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On the class of infinitely divisible distributions and on its subclasses

by

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1. On the symmetry of infinitely divisible and stable distributions. As it is known [4], the logarithm of the characteristic function $f(t)$ of any infinitely divisible distribution can be written in the Lévy form

$$\log f(t) = i\gamma t - \frac{1}{2}\sigma^2 t^2 + \int_{-\infty}^{0} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) dM(u) + \int_{0}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) dN(u), \tag{1}$$

where $\gamma = \text{const}$, $\sigma \geqslant 0$, $M(u)$ and $N(u)$ are non-decreasing (continuous from the left) functions, defined on $(-\infty, 0)$ and $(0, +\infty)$ respectively, and such that $M(-\infty) = N(+\infty) = 0$ and

$$\int_{-\varepsilon}^{0} u^2 dM(u) + \int_{0}^{\varepsilon} u^2 dN(u) < +\infty$$

for every $\varepsilon > 0$.

Let us introduce the following definition:

DEFINITION. An infinitely divisible distribution [1] X is said to be *completely asymmetric in the class of infinitely divisible distributions* (denoted in the sequel by I) if it is impossible to represent it in the form $X = X_1 + X_2$, where $X_1, X_2 \in I$, X_1 and X_2 are independent and X_1 is symmetric.

The following theorem holds:

THEOREM 1. *Every infinitely divisible distribution X can be written in the form*

$$X \overset{*}{=} a_1 X^{(s)} + a_2 X^{(a)}, \tag{2}$$

where $a_1, a_2 = 0$ or 1, $a_1 + a_2 > 0$, $X^{(s)}, X^{(a)} \in I$, $X^{(s)}$ is symmetric and $X^{(a)}$ is completely asymmetric in the class I.

[1] We assume all considered distributions to be non-degenerate.

Proof. Lévy functions $M(u)$ and $N(u)$ of an infinitely divisible distribution X define on $(-\infty, 0) \cup (0, \infty)$ a non-negative measure μ satisfying the conditions

$$\int_{-1}^{0} u^2 \mu(du) + \int_{0}^{1} u^2 \mu(du) < +\infty, \qquad \mu(-\infty, 1) < \infty, \qquad \mu(1, \infty) < \infty$$

and such that

$$\log f(t) = i\gamma t - \frac{1}{2}\sigma^2 t^2 + \int_{u\neq 0} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\mu(du). \tag{1'}$$

Let us define the measure μ^* by the formula

$$\mu^*(A) \overset{\text{def}}{=} \mu(-A)$$

where the set $-A$ denotes the symmetric reflection of A with respect to $u = 0$. Let us write further $\mu + \mu^* = \nu$. Obviously the measures μ and μ^* are absolutely continuous with respect to the measure ν. Therefore, in virtue of Radon-Nikodym theorem, we may write

$$\mu(A) = \int_A \varrho(u)\nu(du)$$

where $\varrho(u) \geqslant 0$ is a measurable function. Let us write

$$\varrho_s(u) = \min(\varrho(u), \varrho(-u)), \qquad \varrho_a(u) = \varrho(u) - \varrho_s(u).$$

We have of course $\varrho(u) = \varrho_s(u) + \varrho_a(u)$, $\varrho_s(u) \geqslant 0$, $\varrho_a(u) \geqslant 0$, $\varrho_s(u) = \varrho_s(-u)$ and for every u either $\varrho_a(u) = 0$ or $\varrho_s(-u) = 0$. The functions $\varrho_s(u)$ and $\varrho_a(u)$ define the measures μ_s and μ_a respectively by the formulas

$$\mu_s(A) = \int_A \varrho_s(u)\nu(du), \qquad \mu_a(A) = \int_A \varrho_a(u)\nu(du).$$

We have of course

$$\mu = \mu_s + \mu_a.$$

The measure μ_s is symmetric ($\mu_s = \mu_s^*$) and the measure μ_a is completely asymmetric (if $0 \leqslant \mu_1 \leqslant \mu_a$ and $\mu_1 = \mu_1^*$, then $\mu_1 \equiv 0$). The measures μ_s and μ_a define the distributions $X^{(s)}$ and $X^{(a)}$ by means of (1′) (the term $i\gamma t$ we include to $X^{(a)}$ and the term $-\frac{1}{2}\sigma^2 t^2$ to $X^{(s)}$). Theorem 1 is thus proved.

If the distribution X from Theorem 1 is completely asymmetric in I, we have $a_1 = 0$. If X is symmetric, we have $a_2 = 0$.

In the sequel we shall use the following theorem due to Braumann [2]:

BRAUMANN'S THEOREM. *Infinitely divisible distribution X is symmetric with respect to a if and only if for every $u > 0$ such that u is a continuity point of $N(u)$ and $-u$ is a continuity point of $M(u)$ we have $M(-u) = -N(u)$. Besides $a = \gamma$.*

Observe that every non-normal, symmetric, infinitely divisible distribution X is a composition of two distributions completely asymmetric in I. Indeed, if X is defined by Lévy functions $M(u)$ and $N(u) = -M(-u) \not\equiv 0$, we can write $X = X_1 + X_2$, where X_1 is defined by $M_1(u) = M(u)$, $N_1(u) \equiv 0$ and X_2 is defined by $M_2(u) \equiv 0$, $N_2(u) = N(u)$.

We may make the following remark:

REMARK. Every stable distribution X can be written in the form (2), where $X^{(s)}$ and $X^{(a)}$ are stable.

Indeed, for a non-normal stable distribution X we have $M(u) = c_1/|u|^\alpha$, $N(u) = -c_2/u^\alpha$, where $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_1 + c_2 > 0$, $0 < \alpha < 2$. If $c_1 \geqslant c_2$, the distribution $X^{(s)}$ is defined by Lévy functions $M^{(s)}(u) = c_2/|u|^\alpha$, $N^{(s)}(u) = -c_2/u^\alpha$ and the distribution $X^{(a)}$ is defined by $M^{(a)}(u) = (c_1 - c_2)/|u|^\alpha$, $N^{(a)}(u) \equiv 0$. Similarly in the case of $c_1 < c_2$. In either case $X^{(s)}$ and $X^{(a)}$ are stable.

It is easy to observe that for $c_1 = c_2$ the stable distribution X is symmetric (in formula (2) $a_1 = 1$, $a_2 = 0$) and for $c_1 \neq c_2$ the distribution X is asymmetric; if $c_1 = 0$ or $c_2 = 0$, the distribution X is completely asymmetric in I $(a_1 = 0, a_2 = 1)$.

The logarithm of the characteristic function of a stable distribution can be written in the form (1) with $M(u) = c_1/|u|^\alpha$, $N(u) = -c_2/u^\alpha$ or in the form

$$\log f(t) = i\gamma t - c|t|^\alpha \left[1 + i\beta \frac{t}{|t|}\omega(t, \alpha)\right], \tag{3}$$

where $0 < \alpha \leqslant 2$, $c \geqslant 0$, $-1 \leqslant \beta \leqslant 1$. Coefficients c_1, c_2 and β satisfy relations

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad \text{if} \quad \alpha \neq 1, \alpha \neq 2, \tag{4}$$

$$\beta = \frac{c_2 - c_1}{c_1 + c_2}, \quad \text{if} \quad \alpha = 1. \tag{4'}$$

Formula (3) can be written in the form

$$\log f(t) = i\gamma t - b_1|t|^\alpha - b_2|t|^\alpha \left(1 \pm \frac{t}{|t|} i\omega(t, \alpha)\right), \tag{3'}$$

where $b_1 = (1 - |\beta|)c \geqslant 0$, $b_2 = |\beta|c \geqslant 0$.

$b_1 = 0$ is equivalent to $|\beta| = 1$ (or $c = 0$ which represents a degenerate distribution). In view of (4) and (4') this corresponds to $c_1 = 0$ or $c_2 = 0$ hence to a distribution completely asymmetric in I.

$b_2 = 0$ is equivalent to $\beta = 0$ (or $c = 0$). In view of (4) and (4') this corresponds to $c_1 = c_2$, hence to a symmetric distribution (the last conclusion follows also from the fact that in this case $\log f(t)$ is real).

Formula (3') gives us therefore two separated parts of every stable distribution: the symmetric part and the part completely asymmetric in I.

In Lukacs' book [9], p. 106, we find the following theorem:

The stable distributions with $0 < a < 1$ *and* $|\beta| = 1$ *are bounded from one side (from the right, if* $\beta = +1$, *from the left, if* $\beta = -1$).

We shall show that the same is true also for $1 < a < 2$ and $|\beta| = 1$. We shall base the proof on the following theorem (Baxter and Shapiro [1]):

Infinitely divisible distribution X *is bounded from the right if and only if*

$$N(u) \equiv 0, \quad \sigma^2 = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} M(u)\,du < +\infty.$$

X is bounded from the left if and only if

$$M(u) \equiv 0, \quad \sigma^2 = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} N(u)\,du > -\infty.$$

For $1 < a < 2$ we have $\beta = 1$ if and only if $c_2 = 0$ (i.e. $N(u) \equiv 0$), and $\beta = -1$ if and only if $c_1 = 0$ (i.e. $M(u) \equiv 0$). Take $\beta = -1$. We have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} N(u)\,du = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{-c_2}{u^a}\,du = \frac{-c_2}{-a+1} > -\infty.$$

Thus, when $1 < a < 2$ and $\beta = -1$, the stable distribution is bounded from the left. Quite similarly we prove the boundedness from the right for $1 < a < 2$ and $\beta = 1$.

It is easy to verify that:

The stable distributions with $a = 1$ *and* $|\beta| = 1$ *are unbounded from both sides. These are the only completely asymmetric in* I *stable distributions which are unbounded from both sides.*

Observe still that the stable distributions with $|\beta| \neq 1$ are unbounded from both sides (for when $|\beta| \neq 1$, in view of (4) and (4') we have $c_1 \neq 0$ and $c_2 \neq 0$ or $M(u) \not\equiv 0$, $N(u) \not\equiv 0$).

Let us summarize these results in the form of a table:

a	$\lvert\beta\rvert$	boundedness	symmetry
2		unbounded from both sides	symmetric (normal distribution)
$\neq 2$	$\neq 1$	unbounded from both sides	symmetric ($\beta = 0$) or includes both the symmetric and the completely asymmetric parts ($\beta \neq 0$)
$\neq 2$ $\neq 1$	1	bounded from one side	completely asymmetric
1	1	unbounded from both sides	completely asymmetric

2. On the decompositions. It is known that the class I is closed under compositions but is not closed under decompositions ([4], § 17). In my paper [7] I observed that the class L is closed under compositions and in [8] I observed that the class of stable distributions is not closed under compositions. It is easy to observe that the class L and the class of stable distributions are not closed under decompositions. Indeed, we have $X_0 = X_1 + X_2$, where Lévy functions $M_i(u)$, $N_i(u)$ of X_i $(i = 0, 1, 2)$ are defined as follows: $M_i(u) \equiv 0$ $(i = 0, 1, 2)$, $N_2(u) = N_0(u) - N_1(u)$,

$$N_0(u) = \begin{cases} \ln u & \text{for} \quad 0 < u \leqslant 1, \\ 0 & \text{for} \quad u > 1; \end{cases}$$

$$N_1(u) = \begin{cases} \frac{1}{2}\ln u + \frac{1}{4}u^2 - \frac{1}{4} & \text{for} \quad 0 < u \leqslant 1, \\ 0 & \text{for} \quad u > 1. \end{cases}$$

Obviously $X_0, X_2 \in L$, $X_1 \notin L$. Similarly we have $X_0 = X_1 + X_2$, where $N_i(u) \equiv 0$ $(i = 0, 1, 2)$, $M_2(u) = M_0(u) - M_1(u)$, $M_0(u) = -1/u$,

$$M_1(u) = \begin{cases} 0 & \text{for} \quad u \leqslant -1, \\ u+1 & \text{for} \quad u > -1. \end{cases}$$

Obviously distribution X_0 is stable but X_1 and X_2 are not stable.

The probability distribution X is said to be *unimodal* if there exists at least one value a such that the distribution function $F(x)$ of X is convex for $x < a$ and concave for $x > a$. Chung [3] has proved that the class of unimodal distributions is not closed under compositions. I shall prove

THEOREM 2. *The class of unimodal distributions is not closed under decompositions.*

The proof will be based on a simple lemma:

LEMMA. *If the distribution X is unimodal, then the distribution $-X$ is also unimodal.*

Let X be unimodal. Therefore its distribution function $F(x)$ is convex for $x < a$ and concave for $x > a$. For $x \neq -a$ the distribution function of $-X$ is equal to $F_1(x) = 1-F(-x)$ and so it is convex for $x < -a$ and concave for $x > -a$.

Proof of Theorem 2. Ibragimov [5] has proved that not all distributions from the class L are unimodal. Let us take a non-unimodal distribution X_1 from the class L. In virtue of the lemma, distribution $X_2 = -X_1$ is also non-unimodal and it belongs of course to the class L. The distribution $X = X_1+X_2$ is clearly symmetric and $X \epsilon L$. Thus the distribution X is unimodal, because every symmetric distribution from the class L is unimodal (Wintner [10]). Thus we have a decomposition of a unimodal distribution from the class L into two non-unimodal distributions from the class L. Theorem 2 is thus proved.

The following corollary results immediately from the proof of Theorem 2:

COROLLARY. *The class of unimodal distributions from the class L is not closed under decompositions in L.*

Ibragimov [6] has introduced the notion of strongly unimodal distributions: a distribution is strongly unimodal, if its composition with any unimodal distribution is unimodal. The class of strongly unimodal distributions is closed under compositions (see [6]). I shall show that this class is not closed under decompositions. I shall use the following theorem due to Ibragimov:

IBRAGIMOV'S THEOREM. *A non-degenerate unimodal distribution X is strongly unimodal if and only if its distribution function $F(x)$ is continuous and the function $\omega(x) = \log F'(x)$ is concave at the set E of points where neither the right nor the left derivative of the function $F(x)$ is equal to zero.*

Let us consider the gamma distribution X with density function

$$p(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0, \\ \dfrac{b^p}{\Gamma(p)} x^{p-1} e^{-bx} & \text{for} \quad x > 0, \end{cases} \qquad p > 0,\ b > 0.$$

It is easy to verify that it is a unimodal distribution. If $p > 1$, its distribution function $F(x)$ is convex for $x < (p-1)/b$ and concave for $x > (p-1)/b$. If $p \leqslant 1$, $F(x)$ is convex for $x < 0$ and concave for $x > 0$.

The set E defined in Ibragimov's theorem consists of all the points $x > 0$. We have

$$\omega''(x) = \frac{1-p}{x^2}.$$

Thus, the function $\omega(x)$ is concave for $p \geqslant 1$ and convex for $p \leqslant 1$. Therefore the gamma distribution is strongly unimodal for $p \geqslant 1$ and non-strongly unimodal for $p < 1$.

Let us now take two independent gamma distributions X_1 and X_2 with the same values of parameters $b_1 = b_2 = b$ and $p_1 = p_2$, $\frac{1}{2} < p_1 < 1$. The distribution $X = X_1+X_2$ is a gamma distribution with parameter $p = 2p_1 > 1$ and so it is a strongly unimodal distribution. Thus we have proved:

THEOREM 3. *The class of strongly unimodal distributions is not closed under decompositions (even if the components are assumed to be identically distributed).*

Since gamma distributions belong to the class L we have immediately the following corollary:

COROLLARY. *The class of strongly unimodal distributions from the class L is not closed under decompositions in the class L (even if the components are assumed to be identically distributed).*

We have seen that the composition of two non-strongly unimodal distributions may be strongly unimodal. It may also be, of course, non-strongly unimodal. It suffices to take the composition of two gamma distributions with parameters $p_1 < \frac{1}{2}$ and $p_2 < \frac{1}{2}$. It is easy to observe that the composition of a strongly-unimodal and of a non-strongly unimodal distributions may be strongly unimodal. It suffices to take the composition of two gamma distributions: one with $p < 1$ and the second with $p > 1$. But the composition of a strongly unimodal and of a non-strongly unimodal distributions may also be non-strongly unimodal. It suffices to take the composition of the uniform distribution with density function

$$p_1(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x > 1, \end{cases}$$

which is strongly unimodal and of the distribution with density function

$$p_2(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0, \\ 1/2\sqrt{x} & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x > 1, \end{cases}$$

which is non-strongly unimodal. The composition of these distributions has the density function

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ \sqrt{x} & \text{for} \quad 0 < x < 1, \\ 1-\sqrt{x-1} & \text{for} \quad 1 < x < 2, \\ 0 & \text{for} \quad x > 2. \end{cases}$$

It is easy to verify that this distribution is non-strongly unimodal.

It must be mentioned that Lukacs and also Dugué gave an example of decomposition of Cauchy distribution into two non-stable distributions (see e. g. [9], § 9.2).

3. On the distributions of the same sort and of the same type. The logarithm of characteristic function $f(t)$ of an infinitely divisible distribution X with finite variance can be written in Kolmogorov's form [4]:

$$\log f(t) = i\gamma t + \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu)\frac{1}{u^2}\, dK(u),$$

where $\gamma = \text{const}$, $K(u)$ (Kolmogorov function) is a non-decreasing bounded function, $K(-\infty) = 0$. If $X_1 = aX + b$, $a > 0$, we have $f_1(t) = f(at)e^{ibt}$ and so

$$\log f_1(t) = it(a\gamma + b) + \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu)\frac{1}{u^2}\, a^2 dK\left(\frac{u}{a}\right).$$

Thus the distribution $X_1 = aX + b$ $(a > 0)$ is defined by $\gamma_1 = a\gamma + b$ and $K_1(u) = a^2 K(u/a)$.

In my paper [8] I introduced the notion of distributions of the same sort. If infinitely divisible distributions X and X_1 have finite variances, then they are of the same sort if and only if there exists a constant $\alpha > 0$ such that $K_1(u) = \alpha K(u)$ for all u, where $K(u)$ and $K_1(u)$ are Kolmogorov functions of X and X_1, respectively. The question arises, when two infinitely divisible distributions X and X_1 with finite variances are simultaneously of the same type and of the same sort. We must then have for all u

$$a^2 K\left(\frac{u}{a}\right) = \alpha K(u), \qquad a > 0,\ \alpha > 0.$$

Substituting $u = 0$ we obtain $\alpha = a^2$, hence

$$K(u) = K\left(\frac{u}{a}\right), \qquad a > 0. \tag{5}$$

Suppose now that (5) holds for some $a < 1$. Let us take an arbitrary $u_0 > 0$ and form the increasing sequence $u_n = u_{n-1}/a$. From (5) we have $K(u_{n-1}) = K(u_n)$ and since $\lim\limits_{u\to\infty} u_n = +\infty$ and $K(u)$ does not decrease, $K(u) = \text{const}$ for $u \geqslant u_0$. Similarly, taking the decreasing sequence $u_{-n} = au_{-n+1}$, which converges to zero, we state that $K(u) = \text{const}$ for $0 < u \leqslant u_0$. Therefore $K(u) = \text{const}$ for $u > 0$. Similarly (taking $u_0 < 0$) we have $K(u) = \text{const}$ for $u < 0$. The case $a > 1$ is quite analogous. We have then

THEOREM 4. *The class of normal distributions is the only class of distributions with finite variances which are of the same sort and simultaneously form one type of distributions.*

This theorem gives us a new characterization of normal distributions (another characterization I gave in my paper [8]).

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