphere $Z_*^n$ has a hyperplane of support to each of its points.

Substituting (10) in (26) we obtain

$$(28) \qquad \Big(\sum_{i=1}^{n} \lambda_0^i a_i(t), r(t)\Big) = \sum_{i=1}^{n} \lambda_0^i \big(a_i(t), r(t)\big) = \sum_{i=1}^{n} \lambda_0^i l_i(t) = 0,$$

where $l_i(t) = \big(a_i(t), r(t)\big)$.

The necessary condition for the existence of a non-unique time optimal control is that equation (28) hold for each $t$ belonging to some positive measure set $D$. It is equivalent to the requirement that functions $l_i$ be linearly dependent on the set $D$.

It is well known [16] that the functions $l_i$ are linearly dependent on the set D if and only if the rank of the following Gramian matrix is less than $n$:

$$(29) \qquad \operatorname{rank} \int_D \big[\big(l_i(t), l_j(t)\big)\big]\, dt = q < n.$$

Using the definitions of the functions $l_i$ and the matrix $A$ we can formulate the following

THEOREM 1. *The necessary condition for the time optimal control of system* (1) *to be non-unique is that there exist a measurable set $D \subset [t_0, t_1]$, $\operatorname{mes} D > 0$, and a measurable vector function $r$ with its values $r(t)$ parallel to some edge $R^1(t)$ of the control region $U(t)$ such that*

$$(30) \qquad \operatorname{rank} \int_D \Phi(t_0, t) G(t) r(t) r^T(t) G^T(t) \Phi^T(t_0, t)\, dt = q < n,$$

*where the superscript $T$ denotes transposition.*

To obtain the sufficient condition for the time optimal control to be non-unique we substitute (10) in (25) and get

$$(31)\qquad \frac{M_j(t)}{M(t)} = \frac{\left|s_1'(t), s_2'(t), \ldots, s_{j-1}'(t), \sum_{i=1}^{n} \lambda_0^i a_i', s_{j+1}'(t), \ldots, s_{m-1}'(t)\right|}{M(t)}$$

$$= \sum_{i=1}^{n} \lambda_0^i \frac{\left|s_1'(t), s_2'(t), \ldots, s_{j-1}'(t), a_i'(t), s_{j+1}'(t), \ldots, s_{m-1}'(t)\right|}{M(t)}$$

$$= \sum_{i=1}^{n} \lambda_0^i N_j^i(t) \geqslant 0 \qquad (j = 1, 2, \ldots, m-1),$$

where $a_i'(t) = [a_i^1(t), a_i^2(t), \ldots, a_i^{m-1}(t)]$.

It is well known [16] that the vector $\lambda_0$ satisfies equation (28) on the set $D$ if and only if it satisfies the equation

$$(32)\qquad \lambda_0 \int_D \Phi(t_0, t) G(t) r(t) r^T(t) G^T(t) \Phi^T(t_0, t)\, dt = \theta.$$

Hence we have

THEOREM 2. *A non-unique time optimal control of system* (1) *can exist if and only if condition* (30) *is satisfied and if, in addition, there exists a positive measure set $D' \subset D$ such that for every $t \in D'$ inequalities* (31) *are fulfilled, where $\lambda_0^i$ denotes one of the non-trivial solutions* (*the same for all $t \in D'$*) *of equation* (32).

From Corollary 1 we obtain the following

COROLLARY 2. *If the control region $U(t)$ is a parallelepiped, inequality* (30) *is a necessary and sufficient condition for the existence of a non-unique time optimal control.*

In particular, if the module of each component of the vector $u(t)$ is bounded ($|u^j(t)| \leqslant 1$), the control region $U(t)$ is a cube and the vectors $r(t)$ have the form $[0, \ldots, 0, 1, 0, \ldots, 0]$. For this particular case result (30) was obtained by La Salle [12]. The system described by equation (1) was called by La Salle a *normal system* if the rank of matrix (30) is equal to $n$ for every vector $r_i$ of the form $[0, \ldots, 0, 1, 0, \ldots, 0]$.

Let us consider now the case where system (1) is stationary, i.e. the matrices $F$ and $G$ are constant, and the control region $U$ does not depend upon $t$.

COROLLARY 3. *If system* (1) *is stationary and the control region $U$ does not depend upon $t$, the necessary condition for the time optimal control to be non-unique has the form*

$$(33)\qquad \operatorname{rank}[Gr, FGr, F^2Gr, \ldots, F^{n-1}Gr] = q < n.$$



Proof. If the system is stationary, then $\Phi(t_0, t) = [\exp F(t_0 - t)]$ and from (28) we get

$$(34)\qquad \lambda_0 [\exp F(t_0 - t)] Gr = \theta.$$

Equation (34) must be satisfied on the positive measure set $D$. Note that the left-hand side of this equation is an analytical function, whence it must be identically equal to zero.

We successively differentiate equation (34). In virtue of the fact that the matrices $F$ and $[\exp F(t_0 - t_1)]$ are interchangeable we obtain after $k$ successive differentiations

$$\lambda_0 [\exp F(t_0 - t)] F^k Gr = \theta.$$

Then the vector $\lambda_0$ must satisfy the following equation:

$$(35)\qquad \lambda_0 [\exp F(t_0 - t)] [Gr, FGr, F^2Gr, \ldots, F^kGr, \ldots] = \theta.$$

Since the matrix $[\exp F(t_0 - t)]$ is non-singular, equation (35) has a non-trivial solution if and only if the rank of the matrix $[Gr, FGr, F^2Gr, \ldots, F^kGr, \ldots]$ is less than $n$. On the other hand, it is easy to see that it is fulfilled if and only if rank $[Gr, FGr, \ldots, F^{n-1}Gr] = q < n$. Geometrically this means that each vector of the form $F^iGr$ belongs to the $q$-dimensional subspace spanned on the vectors $Gr, FGr, \ldots, F^{n-1}Gr$.

Using the definition of the matrix $[\exp F(t_0 - t)]$ we rewrite equation (34) in the form

$$(34a)\qquad \lambda_0 \sum_{i=0}^{\infty} \frac{(t_0 - t)^i}{i!} F^i Gr = 0.$$

According to the above considerations if the rank of matrix (33) is equal to $q < n$, then every vector $F^iGr$ belongs to the same $q$-dimensional subspace. This subspace is closed [4], and thus the vector

$$\sum_{i=0}^{\infty} \frac{(t_0 - t)^i}{i!} F^i Gr$$

belongs to it for each $t$. Hence equation (34) has a non-trivial solution the same for each $t$.

Corollary 4 gives the well-known general position condition introduced in [18]. Hence Theorem 2 can be treated as a generalization of the general position condition for the cases where the system is non-stationary and the control region $U(t)$ depends upon the parameter $t$.

The maximal number of linearly independent solutions of equation (34) cannot exceed $n$. Thus we can formulate the following

Remark 2. Suppose that system (1) is stationary and that the control region $U$ does not depend upon $t$. Then the maximal number of linearly independent elements $a_0 = \lambda_0 A$ for which there are more than one solution of equation (11) cannot exceed $n\gamma$, where $\gamma$ is the number of edges of the parallelepiped $U$.

**4. Determination of the time optimal control in the case of non-uniqueness.** Suppose that for the terminal position $c^1(t_1)$ equation (11) has more than one solution, i.e. that there is such a positive measure set $D\subset[t_0, t]$ that for each $t\in D$ the hyperplane $H^{m-1}(t)$ determined by the vector

$$a_0(t) = \sum_{i=1}^{n} \lambda_{0_1}^i a_i = \lambda_{0_1} A$$

(where $\lambda_{0_1}^i$ denote the values of parameters $\lambda^i$ maximizing (7) for $c^1(t_1)$ and $T = t_1$) contains some face $R^{k(t)}(t)$ of the control region $U(t)$. We assume that $D$ is the largest set possessing such a property.

We want to find the set $\Gamma(t_1)$ of all points $c(t_1)$ which can be reached in the time $t_1-t_0$ from the initial point $\theta$ using all functions satisfying (11) as controls. It is obvious that the following relation must hold:

$$c_1(t_1)\in\Gamma(t_1).$$

The set $\Gamma(t_1)$ is isomorphic with the set of all $(n-1)$-dimensional hyperplanes $G^{n-1}$ supporting to the unit sphere $Z_*^n$ at the point $a_0$.

To find $\Gamma(t_1)$ we put $t_1$ instead of $t$ in equation (2a) and we rewrite it in the following form:

$$c(t_1) = \int_D A(\tau)u(\tau)d\tau + \int_{[t_0,t_1]\setminus D} A(\tau)u(\tau)d\tau. \tag{35}$$

Let $u_0(t)$ denote the centre of gravity of the face $R^{k(t)}(t)$. The vector function $u_0$ is a measurable function of $t$ on the set $D$ [15].

We introduce the following notation:

$$v(t) = u(t)-u_0(t). \tag{36}$$

Substituting (36) in (35) we obtain

$$c(t_1) = \int A(\tau)v(\tau)d\tau + \int_D A(\tau)u_0(\tau)d\tau + \int_{[t_0,t_1]\setminus D} A(\tau)u(\tau)d\tau. \tag{35a}$$

Since the control function $u$ on the set $[t_0, t_1]\setminus D$ is given uniquely by equation (13), the third term of the right-hand side of equation (35a) is a given $n$-dimensional vector. Similarly, the second term is a given vector. Let us write

$$\begin{aligned} d(t_1) &= \int_D A(\tau)v(\tau)d\tau, \\ d_0(t_1) &= \int_D A(\tau)u_0(\tau)d\tau + \int_{[t_0,t_1]\setminus D} A(\tau)u(\tau)d\tau. \end{aligned} \tag{37}$$



It is clear that to find $\Gamma(t_1)$ we must find the set $\Delta(t_1)$ of elements $d(t_1)$ corresponding to all measurable functions $v$ satisfying the condition $v(t)\in R^{k(t)}(t)$ on the set $D$. The set $\Gamma(t_1)$ we obtain from the formula

$$\Gamma(t_1) = \Delta(t_1)+d_0(t_1). \tag{38}$$

We prove some properties of the set $\Delta(t_1)$.

The set $R^{k(t)}(t)$ can be treated [16] as a unit sphere inducing at the time $t$ the Minkowskian norm $\|\cdot\|_1$ in the space $E_{1t}$ of $n$-dimensional vectors $v(t)$ given by (36).

In the same way as in section 2, we introduce the function space $B_1(D) = L_{E_{1t}}^{\infty}(D)$ of measurable functions $v$ defined on the set $D$ and such that

$$|v|_1 = \operatorname*{ess\,sup}_{t\in D}\|v(t)\|_1 < +\infty. \tag{39}$$

This space is conjugate of the space $B_{1*}(D) = L_{E_{1t}^*}^1(D)$ of $m$-dimensional vector functions $a$ with the norm

$$|a|_1 = \int_D \|a(t)\|_1^* dt < +\infty, \tag{40}$$

where $\|\cdot\|_1^*$ denotes the norm in the space conjugate of $E_{1t}$.

Let us consider the subspace $B_{1*}^n(D)\subset B_{1*}(D)$ spanned on $n$ vector functions $a_i$. Using the same argument as in [1], one can see that the set $\Delta(t_1)$ is identical with the unit sphere in the space conjugate of the subspace $B_{1*}^n(D)$.

Let us assume that $n-q$ is the maximal number of linearly independent vectors $\lambda_{0j}$ $(j = 1, 2, \ldots, n-q)$ such that $|\lambda_{0j}A| = 1$ and

$$|\lambda_{0j}A|_1 = \Big|\sum_{i=1}^{n}\lambda_{0j}^i a_i\Big|_1 = 0 \tag{41}$$

(at least one such vector exists, namely the vector $\lambda_{01}$ maximizing (7)).

It is obvious that the unit sphere $Z_{1*}^n$ in the subspace $B_{1*}^n(D)$ belongs to the $q$-dimensional hyperplane orthogonal to vectors $\lambda_{0j}$ $(j = 1, 2, \ldots, n-q)$ and is $q$-dimensional itself.

Then the set $\Delta(t_1)$ is also $q$-dimensional and belongs to the hyperplane orthogonal to the vector $\lambda_{0j}$ $(j = 1, 2, \ldots, n-q)$. In addition $\Delta(t_1)$ is convex, closed and compact as a unit sphere in the finite-dimensional space.

The same properties has the set $\Gamma(t_1)$.

It is obvious that the vector $\lambda_{0j}$ fulfills (41) if and only if it satisfies formulas (32) and (31) for each measurable vector function $r$ with its values $r(t)$ parallel to any edge of the face $R^{k(t)}(t)$.

Hence we have the following

LEMMA 5. *By $\lambda_{0j}$ we denote a non-zero vector satisfying* (32) *for every measurable vector function $r$ the values $r(t)$ of which are parallel to some edge of the face $R^{k(t)}(t)$. Let $n-q$ be the maximal number of linearly independent vectors $\lambda_{0j}$. Further, let inequalities* (31) *be satisfied for every $t \in D$. Then the set $\Gamma(t_1)$ is $q$-dimensional and belongs to the hyperplane orthogonal to the vectors $\lambda_{0j}$ $(j = 1, 2, \ldots, n-q)$. Moreover, the set $\Gamma(t_1)$ is convex and compact.*

COROLLARY 4. *Every function $u$ satisfying equation* (11) *is a time optimal control which transfers the system from $\theta$ to $c^1(t_1)$ if and only if there are $n$ linearly independent vectors $\lambda_{0j}$ $(j = 1, 2, \ldots, n)$ satisfying the conditions given in Lemma* 5.

Proof. The vector $\lambda_{0j}$ satisfies (41). Hence the unit sphere $Z^n_{1*}(D)$ is reduced to one point. Thus the set $\Gamma(t_1)$ is also reduced to one point. Obviously this must be the point $c_1(t_1)$, q.e.d.

Let us assume now that we change the time of control. By $\Gamma(t)$ we denote the set of all points $c(t)$ which can be reached from $\theta$ in the time $t-t_0$ $(t > t_0)$ by using all admissible control functions satisfying (13).

The set $D(t)$ on which the solution of equation (11) is non-unique is a non-decreasing function of $t$. On the other hand, the number of linearly independent vectors $\lambda_{0j}$ satisfying (32) is a non-increasing function of $D(t)$. Then in virtue of Lemma 5 the dimension of the set $\Gamma(t)$ is a non-decreasing function of $t$. Each of the sets $\Gamma(t)$ is convex and compact and belongs to an $(n-1)$-dimensional hyperplane orthogonal to the vector $\lambda_{01}$.

We now define the set

$$\Gamma(t_0, t_1) = \bigcup_{t_0 \leqslant t \leqslant t_1} \Gamma(t)$$

of all points $c(t)$ which can be reached from $\theta$ in the time $t_0 \leqslant t \leqslant t_1$ by using all admissible control functions satisfying (13). This set is equivalent to the set of the respective segments of all trajectories corresponding to those control functions and starting from $\theta$. It is obvious that $\Gamma(t_0, t_1)$ is a closed and connected set.

By analogy to $\Gamma(t)$ we denote by $\Gamma'(t)$ the set of all points $c(t)$ from which the point $c_1(t_1)$ can be reached in the time $t_1 - t$ $(t < t_1)$ by using all admissible control functions satisfying formula (13). It is easy to see that the set $\Gamma'(t)$ has analogical properties to those of the set $\Gamma(t)$. In particular, it belongs to the $(n-1)$-dimensional hyperplane orthogonal to the vector $\lambda_{01}$.

Between $\Gamma(t_1)$ and $\Gamma'(t_0)$ the following relation takes place:

$$\Gamma'(t_0) = c_1(t_1) - \Gamma(t_1). \tag{42}$$



We write further $\Gamma'(t_0, t_1) = \bigcup_{t_0 \leqslant t \leqslant t_1} \Gamma'(t)$.

THEOREM 3. *Let us assume that there exists no positive measure set $\delta \in [t_0, t_1]$ such that the function $\|a_0(t)\|^* = \|\lambda_{01} A(t)\|^*$ is identically equal to zero on that set. Thus the measurable function $u$ satisfying* (13) *is the time optimal control transferring $\theta$ to $c_1(t)$ if and only if it does not lead the trajectory of the system outside the set $\Gamma(t_0, t_1) \cap \Gamma'(t_0, t_1)$.*

Proof. It is obvious that the function $u$ satisfying (13) is an optimal control if and only if the point $c(t)$ of the corresponding trajectory belongs to the set $\Gamma(t) \cap \Gamma'(t)$ for every $t \in [t_0, t_1]$. Each of these points belongs to $\Gamma(t_0, t_1) \cap \Gamma'(t_0, t_1)$ Thus to prove our theorem it is enough to show that only those points belong to $\Gamma(t_0, t_1) \cap \Gamma'(t_0, t_1)$, i.e. that for every different $t, t' \in [t_0, t_1]$ the set $\Gamma(t) \cap \Gamma(t')$ is void.

In virtue of Lemma 5 and of the further considerations the sets $\Gamma(t)$ and $\Gamma'(t')$ belong to the $(n-1)$-dimensional hyperplane orthogonal to the vector $\lambda_{01}$ for each $t$ and $t'$. Moreover, for $t \in [t_0, t_1]$ both $\Gamma(t)$ and $\Gamma'(t)$ must belong to the same hyperplane since the time optimal control exists.

Then we must show that for every different $t, t' \in [t_0, t_1]$ the sets $\Gamma(t)$ and $\Gamma(t')$ belong to different $(n-1)$-dimensional hyperplanes orthogonal to $\lambda_{01}$. But these sets belong to the same hyperplane if and only if $(a_0(t), u(t)) = (\lambda_{01} A(t), u(t))$ equals to zero almost everywhere on the interval $[t, t']$ for any $u(t) \in R^{k(t)}(t)$. Then from equation (13) we obtain $\|a_0(t)\|^* = \|\lambda_{01} A(t)\|^* = 0$ almost everywhere on the interval $[t, t']$, which contradicts the assumption, q.e.d.

**5. The bang-bang principle generalization.** La Salle investigated the optimal control problem of the linear system with the control region in the form of a cube and he proved the following theorem:

If there is any time optimal control transferring the trajectory of system (1) from $\theta$ to $c_1(t_1)$, then there is also a control function with the same properties which assumes its value at the vertices of the control region for almost every $t \in [t_0, t_1]$. Such a control function is the most convenient from the technical point of view. It is called the *bang-bang control*, and the theorem is called the *bang-bang principle*.

We show that this principle is also true for the more general case of the control region $U(t)$ of the form considered here.

In virtue of equation (13) and its geometrical interpretation (see section 2) it is clear that if the time optimal control is unique, it must be the bang-bang control. Then it is enough to consider the case where the time optimal control is non-unique.

Let us assume that the required terminal point $c_1(t_1)$, which must be reached from $\theta$, belongs to the set $\Gamma(t_1)$ defined in section 4.

LEMMA 6. *If the point $c_1(t_1)$ is an extremal point of the set $\Gamma(t_1)$, then there is a bang-bang time optimal control transferring the trajectory of the system from $\theta$ to $c_1(t_1)$.*

Proof. It follows from the definition of the set $\Gamma(t_1)$ that there is at least one admissible control function $u_e$ which transfers the system from $\theta$ to $c_1(t_1)$ in the minimal time $t_1 - t_0$, i.e. the control function $u_e$ satisfies equation (2a) for $t = t_1$ and $c(t_1) = c_1(t_1)$. Let us assume that there is a positive measure set $D_e \subset D$ such that for every $t \in D_e$ the value $u_e(t)$ of the control function $u_e$ does not correspond to any vertex of the polyhedron $R^{k(t)}(t)$ and the control function $u_e$ is of the bang-bang type on the set $[t_0, t_1] \setminus D_e$.

We will consider the set $D_e$ only. Let $R^{h(t)}(t) \subset R^{k(t)}(t)$ be the largest closed convex polyhedron such that $u_e(t)$ belongs to the interior (2) of $R^{h(t)}(t)$ and all vertices of $R^{h(t)}(t)$ correspond to certain vertices of $R^{k(t)}(t)$.

We will show that each admissible control function $u$ such that $u(t) \in R^{h(t)}(t)$ for $t \in D_e$ and $u(t) = u_e(t)$ for $t \in [t_0, t_1] \setminus D_e$ satisfies equation (2a) for $c(t_1) = c_1(t_1)$.

Let us assume that it is not true, i.e. that there is an admissible control function $\tilde{u}$ with those properties which satisfies equation (2a) for $c(t_1) = \tilde{c}(t_1) \neq c_1(t_1)$.

By construction, $u_e(t)$ belongs to the interior of $R^{h(t)}(t)$. Then there is a number $\beta > 0$ such that the minimal distance from $u_e(t)$ to the boundary of $R^{h(t)}(t)$ is larger than $\beta$ for every $t \in D_e^\beta$, where $D_e^\beta \subset D_e$ and

$$\lim_{\beta \to 0} \operatorname{mes}(D_e \setminus D_e^\beta) = 0.$$

Therefore one can find $\beta > 0$ such that

$$\int_{D_e^\beta} A(t) u_e(t)\, dt \neq \int_{D_e^\beta} A(t) \tilde{u}(t)\, dt.$$

Let $u'$ denote an admissible control function such that $u'(t) = \tilde{u}(t)$ for $t \in D_e^\beta$ and $u'(t) = u_e(t)$ for $t \in [t_0, t_1] \setminus D_e^\beta$. The control function $u'$ satisfies (2a) for $c(t_1) = c'(t_1) \in \Gamma(t_1)$, where $c'(t_1) \neq c_1(t_1)$. We write

$$\varkappa = \beta \cdot [\max_{t \in D_e^\beta} |u'(t) - u_e(t)|]^{-1},$$

where $|\cdot|$ denotes the Euclidean distance. Since

$$0 < \max_{t \in D_e^\beta} |u'(t) - u_e(t)| < +\infty,$$

we have $0 < \varkappa < +\infty$.

(2) Here and in the sequel by the *interior* and the *boundary* of a set we will understand the interior and the boundary of this set in relation to the minimal linear subspace containing it.



Let us define the control function $u''$ as follows: $u''(t) = u_e(t)$ for $t \in [t_0, t_1] \setminus D_e^\beta$ and $u''(t) = (1+\varkappa) u_e(t) - \varkappa u'(t)$ for $t \in D_e^\beta$. It is easy to see that $u''(t) \in R^{h(t)}(t)$ for $t \in D_e^\beta$. Then $u''$ is an admissible control function and it satisfies (2a) for $c(t_1) = c''(t_1) \in \Gamma(t_1)$ where $c''(t) \neq c'(t)$. On the other hand,

$$c_1(t_1) = \frac{\varkappa}{1+\varkappa} c'(t_1) + \frac{1}{1+\varkappa} c''(t_1),$$

which contradicts the assumption that $c_1(t_1)$ is an extremal element of the set $\Gamma(t_1)$.

Hence each admissible control function $u$ such that $u(t) \in R^{h(t)}(t)$ for $t \in D_e$ and $u(t) = u_e(t)$ for $t \in [t_0, t_1] \setminus D_e$ transfers the trajectory of the system from $\theta$ to $c_1(t_1)$. From the set of these control functions we can choose an admissible bang-bang control, q.e.d.

THEOREM 4. *For each $c_1(t_1) \in \Gamma(t_1)$ there is a bang-bang time optimal control transferring the trajectory of the system from $\theta$ to $c_1(t_1)$.*

Proof. For the extremal elements of the set $\Gamma(t_1)$ the theorem is satisfied in virtue of Lemma 6. Then we can assume that $c_1(t_1)$ is not an extremal element of the set $\Gamma(t_1)$, i.e. it can be represented as a convex combination of some extremal elements of the set $\Gamma(t_1)$. Let the minimal number of such elements equal to $N_0$. In virtue of Lemma 5, $\dim[\Gamma(t_1)] = q$. Hence we have, $N_0 \leqslant q+1$.

Let

$$c_1(t_1) = \sum_{a=1}^{N_0} \varkappa_a c_a(t_1),$$

where $c_a(t_1)$ are the extremal elements of $\Gamma(t_1)$ and $0 < \varkappa_a < 1$, $\sum_{a=1}^{N_0} \varkappa_a = 1$.

By Lemma 6 for each $a = 1, \ldots, N_0$ there is a bang-bang control $u_a$ which transfers the trajectory of the system from $\theta$ to $c_a(t_1)$. Let $R_0(t) = \operatorname{conv}\{u_a(t)\}$. It is obvious that $R_0(t) \subset R^{k(t)}(t)$. Analogically to the sets $\Gamma(t)$, by $\Gamma_0(t)$ we denote the set of all points $c(t)$ which can be reached from $\theta$ in the time $t - t_0$ by using all admissible control functions $u$ satisfying the condition $u(t) \in R_0(t)$ for almost every $t \in [t_0, t_1]$.

By the same argument as in the proof of Lemma 5 one can show that each set $\Gamma_0(t)$ $(t_0 \leqslant t \leqslant t_1)$ belongs to a $q_0$-dimensional hyperplane orthogonal to vectors $\lambda_{0j}$ $(j = 1, 2, \ldots, n - q_0)$, where $\lambda_{0j}$ are linearly independent vectors satisfying (32) and (31) for each measurable vector function $r$ which has its values $r(t)$ parallel to any edge of $R_0(t)$. It is obvious that $q_0 \leqslant q$. Moreover, $\dim[\Gamma_0(t_1)] = q_0$. It follows from the construction of the set $\Gamma_0(t)$ that $c_1(t_1)$ belongs to the interior of $\Gamma_0(t_1)$.

In the same way as in section 4 we denote by $\Gamma_0'(t)$ the set of all points $c(t)$ from which the point $c_1(t_1)$ can be reached in the time $t_1 - t$

by using all admissible control functions $u$ satisfying the condition $u(\tau)\in R_0(\tau)$ for almost every $\tau\in[t_0, t]$. For each $t\in[t_0, t_1]$ the set $\Gamma_0'(t)$ belongs to the same $q_0$-dimensional hyperplane as the set $\Gamma_0(t)$.

Let

$$\Gamma_0'(t_0, t_1) = \bigcup_{t_0\leqslant t\leqslant t} \Gamma_0'(t).$$

It is obvious that $\Gamma_0'(t_0, t_1)$ is a closed and connected set.

Similarly to (42) we have $\Gamma_0'(t_0) = c_1(t_1) - \Gamma_0(t_1)$. Hence $\dim[\Gamma'(t_0)] = q_0$ and $\theta$ is an interior point of the set $\Gamma_0'(t_0)$ since $c_1(t_1)$ is an interior point of $\Gamma_0(t_1)$. Since trajectories are continuous then for $t > t_0$ and sufficiently close to $t_0$ we have $\dim[\Gamma'(t)] = q_0$. Therefore, taking into consideration that for $t\in[t_0, t_1]$ the set $\Gamma_0(t)$ belongs to the same $q_0$-dimensional hyperplane as $\Gamma_0'(t)$, one can see that for $t > t_0$ and sufficiently close to $t_0$ the set $\Gamma_0(t)$ belongs to $\Gamma_0'(t)$.

We choose an arbitrary bang-bang control $u$ such that $u(t)\in R_0(t)$ and the points $c(t)$ corresponding to this control are extremal points of $\Gamma_0(t)$. For $t$ sufficiently close to $t_0$ these points $c(t)$ belong to $\Gamma_0'(t_0, t_1)$. But since $c_1(t_1)$ is not an extremal element of the set $\Gamma_0(t_1)$, the control $u$ cannot transfer the trajectory from $\theta$ to $c_1(t_1)$. Thus there must exist a time $T_1\in(t_0, t_1)$ such that $c(t)\notin\Gamma_0'(t_0, t_1)$ for $t > T_1$. Since trajectories are continuous and the set $\Gamma_0'(t_0, t_1)$ is closed and connected, the respective point $c(T_1)$ must belong to the boundary of the set $\Gamma_0'(t_0, t_1)$. If the point $c(T_1)$ is an extremal point of the set $\Gamma_0'(T_1)$ then using the same argument as in the proof of Lemma 6 one can show that there exists a bang-bang control leading the trajectory of the system to $c_1(t_1)$. If $c(T_1)$ is not an extremal point of $\Gamma_0'(T_1)$, we can repeat the construction given above. Note that since $c(T_1)$ belongs to the boundary of $\Gamma_0'(T_1)$ and $\dim\Gamma_0'(T_1)\leqslant q_0$, the point $c(T_1)$ can be represented as a convex combination of $N_1$ extremal elements of $\Gamma_0'(T_1)$, where $N_1\leqslant q_0\leqslant q$. Using the next segment of the bang-bang control we reach the point $c(T_2)(T_1 < T_2 < t_1)$ belonging to the boundary of the respective set $\Gamma_1'(T_2)$ defined in the same way as $\Gamma_0'(T_1)$. We repeat this construction successively. After each step the number $N_i$ decreases at least by one. Thus we obtain $N_i = 1$ after $i\leqslant q_0\leqslant q$ steps. It means that the respective point $c(T_i)$ is an extremal element of the set $\Gamma_{i-1}'(T_i)$. Hence there is a bang-bang control leading the trajectory of the system to $c_1(t_1)$, q.e.d.

Note that Theorem 5 was obtained in another way and in a more general case by Olech [17].

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