is cross-Lipschitzian but not Lipschitzian over any open ball about the origin.

For linear mappings our result may be phrased so as to say:

COROLLARY 3. *For any densely defined linear mapping* $T:\mathscr{H}\to\mathscr{H}$,

$$K\inf_{\lambda}\|T-\lambda I\|\leqslant\|T\|^{\perp}\leqslant\inf_{\lambda}\|T-\lambda I\|, \tag{41}$$

where

$$\|T\|^{\perp}=\sup_{\|x\|\leqslant 1}\{\|Tx\|^2-|(Tx,x)|^2\}^{1/2},$$

and K *is a positive constant independent of* T.

The best value of K is not known to the authors, they can only say that it is not smaller than $5^{-1/2}$. It may be shown that the value is one if the dimension of the space is 2 or if the mapping is normal. Is it always so?

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Properties of the orthonormal Franklin system, II

by

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1. Introduction. This is to continue the investigations undertaken in the paper [1]. Most of the results were announced without proofs in [2].

In Sections 3 and 4 sharp estimates from above and from below for the single Franklin functions and for the Dirichlet kernel of the Franklin system are obtained. Actually, we work out an explicite formula for the Dirichlet kernel.

Theorem 4 shows that the Fourier-Franklin series of an integrable function converges at each weak Lebesgue point. Using Theorem 3 and Lemma 8 one could deduce this result from the general criterion for singular integrals of Krein and Levin [10]. However, with the help of generalized Natanson Lemma, proved by Taberski in [15], the straightforward proof of the Theorem 4 becomes very simple and therefore it is presented here.

The next part of this paper deals with the best approximation and with the approximation by the partial sums of the Fourier-Franklin expansions in the $L_p\langle 0,1\rangle$ spaces. Most of the corresponding results for the space $C\langle 0,1\rangle$ were discussed in [1]. Theorem 9 shows that there is a non-trivial difference in the order of approximation of smooth functions by the partial sums of the Fourier-Franklin and Haar-Fourier expansions.

Theorem 12 extends the results obtained in [3] for the case $p=\infty$ to the Lipschitz classes in $L_p\langle 0,1\rangle$. It shows that there is a constructive linear isomorphism between any two L_p Lipschitz classes with the exponents α, $0<\alpha<1$. Again, the limit case $\alpha=1$, like for $p=\infty$ [3], is singular. We do not know whether the isomorphism exists for $1<p<\infty$ and $\alpha=1$. If $p=\infty$ and $\alpha=1$ then it exists but the known proof is not constructive [12].

Theorem 6 is a generalization of the main inequality proved in [1]. It plays an important role in the proofs of the absolute convergence theorems of Section 7.

Comparing the results of two parts of this work with corresponding results for the Haar system (see e.g. [5], [6], [7], [8], [16] and [17]) we find the far going similarity between the Haar and Franklin systems.

The Haar functions, the integrated Haar functions plus a constant function, i.e. the Schauder functions (cf. [5], Theorem 3), the orthonormalized Schauder functions, i.e. the Franklin functions, and finally the integrated Franklin functions plus a constant function (cf. Theorem 20) form the Schauder bases for the Banach space $C\langle 0,1\rangle$. This suggests strongly the possibility of continuing this process.

2. Notation. We are going to continue the notation developed in [1]. The partial sums of the function $x \epsilon L_1\langle 0,1\rangle$ of the Schauder, Haar and Franklin expansions are denoted by $\sigma_n(x;t)$, $H_n(x;t)$ and $S_n(x;t)$, respectively. Quite often we drope the argument t in the symbol denoting a function.

For a given $x \epsilon L_p\langle 0,1\rangle$, $1 \leqslant p \leqslant \infty$, we put

$$\|x\|_p = \Big(\int_0^1 |x(t)|^p\,dt\Big)^{1/p}, \qquad \|x\|_\infty = \|x\|.$$

The best approximation is defined as

$$E_n^{(p)}(x) = \inf_{\varphi} \|x-\varphi\|_p, \qquad E_n^{(\infty)}(x) = E_n(x),$$

where φ denotes a polynomial of the n-th degree corresponding to the partition $0 = t_0 < \ldots < t_n = 1$.

The *ordinary differences* and the *divided differences* are defined as follows:

$$\Delta_h x(t) = x(t+h)-x(t),$$

$$\Delta_h^2 x(t) = x(t+2h)-2x(t+h)+x(t),$$

$$[a_1, a_2; x] = \frac{x(a_2)-x(a_1)}{a_2-a_1},$$

$$[a_1, a_2, a_3; x] = \frac{[a_2, a_3; x]-[a_1, a_2; x]}{a_3-a_1}.$$

We have the obvious relation

$$(2h^2)^{-1}\Delta_h^2 x(t) = [t, t+h, t+2h; x].$$

The *moduli of continuity of the first and second order* for the functions in $L_p\langle 0,1\rangle$ we define as follows:

$$\omega_1^{(p)}(\delta; x) = \sup_{0<h\leqslant\delta}\Big(\int_0^{1-h} |\Delta_h x(s)|^p ds\Big)^{1/p}, \qquad 0<\delta<1;$$

$$\omega_2^{(p)}(\delta; x) = \sup_{0<h\leqslant\delta}\Big(\int_0^{1-2h} |\Delta_h^2 x(s)|^p ds\Big)^{1/p}, \qquad 0<2\delta<1.$$

We put also $\omega_k^{(\infty)}(\delta; x) = \omega_k(\delta; x)$ for $k = 1, 2$.

Now let us introduce the following Banach spaces ($0<\alpha\leqslant 1$):

$\langle L_p^{(\alpha)}, \|\ \|_p^{(\alpha)}\rangle$ is the space of all $x \epsilon L_p\langle 0,1\rangle$ such that $\omega_1^{(p)}(\delta; x) = O(\delta^\alpha)$, with the norm

$$\|x\|_p^{(\alpha)} = \max\Big[\|x\|_p, \sup_{0<h<1}\frac{1}{h^\alpha}\Big(\int_0^{1-h} |\Delta_h x(s)|^p ds\Big)^{1/p}\Big];$$

$\langle L_p^{(\alpha,0)}, \|\ \|_p^{(\alpha)}\rangle$ is a subspace of the last space of all x such that $\omega_1^{(p)}(\delta; x) = o(\delta^\alpha)$;

$\langle m_p, \|\ \|\rangle$ — the space of all sequences $a = (a_0, a_1, \ldots)$ such that ($1 \leqslant p \leqslant \infty$)

$$\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p} = O(1),$$

with the norm

$$\|a\| = \sup\Big[|a_0|, |a_1|, \Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p}, m = 0, 1, \ldots\Big];$$

$\langle m_p^{(0)}, \|\ \|\rangle$ — is a subspace of the last space of all a such that

$$\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p} = o(1).$$

All these spaces are complete. $\langle L_p^{(\alpha)}, \|\ \|_p^{(\alpha)}\rangle$ and $\langle m_p, \|\ \|\rangle$ are non-separable; $\langle L_p^{(\alpha,0)}, \|\ \|_p^{(\alpha)}\rangle$ and $\langle m_p^{(0)}, \|\ \|\rangle$ are separable subspaces, respectively.

3. Estimates for the single Franklin functions. To prove the main inequalities of this section we need the following three lemmas.

LEMMA 1. *Let $n = 2^m+\nu$, $m \geqslant 1$, $1 \leqslant \nu \leqslant 2^m$ and let $\eta = \frac{1}{2}(1-\eta_{2\nu-2}-\eta_{2\nu-1})$. Then $\eta > \frac{1}{4}$ and*

$$\frac{1}{9} < \min(\eta_{2\nu-2}, \eta_{2\nu-1}) < \max(\eta, \eta_{2\nu-2}, \eta_{2\nu-1}) < \frac{1}{3}.$$

Proof. In the proof we distinguish three cases corresponding to equations (16.I)-(16. III) of [1].

First case. Lemma 3 of [1] and formula (18) of [1] give that $\eta > \eta_{2\nu-2} > \eta_{2\nu-1}$.

Now, the inequality $\frac{1}{2}(\eta_{2\nu-2}+\eta_{2\nu-1}) > \frac{1}{6}$ (cf. [1], p. 151) implies that $\eta < \frac{1}{3}$. On the other hand, equations (16. I) of [1] give

$$\frac{1}{2}\eta_{2\nu-3}+3\eta_{2\nu-2}+\eta_{2\nu-1} = \frac{3}{4},$$

$$\eta_{2\nu-2}+4\eta_{2\nu-1}+\eta_{2\nu} = \frac{3}{4};$$

hence it follows that

$$\eta_{2\nu-1} = \frac{3}{22} - \frac{3}{11}\eta_{2\nu} + \frac{1}{22}\eta_{2\nu-3}.$$

According to (17) of [1], $-\eta_{2\nu} > 0$ and $\eta_{2\nu-3} > -\eta_{2\nu-2} > -\eta > -\frac{1}{3}$. Thus, $\eta_{2\nu-1} > \frac{3}{22} - \frac{1}{66} > \frac{1}{9}$.

A similar argument and the equation (see [1], p. 151)

$$\frac{1}{2}(\eta_{2\nu-1} + \eta_{2\nu-2}) = \frac{15}{88} - \frac{1}{11}\eta_{2\nu} - \frac{3}{44}\eta_{2\nu-3}$$

give

$$\frac{1}{2}(\eta_{2\nu-1} + \eta_{2\nu-2}) < \frac{15}{88} + \frac{1}{3}\frac{1}{11} + \frac{1}{3}\frac{3}{44} < \frac{1}{4},$$

hence $\eta > \frac{1}{4}$.

Second case. It is proved on p. 151 of [1] that $\eta_0 > \eta > \eta_1$. Since $\eta_2 < 0$ and (see (17) and p. 152 of [1])

$$\eta_0 = \frac{9}{28} + \frac{1}{7}\eta_2,$$

it follows that $\eta_0 < \frac{1}{3}$.

Now, let $m = 1$. Then, solving equations (16.II) of [1], we obtain $\eta_1 = \frac{1}{8} > \frac{1}{9}$.

For $m > 1$ we argue as follows. On the one hand (see [1], p. 152),

$$\eta_1 = \frac{1}{7}\left(\frac{3}{4} - 2\eta_2\right),$$

and, on the other hand, by (16.II) of [1] we have

$$\eta_1 + 4\eta_2 + \eta_3 = 0.$$

This combined with (17) of [1], gives $\eta_1 + 4\eta_2 < 0$ or $-2\eta_2 > \frac{1}{2}\eta_1$. Thus,

$$\eta_1 > \frac{1}{7}\left(\frac{3}{4} + \frac{1}{2}\eta_1\right), \quad \text{hence} \quad \eta_1 > \frac{3}{26} > \frac{1}{9}.$$

Using the equations given on p. 152 of [1], we obtain

$$\eta_1 + \eta_0 = \frac{3}{28} + \frac{9}{28} - \frac{1}{7}\eta_2 < \frac{12}{28} + \frac{1}{3}\frac{1}{7} < \frac{1}{2},$$

hence $\eta > \frac{1}{4}$.

Third case. According to (17) of [1], $\eta_{n-3} < 0$. It is shown on p. 152 of [1] that $\eta_{n-2} < \eta < \eta_{n-1}$ and that

$$\eta_{n-2} = \frac{3}{20} - \frac{1}{5}\eta_{n-3},$$

$$\eta_{n-1} = \frac{3}{10} + \frac{1}{10}\eta_{n-3}.$$

These equalities give

$$\eta_{n-2} > \frac{3}{20} > \frac{1}{9}, \quad \eta_{n-1} < \frac{3}{10} < \frac{1}{3}$$

and

$$\eta_{n-2} + \eta_{n-1} = \frac{3}{20} + \frac{3}{10} - \frac{1}{10}\eta_{n-3} < \frac{9}{20} + \frac{1}{30} < \frac{1}{2}.$$

The last inequality gives that $\eta > \frac{1}{4}$, and the proof is complete.

LEMMA 2. *Suppose that the sequence* $\{x_0, \ldots, x_{m+1}\}$ *satisfies the following equations:*

$$2x_0 + x_1 = 0,$$

$$x_{k-1} + 4x_k + x_{k+1} = 0, \quad k = 1, 2, \ldots, m.$$

Then,

$$x_k = (-1)^{k+m+1}\frac{\operatorname{ch} \alpha k}{\operatorname{ch}\alpha(m+1)}x_{m+1}$$

for $k = 0, \ldots, m$, *where* α *is the positive solution of the equation* $\operatorname{ch}\alpha = 2$.

LEMMA 3. *Let the sequence* $\{x_{n-m-1}, \ldots, x_n\}$ *satisfy the equations*

$$x_{k-1} + 4x_k + x_{k+1} = 0, \quad k = n-m, \ldots, n-1;$$

$$x_{n-1} + 2x_n = 0.$$

Then,

$$x_k = (-1)^{n+m+k+1}\frac{\operatorname{ch}\alpha(n-k)}{\operatorname{ch}\alpha(m+1)}x_{n-m-1}$$

for $k = n-m, \ldots, n$; α *is defined as in Lemma* 2.

Both these lemmas are a simple consequence of the elementary theory of linear difference equations (cf. [9], p. 241).

THEOREM 1. *Let* $n = 2^m + \nu$, $m \geqslant 0$, $1 \leqslant \nu \leqslant 2^m$, *and let* $\{t_0, \ldots, t_n\}$ *be defined as in* (10) *of* [1]. *Then,*

$$c_1 2^{m/2} e^{-\alpha|k-(2\nu-1)|} < (-1)^{k+1} f_n(t_k) < c_2 2^{m/2} e^{-\alpha|k-(2\nu-1)|},$$

where $e^a = 2+3^{1/2}$ (*i.e.* $a>0$ *and* $\operatorname{ch} a = 2$) *and*

$$c_1 = \frac{2+3^{1/2}}{3\cdot 3^{1/2}}, \quad c_2 = 4\cdot 3^{1/2}\,(2+3^{1/2}).$$

Proof. Using (26) of [1], one checks easily the statement for $m=0$, i.e. for $n=2$. Let $m \geqslant 1$. We know that (see Section 4 of [1])

$$2^{m/2}\lambda_{nn}^{-1} f_n(t_k) = \begin{cases} -\eta_k & \text{for } k = 0, \ldots, 2\nu-2; \\ \eta & \text{for } k = 2\nu-1; \\ -\eta_{k-1} & \text{for } k = 2\nu, \ldots, n. \end{cases} \tag{1}$$

The bounds for λ_{nn} we obtain from Lemma 7 of [1]. Namely,

$$2\cdot 3^{1/2}\cdot 2^m \leqslant \lambda_{nn} \leqslant 6\cdot 3^{1/2}\cdot 2^m. \tag{2}$$

The required inequalities for the η's we are going to establish separately in each of the three cases corresponding to the sets of equations (16) of [1].

First case. Applying Lemmas 2 and 3 to equations (16.I) of [1], we obtain

$$\eta_k = (-1)^k \frac{\operatorname{ch} ak}{\operatorname{ch} a(2\nu-2)}\,\eta_{2\nu-2} \quad \text{for} \quad k = 0, \ldots, 2\nu-3;$$

and

$$\eta_k = (-1)^{k+1} \frac{\operatorname{ch} a(n-1-k)}{\operatorname{ch} a(n-2\nu)}\,\eta_{2\nu-1} \quad \text{for} \quad k = 2\nu, \ldots, n-1.$$

This and Lemma 1 give

$$\frac{1}{9}\,\frac{\operatorname{ch} ak}{\operatorname{ch} a(2\nu-2)} < (-1)^k \eta_k < \frac{1}{3}\,\frac{\operatorname{ch} ak}{\operatorname{ch} a(2\nu-2)}$$

for $k = 0, \ldots, 2\nu-2$; and

$$\frac{1}{9}\,\frac{\operatorname{ch} a(n-k)}{\operatorname{ch} a(n-2\nu)} < (-1)^k \eta_{k-1} < \frac{1}{3}\,\frac{\operatorname{ch} a(n-k)}{\operatorname{ch} a(n-2\nu)}$$

for $k = 2\nu, \ldots, n$. Now, the inequalities $\frac{1}{2}e^t < \operatorname{ch} t < e^t$ $(t>0)$ imply

$$\frac{e^a}{18} e^{-a|k-(2\nu-1)|} < (-1)^k \eta_k < \frac{2e^a}{3} e^{-a|k-(2\nu-1)|} \tag{3}$$

for $k = 0, \ldots, 2\nu-2$; and

$$\frac{e^a}{18} e^{-a|k-(2\nu-1)|} < (-1)^k \eta_{k-1} < \frac{2e^a}{3} e^{-a|k-(2\nu-1)|} \tag{4}$$

for $k = 2\nu, \ldots, n$.

Moreover, Lemma 1 gives

$$\frac{e^a}{18} < \frac{1}{4} < \eta < \frac{1}{3} < \frac{2e^a}{3}. \tag{5}$$

Combining (1), (2), (3), (4) and (5) we complete the proof in the first case.

Second case. Equations (16.II) of [1], and Lemma 3 give

$$\eta_{k-1} = (-1)^k \frac{\operatorname{ch} a(n-k)}{\operatorname{ch} a(n-2)}\,\eta_1, \quad k = 2, \ldots, n,$$

Thus, Lemma 1 gives inequalities (5) and also

$$\begin{aligned} &\frac{e^a}{18} e^{-a|k-1|} < (-1)^k \eta_k < \frac{2e^a}{3} e^{-a|k-1|}, \quad k = 0; \\ &\frac{e^a}{18} e^{-a|k-1|} < (-1)^k \eta_{k-1} < \frac{2e^a}{3} e^{-a|k-1|}, \quad k = 2, \ldots, n. \end{aligned} \tag{6}$$

Now, (6), (5), (2) and (1) give the required result.

Third case. Lemma 2 and (16.III) of [1] give

$$\eta_k = (-1)^k \frac{\operatorname{ch} ak}{\operatorname{ch} a(n-2)}\,\eta_{n-2}$$

for $k = 0, \ldots, n-2$. Lemma 1 gives (5) and

$$\begin{aligned} &\frac{e^a}{18} e^{-a|k-(2\nu-1)|} < (-1)^k \eta_k < \frac{2e^a}{3} e^{-a|k-(2\nu-1)|}, \quad k = 0, \ldots, n-2; \\ &\frac{e^a}{18} e^{-a|k-(2\nu-1)|} < (-1)^k \eta_{k-1} < \frac{2e^a}{3} e^{-a|k-(2\nu-1)|}, \quad k = n. \end{aligned} \tag{7}$$

Finally, (7), (5) and (2) applied to (1) complete the proof of the theorem.

4. Dirichlet kernel and the pointwise convergence of the Fourier-Franklin series. Let us denote by $K_n(t,s)$ the Dirichlet kernel of the orthonormal Franklin system, i.e. let for $t, s \in \langle 0,1\rangle$

$$K_n(t,s) = \sum_{i=0}^{n} f_i(t) f_i(s), \quad n = 0, 1, \ldots$$

The n-th partial sum can be written now for $x \in L_1\langle 0,1\rangle$ in the form

$$S_n(x;t) = \int_0^1 K_n(t,s)\,x(s)\,ds. \tag{8}$$

Let φ be a polynomial of the n-th degree corresponding to the partition $\{t_0, \ldots, t_n\}$ defined in (10) of [1]. Then the definition of the Franklin functions and (8) give

$$\varphi(t_i) = \int_0^1 K_n(t_i, s)\varphi(s)\,ds, \quad i = 0, \ldots, n. \tag{9}$$

We introduce for fixed n the following notation:

$$\xi_i = \varphi(t_i), \quad a_{i,j} = K_n(t_i, t_j), \quad \delta_i = t_i - t_{i-1},$$

$$I_i = \langle t_{i-1}, t_i) \text{ for } i = 1, \ldots, n-1; \quad I_n = \langle t_{n-1}, t_n\rangle;$$

$$\alpha_i(t) = \frac{t-t_{i-1}}{t_i - t_{i-1}} \quad \text{and} \quad \beta_i(t) = \frac{t_i - t}{t_i - t_{i-1}} \quad \text{for} \quad t \in I_i.$$

In this section the letter a for $\log(2+3^{1/2})$ will be used, systematically.

Lemma 4. *For fixed n and i such that $n = 2^m + k$, $1 \leqslant k \leqslant 2^m$, $0 \leqslant i \leqslant n$, the following equations are satisfied:*

$$\left\{\begin{aligned} 2a_{0,i} + a_{1,i} &= \frac{6}{\delta}\delta_{0i}, \\ a_{j-1,i} + 4a_{j,i} + a_{j+1,i} &= \frac{6}{\delta}\delta_{ji}, \quad j = 1, \ldots, 2k-1; \\ \tfrac{1}{2}a_{j-1,i} + 3a_{j,i} + a_{j+1,i} &= \frac{3}{\delta}\delta_{ji}, \quad j = 2k; \\ a_{j-1,i} + 4a_{j,i} + a_{j+1,i} &= \frac{3}{\delta}\delta_{ji}, \quad j = 2k+1, \ldots, n-1; \\ a_{n-1,i} + 2a_{n,i} &= \frac{3}{\delta}\delta_{ni}; \end{aligned}\right. \tag{10}$$

where $\delta = 2^{-(m+1)}$ and δ_{ij} equals 1 if $i = j$ and 0 if $i \neq j$.

Proof. We notice that for $t \in I_h$

$$\varphi(t) = \alpha_h(t)\xi_h + \beta_h(t)\xi_{h-1}, \tag{11}$$

and that for $(t, s) \in I_j \times I_h$

$$\begin{aligned} K_n(t, s) = {} & \alpha_j(t)\alpha_h(s)a_{j,h} + \beta_j(t)\alpha_h(s)a_{j-1,h} + \alpha_j(t)\beta_h(s)a_{j,h-1} + \\ & + \beta_j(t)\beta_h(s)a_{j-1,h-1}. \end{aligned} \tag{12}$$

Moreover, one checks easily that

$$\int_{I_h} \alpha_h^2(t)\,dt = \int_{I_h} \beta_h^2(t)\,dt = \frac{\delta_h}{3},$$

$$\int_{I_h} \alpha_h(t)\beta_h(t)\,dt = \frac{\delta_h}{6}, \quad h = 1, \ldots, n.$$

This, (12), (11) and (9) give

$$\sum_{h=0}^{n} \xi_h \frac{\delta_h}{6}(a_{h-1,j} + 2a_{h,j}) + \frac{\delta_{h+1}}{6}(2a_{h,j} + a_{h+1,j}) = \xi_j, \tag{13}$$

where it is assumed that $\delta_0 = \delta_{n+1} = 0$. Since φ was arbitrary, we can specify the numbers $\xi_0, \ldots, \xi_n$ as follows: $\xi_j = \delta_{jl}$. Now (13) gives for $j = i$ and $l = 0, \ldots, n$

$$\frac{\delta_l}{6}(a_{l-1,i} + 2a_{l,i}) + \frac{\delta_{l+1}}{6}(2a_{l,i} + a_{l+1,i}) = \delta_{il}.$$

Remembering that $\delta_l = \delta$ for $l = 1, \ldots, 2k$, and $\delta_l = 2\delta$ for $l = 2k+1, \ldots, n$, we reduce easily the last equations to the form (10).

Lemma 5. *Let $\{A_0, \ldots, A_n\}$ be a given sequence of real numbers and let $n \geqslant 2$. Suppose that the numbers $\{a_0, \ldots, a_n\}$ satisfy the following equations:*

$$\left\{\begin{aligned} 2a_0 + a_1 &= A_0; \\ a_{i-1} + 4a_i + a_{i+1} &= A_i, \quad i = 1, \ldots, n-1; \\ a_{n-1} + 2a_n &= A_n. \end{aligned}\right. \tag{14}$$

Then,

$$a_i = \frac{1}{\operatorname{sh} a \operatorname{sh} an} \sum_{j=0}^{n} (-1)^{i+j} A_j \operatorname{ch} a \min(i, j) \operatorname{ch} a \min(n-i, n-j) \tag{15}$$

for $i = 0, \ldots, n$.

Proof. To derive (15) we use the well-known technic in the theory of linear difference equations with constant coefficients (e.g. Chapter III of [9]). For the sake of completeness we state the results of the main steps in solving (14).

The general solution of the homogeneous equation $x_{i-1} + 4x_i + x_{i+1} = 0$ is the sequence

$$(-1)^i(c_1 \operatorname{ch} ai + c_2 \operatorname{sh} ai) \tag{16}$$

where c_1 and c_2 are arbitrary constants.

A particular solution of the non-homogeneous equation $x_{i-1}+4x_i+x_{i+1} = A_i$ (we may assume that $A_i = 0$ for $i<0$ and $i>n$) is the sequence

$$\frac{1}{\operatorname{sh}\alpha}\sum_{j=0}^{i}(-1)^{i+j+1}A_j\operatorname{sh}\alpha(i-j). \tag{17}$$

Thus, the sum of (16) and (17) gives the general solution of the non-homogeneous equation.

The first boundary condition is $2x_0+x_1 = A_0$ and it implies that $c_2 = 0$. The second boundary condition $x_{n-1}+2x_n = A_n$ gives

$$c_1 = \frac{1}{\operatorname{sh}\alpha}\sum_{j=0}^{n}(-1)^j A_j\frac{\operatorname{ch}\alpha(n-j)}{\operatorname{sh}\alpha n}.$$

Combining these facts we get

$$a_i = \frac{1}{\operatorname{sh}\alpha}\sum_{j=0}^{n}(-1)^{i+j}A_j\frac{\operatorname{ch}\alpha i\operatorname{ch}\alpha(n-j)}{\operatorname{sh}\alpha n}+\frac{1}{\operatorname{sh}\alpha}\sum_{j=0}^{i}(-1)^{i+j+1}A_j\operatorname{sh}\alpha(i-j).$$

Applying to this formula for the indices $j\leqslant i$ the identity

$$\operatorname{ch}\alpha i\operatorname{ch}\alpha(n-j)-\operatorname{sh}\alpha(i-j)\operatorname{sh}\alpha n = \operatorname{ch}\alpha j\operatorname{ch}\alpha(n-i),$$

we obtain (15).

THEOREM 2. *Let $n = 2^m+k$, $m\geqslant 0$ and $1\leqslant k\leqslant 2^m$. Then*

$$K_n(t_i,t_j) = 3^{1/2}\cdot 2^{m+1}\gamma_n(-1)^{i+j}\varepsilon_{ij},$$

where

$$\gamma_n^{-1} = \operatorname{sh}\alpha n+\operatorname{sh}\alpha(n-2k)\operatorname{ch}\alpha 2k$$

and

$$\varepsilon_{ij} = \begin{cases} 2\operatorname{ch}\alpha\min(i,j)[\operatorname{ch}\alpha\min(n-i,n-j)+ \\ \quad +\operatorname{sh}\alpha(n-2k)\operatorname{sh}\alpha\min(2k-i,2k-j)] \ \text{for } 0\leqslant i,j\leqslant 2k-1; \\ \operatorname{ch}\alpha\min(n-i,n-j)[\operatorname{ch}\alpha\min(i,j)+\operatorname{ch}\alpha 2k\operatorname{ch}\alpha\min(i-2k,j-2k)] \\ \quad \text{for } i,j\geqslant 2k; \\ 2\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j) \ \text{for } i,j \text{ such that } \min(i,j) \\ \quad \leqslant 2k-1<\max(i,j). \end{cases}$$

Proof. Let us fix for the proof the index j and let $a_i = a_{i,j}$. According to Lemma 4 the numbers $a_0,\dots,a_n$ satisfy equations (14) with

$$A_i = \frac{6}{\delta}\delta_{ij}\varepsilon_i+\delta_{2k,i}\left(\frac{1}{2}a_{2k-1}+a_{2k}\right),$$

where ε_i equals 1 for $0\leqslant i\leqslant 2k-1$ and $\frac{1}{2}$ for $2k\leqslant i\leqslant n$. Substituting this into (15) we get for $i = 0,\dots,n$

$$\begin{aligned} a_i = {} & \frac{6\varepsilon_j(-1)^{i+j}}{\delta\operatorname{sh}\alpha\operatorname{sh}\alpha n}\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j)+ \\ & +\frac{(-1)^i}{\operatorname{sh}\alpha\operatorname{sh}\alpha n}\left(\frac{1}{2}a_{2k-1}+a_{2k}\right)\operatorname{ch}\alpha\min(i,2k)\operatorname{ch}\alpha\min(n-i,n-2k). \end{aligned} \tag{18}$$

We introduce the following notation:

$$A = \frac{1}{2}a_{2k-1}+a_{2k},\qquad B = 1-\frac{\operatorname{ch}\alpha(n-2k)\operatorname{sh}\alpha 2k}{2\operatorname{sh}\alpha n},$$

$$A_{ij} = \frac{6}{\delta}(-1)^{i+j}\frac{\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j)}{\operatorname{sh}\alpha\operatorname{sh}\alpha n}.$$

Using (18) with $i = 2k-1, 2k$ we obtain

$$a_{2k-1} = \varepsilon_j A_{2k-1,j}-A\frac{\operatorname{ch}\alpha(2k-1)\operatorname{ch}\alpha(n-2k)}{\operatorname{sh}\alpha\operatorname{sh}\alpha n},$$

$$a_{2k} = \varepsilon_j A_{2k,j}+A\frac{\operatorname{ch}\alpha 2k\operatorname{ch}\alpha(n-2k)}{\operatorname{sh}\alpha\operatorname{sh}\alpha n},$$

whence

$$A = \varepsilon_j\left(\frac{1}{2}A_{2k-1,j}+A_{2k,j}\right)+A\frac{\operatorname{ch}\alpha(n-2k)\operatorname{sh}\alpha 2k}{2\operatorname{sh}\alpha n}.$$

Thus,

$$AB = \varepsilon_j(\tfrac{1}{2}A_{2k-1,j}+A_{2k,j}),$$

and therefore

$$AB = \begin{cases} \dfrac{3}{\delta}(-1)^{j+1}\dfrac{\operatorname{ch}\alpha j\operatorname{sh}\alpha(n-2k)}{\operatorname{sh}\alpha n} & \text{if } j\leqslant 2k-1, \\ \dfrac{3}{2\delta}(-1)^j\dfrac{\operatorname{ch}\alpha(n-j)\operatorname{sh}\alpha 2k}{\operatorname{sh}\alpha n} & \text{if } j\geqslant 2k. \end{cases}$$

This and (18) give

$$a_{ij} = \frac{3(-1)^{i+j}}{\delta\operatorname{sh}\alpha}\gamma_n\varepsilon_{ij},$$

where ε_{ij} and γ_n are defined as in the statement, and the proof is complete.

THEOREM 3. *Let* $n = 2^m+k$, $m \geqslant 0$, $1 \leqslant k \leqslant 2^m$. *Then, for* $i, j = 0, \ldots, n$ *we have the following inequalities*:

$$\frac{1}{2}\cdot 3^{1/2}\cdot n\,\frac{\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j)}{\operatorname{sh}\alpha n} \leqslant (-1)^{j+i}K_n(t_i,t_j) \leqslant 8\cdot 3^{1/2}\cdot n\,\frac{\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j)}{\operatorname{sh}\alpha n},$$

$$0.5\cdot 3^{1/2}\frac{2^m}{(2+3^{1/2})^{|i-j|}} \leqslant (-1)^{i+j}K_n(t_i,t_j) \leqslant 12.5\cdot 3^{1/2}\frac{2^m}{(2+3^{1/2})^{|i-j|}}.$$

Proof. Notice that Theorem 2 gives

$$\operatorname{sh}\alpha n \leqslant \gamma_n^{-1} \leqslant 2\operatorname{sh}\alpha n$$

and

$$\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j) \leqslant \varepsilon_{ij} \leqslant 4\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j).$$

This proves the first part of the theorem.

Since $\operatorname{ch}t > \frac{1}{2}e^t$ and $2\operatorname{sh}t < e^t$, it follows that

$$(-1)^{i+j}K_n(t_i,t_j) \geqslant 3^{1/2}2^{m+1}\frac{\operatorname{ch}\alpha\min(i,j)\operatorname{ch}\alpha\min(n-i,n-j)}{2\operatorname{sh}\alpha n} \geqslant \tfrac{1}{2}\cdot 3^{1/2}\cdot 2^m e^{-\alpha|i-j|}.$$

It is a consequence of the assumptions that $n \geqslant 2$, and this implies

$$\gamma_n \leqslant (\operatorname{sh}\alpha n)^{-1} \leqslant 2e^{-\alpha n}(1-e^{-4\alpha})^{-1} \leqslant 2(1+\frac{1}{4}\alpha^{-1})e^{-\alpha n} < \frac{5}{2}e^{-\alpha n}.$$

Moreover, Theorem 2 gives: $\varepsilon_{ij} < \frac{5}{2}e^{\alpha(n-|i-j|)}$. These inequalities and Theorem 2 complete the proof.

In the proof of the convergence theorem we are going to employ the following generalization of the I. P. Natanson Lemma ([11], p. 243):

LEMMA 6 (Taberski [15]). *Let* $g(t)$ *be a function of bounded variation in every interval* $\langle a+\varepsilon, b\rangle$, $0<\varepsilon<b-a$, *such that*

$$\int_a^b v(s)\,ds < \infty,$$

where

$$v(s) = \operatorname*{var}_{s\leqslant t\leqslant b} g(t) \quad (a \leqslant s < b),\ v(b) = 0.$$

Then, if

$$M = \sup_{0<h\leqslant b-a}\left|\frac{1}{h}\int_a^{a+h} f(t)\,dt\right| < \infty \qquad (f\in L\langle a,b\rangle),$$

the improper Lebesgue integral

$$I = \int_{a+}^b f(t)g(t)\,dt$$

exists and

$$|I| \leqslant M\int_a^b [v(s)+|g(b)|]\,ds.$$

LEMMA 7. *Let* $n \geqslant 2$, $n = 2^m+k$, $1 \leqslant k \leqslant 2^m$. *Then*

$$|K_n(t,s)| \leqslant ane^{-an|t-s|/2}$$

holds for $t, s\in\langle 0,1\rangle$ *with* $a = 12.5\cdot 3^{1/2}(2+3^{1/2})$.

Proof. Notice that

$$|i-j| \geqslant \tfrac{1}{2}n|t_i-t_j|, \qquad i,j = 0,\ldots,n, \tag{19}$$

where $\{t_0,\ldots,t_n\}$ is the partition defined by (10) of [1].

Let $t \leqslant s$. Then there exist i and j such that $t\in I_i$, $s\in I_j$ and $i \leqslant j$. Formula (12) gives

$$|K_n(t,s)| \leqslant \max(|a_{i,j}|, |a_{i-1,j}|, |a_{i,j-1}|, |a_{i-1,j-1}|).$$

Thus, Theorem 3 and (19) imply

$$|K_n(t,s)| \leqslant 12.5\cdot 3^{1/2}e^{2\alpha}2^m e^{-\alpha(j-i+1)} \leqslant ane^{-an(t_j-t_{i-1})/2} \leqslant ane^{-an(s-t)/2}.$$

Since $K_n(t,s) = K_n(s,t)$, the proof is complete.

LEMMA 8. *There exists an absolute constant* M *such that the estimates*

$$\int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds \leqslant M \quad \text{and} \quad \int_0^t \operatorname*{var}_{0\leqslant u\leqslant s} K_n(t,u)\,ds \leqslant M$$

hold for all $n \geqslant 0$ *and* $t\in\langle 0,1\rangle$.

Proof. The proof in the case of the interval $\langle 0,t\rangle$ goes very much like in the case of the interval $\langle t,1\rangle$. Therefore we are going to prove the first inequality only.

Let $t_q \leqslant t < t_{q+1}$. Then,

$$(20)\qquad \int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds \leqslant \int_{t_q}^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds = \sum_{j=q+1}^{n} \int_{t_{j-1}}^{t_j} \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds$$

$$\leqslant \sum_{j=q+1}^{n} \int_{t_{j-1}}^{t_j} \operatorname*{var}_{t_{j-1}\leqslant u\leqslant 1} K_n(t,u)\,ds = \sum_{j=q+1}^{n} \delta_j \sum_{i=j}^{n} \operatorname*{var}_{u\in I_i} K_n(t,u).$$

Formula (12) and Theorem 3 give

$$\operatorname*{var}_{u\in I_i} K_n(t,u) = \int_{t_{i-1}}^{t_i} \left|\frac{\partial}{\partial u} K_n(t,u)\right| du$$

$$= \frac{1}{\delta_i} \int_{t_{i-1}}^{t_i} |a_q(t)\,a_{q,i} + \beta_q(t)\,a_{q-1,i} - a_q(t)\,a_{q,i-1} - \beta_q(t)\,a_{q-1,i-1}|\,du$$

$$\leqslant \max(|a_{q-1,i}|, |a_{q,i-1}|) + \max(|a_{q,i}|, |a_{q-1,i-1}|) = O(ne^{-a(i-q)}),$$

whence

$$\sum_{i=j}^{n} \operatorname*{var}_{u\in I_i} K_n(t,u) = O(ne^{-a(j-q)}).$$

Combinig this with (20) we obtain

$$\int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds = O\Big(\sum_{j=q+1}^{n} e^{-a(j-q)}\Big) = O(1).$$

THEOREM 4. *Let $x\in L_1\langle 0,1\rangle$ and let t, $t\in\langle 0,1\rangle$, be such that*

$$x(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x(s)\,ds.$$

Then,

$$x(t) = \sum_{n=0}^{\infty} a_n f_n(t), \qquad a_n = \int_0^1 x(s) f_n(s)\,ds, \qquad n = 0,1,\ldots$$

Proof. We are going to estimate the integral

$$I_n = \int_0^1 K_n(t,s)[x(s)-x(t)]\,ds$$

by splitting it in suitable way into three parts.

The argument given below can be used easily to prove the theorem in the case when t is one of the end points of $\langle 0,1\rangle$. Therefore we may assume that $t\in(0,1)$.

Let ε be an arbitrary positive number, and let δ be such that $0 < t-\delta < t < t+\delta < 1$ and

$$\sup_{0<|h|<\delta} \left|\frac{1}{h}\int_t^{t+h} [x(s)-x(t)]\,ds\right| < \varepsilon.$$

We decompose I_n as follows:

$$I_n = \int_{|s-t|>\delta} + \int_{t-\delta}^{t} + \int_t^{t+\delta}.$$

Lemma 7 gives the bound for the first integral:

$$\left|\int_{|s-t|>\delta} K_n(t,s)[x(s)-x(t)]\,ds\right| \leqslant \sup_{|s-t|>\delta} |K_n(t,s)|\,(\|x\|_1 + |x(t)|)$$

$$\leqslant a(\|x\|_1 + |x(t)|)\,ne^{-an\delta/2}.$$

Therefore, there exists n_1 such that

$$\left|\int_{|s-t|>\delta}\right| < \varepsilon \quad \text{for} \quad n > n_1.$$

Now, let us estimate the integral over $(t, t+\delta)$. Lemmas 6, 7 and 8 give

$$\left|\int_t^{t+\delta} K_n(t,s)[x(s)-x(t)]\,ds\right|$$

$$\leqslant \varepsilon \int_t^{t+\delta} [\operatorname*{var}_{s\leqslant u\leqslant t+\delta} K_n(t,u) + |K_n(t,t+\delta)|]\,ds$$

$$\leqslant \varepsilon \int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n(t,u)\,ds + \varepsilon\delta|K_n(t,t+\delta)|$$

$$\leqslant \varepsilon M + \varepsilon\delta a n e^{-an\delta/2} < \varepsilon\Big(M + \frac{2a}{ae}\Big) = \varepsilon M_1.$$

Similar argument shows that

$$\left|\int_{t-\delta}^{t} K_n(t,s)[x(s)-x(t)]\,ds\right| \leqslant \varepsilon M_1.$$

Thus, for $n > n_1$ we have

$$|I_n| < \varepsilon(2M_1+1).$$

Since M_1 is a numerical constant, the proof is complete.

5. Inequalities. The following Lemma will be very useful in the next section:

LEMMA 9 (P. L. Uljanov, [17], p. 384). *Let* $x \in L_p\langle a, b\rangle$, $1 \leqslant p < \infty$, $-\infty < a < b < \infty$. *Then*

$$\int_a^b \int_a^b |x(t)-x(s)|^p dt ds \leqslant 2\int_0^{b-a} dh \int_a^{b-h} |\Delta_h x(t)|^p dt$$
$$\leqslant 2\int_0^{b-a} [\omega_1^{(p)}(h; x)]^p dh.$$

THEOREM 5 (1). *Let* φ *be a polynomial of the* n*-th degree corresponding to the partition* $0 = t_0 < \ldots < t_n = 1$ *and let* $\delta = \min_i (t_i - t_{i-1})$. *Then, if* $1 \leqslant p \leqslant \infty$, *we have*

$$\left(\int_0^{1-h} |\Delta_h \varphi(s)|^p ds\right)^{1/p} \leqslant 4\min\left(\frac{1}{2}, \frac{h}{\delta}\right)\|\varphi\|_p \tag{21}$$

for $0 < h < 1$, *and*

$$\left(\int_0^{1-2h} |\Delta_h^2 \varphi(s)|^p ds\right)^{1/p} \leqslant 8\min\left[\frac{1}{2}, \left(\frac{h}{\delta}\right)^{1+1/p}\right]\|\varphi\|_p \tag{22}$$

for $0 < 2h < 1$.

Proof. We may assume that p is finite. Notice that the inequality

$$\int_0^{1-h} |\Delta_h \varphi(s)|^p ds \leqslant 2^p \|\varphi\|_p^p \tag{23}$$

is satisfied always. Moreover, we have the following chain of inequalities:

$$\int_0^{1-h} |\Delta_h \varphi(s)|^p ds = h^p \int_0^{1-h} \left|\frac{1}{h}\int_s^{s+h} \varphi'(t)\,dt\right|^p ds$$
$$\leqslant h^{p-1}\int_0^{1-h} ds \int_s^{s+h} |\varphi'(t)|^p dt = h^{p-1}\int_0^{1-h} ds \int_0^h |\varphi'(s+t)|^p dt$$
$$= h^{p-1}\int_0^h dt \int_0^{1-h} |\varphi'(s+t)|^p ds = h^{p-1}\int_0^h dt \int_t^{1-h+t} |\varphi'(s)|^p ds \leqslant h^p \|\varphi'\|_p^p.$$

(1) The author was kindly informed by Dr P. L. Golubov that a special case ($p=\infty$) of Theorem 2 of [1] was proved earlier by K. M. Shaidukov in [14].

This and Theorem 2 of [1], imply

$$\int_0^{1-h} |\Delta_h \varphi(s)|^p ds \leqslant \left(4\frac{h}{\delta}\right)^p \|\varphi\|_p^p. \tag{24}$$

Combining (23) and (24) we obtain (21).

To prove (22) first of all we notice that

$$\int_0^{1-2h} |\Delta_h^2 \varphi(s)|^p ds \leqslant 4^p \|\varphi\|_p^p. \tag{25}$$

The next step is to show that

$$\int_0^{1-2h} |\Delta_h^2 \varphi(s)|^p ds \leqslant h^p \int_0^{1-h} |\Delta_h \varphi'(s)|^p ds. \tag{26}$$

This can be proved as follows:

$$\int_0^{1-2h} |\Delta_h^2 \varphi(s)|^p ds = h^p \int_0^{1-2h} \left|\frac{1}{h}\int_s^{s+h} \varphi'(t+h)\,dt - \frac{1}{h}\int_s^{s+h} \varphi'(t)\,dt\right|^p ds$$
$$= h^p \int_0^{1-2h} \left|\frac{1}{h}\int_s^{s+h} \Delta_h \varphi'(t)\,dt\right|^p ds \leqslant h^{p-1}\int_0^{1-2h} ds \int_s^{s+h} |\Delta_h \varphi'(t)|^p dt$$
$$= h^{p-1}\int_0^{1-2h} ds \int_0^h |\Delta_h \varphi'(s+t)|^p dt = h^{p-1}\int_0^h dt \int_0^{1-2h} |\Delta_h \varphi'(s+t)|^p ds$$
$$= h^{p-1}\int_0^h dt \int_t^{1-2h+t} |\Delta_h \varphi'(s)|^p ds \leqslant h^{p-1}\int_0^h dt \int_0^{1-h} |\Delta_h \varphi'(s)|^p ds.$$

Now, if $h \leqslant \delta$ and $s_i \in (t_{i-1}, t_i)$, then

$$\int_0^{1-h} |\Delta_h \varphi'(t)|^p dt \tag{27}$$
$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i - h} |\Delta_h \varphi'(t)|^p dt + \sum_{i=1}^{n-1} \int_{t_i - h}^{t_i} |\Delta_h \varphi'(t)|^p dt = \sum_{i=1}^{n-1} \int_{t_i-h}^{t_i} |\Delta_h \varphi'(t)|^p dt$$
$$= \sum_{i=1}^{n-1} \int_{t_i-h}^{t_i} |\varphi'(s_{i+1}) - \varphi'(s_i)|^p dt = h\sum_{i=1}^{n-1} |\varphi'(s_{i+1}) - \varphi'(s_i)|^p$$
$$\leqslant 2^p \frac{h}{\delta}\sum_{i=1}^{n} \delta_i |\varphi'(s_i)|^p = 2^p \frac{h}{\delta}\int_0^1 |\varphi'(t)|^p dt.$$

Applying Theorem 2 of [1] to the last term in (27) we obtain

$$\int_0^{1-h} |\Delta_h \varphi'(t)|^p\,dt \leqslant 8^p \frac{h}{\delta^{p+1}}\, \|\varphi\|_p^p. \tag{28}$$

The combination of (26) and (28) gives

$$\int_0^{1-2h} |\Delta_h^2 \varphi(t)|^p\,dt \leqslant 8^p \left(\frac{h}{\delta}\right)^{p+1} \|\varphi\|_p^p. \tag{29}$$

Finally, (29) and (25) give (22).

Lemma 10. *Let $0 < \alpha \leqslant 1$, $n \geqslant 2$ and let $1 \leqslant p \leqslant \infty$. Then*

$$\frac{1}{16} n^\alpha \|f_n\|_p \leqslant \sup_{0<2h\leqslant 1} h^{-\alpha}\Big(\int_0^{1-2h} |\Delta_h^2 f_n(t)|^p\,dt\Big)^{1/p} \tag{30}$$

$$\leqslant 2 \sup_{0<h\leqslant 1} h^{-\alpha}\Big(\int_0^{1-h} |\Delta_h f_n(t)|^p\,dt\Big)^{1/p} \leqslant 16 n^\alpha \|f_n\|_p.$$

Proof. We notice that Minkowski's inequality gives

$$\int_0^{1-2h} |\Delta_h^2 f_n(t)|^p\,dt \leqslant 2^p \int_0^{1-h} |\Delta_h f_n(t)|^p\,dt.$$

Let $n = 2^m + k$, $m \geqslant 0$, $1 \leqslant k \leqslant 2^m$, and let $\delta = 2^{-(m+1)}$. Then, if $2h \leqslant \delta$, Theorem 5 gives

$$h^{-\alpha}\Big(\int_0^{1-h} |\Delta_h f_n(t)|^p\,dt\Big)^{1/p} \leqslant h^{-\alpha} 4 \frac{h}{\delta} \|f_n\|_p$$

$$\leqslant 8\cdot 2^{\alpha m}\|f_n\|_p \leqslant 8 n^\alpha \|f_n\|_p.$$

If $\delta < 2h \leqslant 2$, then

$$h^{-\alpha}\Big(\int_0^{1-h} |\Delta_h f_n(t)|^p dt\Big)^{1/p} \leqslant 2h^{-\alpha}\|f_n\|_p^p$$

$$\leqslant 2^{1+\alpha}\delta^{-\alpha}\|f_n\|_p^p = 2^{1+2\alpha}2^{\alpha m}\|f_n\|_p^p \leqslant 8n^\alpha\|f_n\|_p^p.$$

It remains to prove the left-hand side inequality in (30). Let $h_0 = \frac{1}{2}\delta$ and let

$$K = \sup_{0<2h\leqslant 1} h^{-\alpha}\Big(\int_0^{1-2h} |\Delta_h^2 f_n(t)|^p\,dt\Big)^{1/p}.$$

The function $\Delta_{h_0}^2 f_n(t)$ vanishes at $t = t_{2k-2}$ and it is linear in $\langle t_{2k-2}, t_{2k-2}+h_0\rangle$. Therefore,

$$K \geqslant h_0^{-\alpha}\Big(\int_0^{1-2h_0} |\Delta_{h_0}^2 f_n(t)|^p\,dt\Big)^{1/p} \geqslant h_0^{-\alpha}\Big(\int_{t_{2k-2}}^{t_{2k-2}+h_0} |\Delta_{h_0}^2 f_n(t)|^p\,dt\Big)^{1/p}$$

$$\geqslant h_0^{-\alpha+1/p}(p+1)^{-1/p}|\Delta_{h_0}^2 f_n(t_{2k-2}+h_0)| = h_0^{-\alpha+1/p}(p+1)^{-1/p}\tfrac{1}{2}|\Delta_\delta^2 f_n(t_{2k-2})|.$$

This and (25) of [1], imply that

$$K \geqslant h_0^{-\alpha+1/p}(p+1)^{-1/p}\frac{\lambda_{nn}}{2\cdot 2^{m/2}},$$

whence by Lemma 5 of [1], we get

$$K \geqslant h_0^{-\alpha+1/p}(p+1)^{-1/p}\frac{1}{2\cdot 2^{m/2}}\,\frac{\|f_n\|_p}{4\|\varphi_n\|_p}$$

$$= h_0^{-\alpha+1/p}\frac{1}{4\cdot 2^{m/2}}2^{m(1/2+1/p)}\|f_n\|_p \geqslant \frac{1}{16}n^\alpha\|f_n\|_p.$$

This completes the proof of (30).

Theorem 6. *Let $1 < p < \infty$, $1/p+1/q = 1$, and let $\{a_2, a_3, \ldots\}$ be a sequence of real numbers. Then for $m \geqslant 0$ the following inequalities are satisfied:*

$$\Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p \leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} |f_n|\Big\|^{1/q}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)^{1/p}\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p}, \tag{31}$$

$$\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p} \leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} |f_n|\Big\|^{1/p}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)^{1/q}\Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p, \tag{32}$$

$$\Big\|\sum_{2^m+1}^{2^{m+1}} |f_n|\Big\|^{1/q}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)^{1/p} \leqslant 2^5 3^{1/2} 2^m 2^{m(1/2-1/p)}. \tag{33}$$

Proof. Let us put

$$p_n(t) = \frac{|f_n(t)|}{b_m(t)}, \qquad b_m(t) = \sum_{2^m+1}^{2^{m+1}} |f_n(t)|.$$

Then (31) is a consequence of the following inequalities:

$$\Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p^p = \int_0^1 |b_m(t)|^p\Big(\sum_{2^m+1}^{2^{m+1}} p_n(t)|a_n|\Big)^p dt \leqslant \int_0^1 |b_m(t)|^p\Big(\sum_{2^m+1}^{2^{m+1}} p_n(t)|a_n|^p\Big) dt$$

$$= \int_0^1 |b_m(t)|^{p-1}\Big(\sum_{2^m+1}^{2^{m+1}} |f_n(t)|\,|a_n|^p\Big) dt \leqslant \|b_m\|^{p-1}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)\sum_{2^m+1}^{2^{m+1}} |a_n|^p.$$

Now, let $\{c_n\}$ be an arbitrary sequence of real numbers. Then the orthogonality of the Franklin functions and Hölder inequality give

$$\Big|\sum_{2^m+1}^{2^{m+1}} a_n c_n\Big| = \Big|\int_0^1 \Big(\sum_{2^m+1}^{2^{m+1}} a_n f_n(t)\Big)\Big(\sum_{2^m+1}^{2^{m+1}} c_n f_n(t)\Big)dt\Big| \leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} a_n f_n\Big\|_p \Big\|\sum_{2^m+1}^{2^{m+1}} c_n f_n\Big\|_q.$$

In particular, using this inequality with $c_n = \operatorname{sgn} a_n |a_n|^{p-1}$ and then applying (31) we obtain

$$\begin{aligned}\sum_{2^m+1}^{2^{m+1}} |a_n|^p &= \Big|\sum_{2^m+1}^{2^{m+1}} a_n c_n\Big| \\ &\leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p \|b_m\|^{1/p}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)^{1/q}\Big(\sum_{2^m+1}^{2^{m+1}} |c_n|^q\Big)^{1/q} \\ &= \Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p \|b_m\|^{1/p}\big(\max_{2^m<n\leqslant 2^{m+1}} \|f_n\|_1\big)^{1/q}\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/q}.\end{aligned}$$

Consequently, dividing each side of these inequalities by

$$\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/q}$$

we obtain (32).

Inequality (33) is a consequence of Theorem 5 of [1] and of Lemmas 5 and 7 of [1]. Indeed, the left-hand side of (33) is less than

$$(2^5 3^{1/2} 2^{m/2})^{1/q}(6\cdot 3^{1/2} 2^{-m/2})^{1/p} = 2^5 3^{1/2}(3\cdot 2^{-4})^{1/p} 2^{(1/q-1/p)m/2} < 2^5 3^{1/2} 2^{m(1/2-1/p)}.$$

Corollary. *Let $1\leqslant p\leqslant\infty$ and let $\{a_n\}$ be an arbitrary sequence of real numbers. Then, for $m\geqslant 1$ we have*

$$\begin{aligned}(34)\quad 2^{-5}3^{-1/2}2^{m(1/2-1/p)}\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p} &\leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} a_n f_n\Big\|_p \leqslant \Big\|\sum_{2^m+1}^{2^{m+1}} |a_n f_n|\Big\|_p \\ &\leqslant 2^5 3^{1/2} 2^{m(1/2-1/p)}\Big(\sum_{2^m+1}^{2^{m+1}} |a_n|^p\Big)^{1/p}.\end{aligned}$$

Remark. Inequality (34) is a generalization of Theorem 5 of [1].

6. Approximation. The following result is essentially due to Uljanov [17]. Since our proof is simpler and gives much better constant (6 instead of 24), it is presented here.

Theorem 7. *Let $x\in L_p\langle 0,1\rangle$, $1\leqslant p<\infty$. Then,*

$$\|x-H_n(x)\|_p \leqslant 6\omega_1^{(p)}\Big(\frac{1}{n};x\Big) \quad \textit{for} \quad n\geqslant 1.$$

Proof. If $n=1$ the result follows immediately from Lemma 9.

Let $n=2^m+k$, $m\geqslant 0$, $1\leqslant k\leqslant 2^m$, and let $\{t_i\}$ be defined by (10) of [1]; $\delta_i = t_i - t_{i-1}$, $\delta = 2^{-(m+1)}$. Then by Lemma 9 we obtain

$$\begin{aligned}\|x-H_n(x)\|_p^p &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |x(s)-H_n(x;s)|^p ds \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i}\Big|\frac{1}{\delta_i}\int_{t_{i-1}}^{t_i}[x(t)-x(s)]dt\Big|^p ds \\ &\leqslant \sum_{i=1}^n \frac{1}{\delta_i}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} |x(t)-x(s)|^p dt ds \\ &\leqslant 2\sum_{i=1}^n \frac{1}{\delta_i}\int_0^{\delta_i} dh \int_{t_{i-1}}^{t_i-h} |\Delta_h x(t)|^p dt \\ &= \frac{2}{\delta}\int_0^{\delta} dh\Big(\sum_{i=1}^{2k}\int_{t_{i-1}}^{t_i-h} |\Delta_h x(t)|^p dt\Big) + \\ &\quad + \frac{1}{\delta}\int_0^{2\delta} dh\Big(\sum_{i=2k+1}^{n}\int_{t_{i-1}}^{t_i-h} |\Delta_h x(t)|^p dt\Big) \\ &\leqslant \frac{2}{\delta}\int_0^{\delta} dh\int_0^{t_{2k}-h} |\Delta_h x(t)|^p dt + \frac{1}{\delta}\int_0^{2\delta} dh\int_{t_{2k}}^{t_n-h} |\Delta_h x(t)|^p dt \\ &\leqslant \frac{2}{\delta}\int_0^{\delta} dh\int_0^{1-h} |\Delta_h x(t)|^p dt + \frac{1}{\delta}\int_0^{2\delta} dh\int_0^{1-h} |\Delta_h x(t)|^p dt \\ &\leqslant 2[\omega_1^{(p)}(\delta;x)]^p + 2[\omega_1^{(p)}(2\delta;x)]^p \\ &\leqslant 2(1+2^p)\Big[\omega_1^{(p)}\Big(\frac{1}{2^{m+1}};x\Big)\Big]^p \leqslant \Big[6\omega_1^{(p)}\Big(\frac{1}{n};x\Big)\Big]^p.\end{aligned}$$

LEMMA 11. *Let $n = 2^m + k$, $m \geqslant 0$, $1 \leqslant k \leqslant 2^m$; let $\{t_i\}$ be the partition corresponding to n and let $\delta_i = t_i - t_{i-1}$, $\delta = 2^{-(m+1)}$. For a given $x \in L_1\langle 0, 1\rangle$ we define a polynomial φ of the n-th degree as follows:*

$$\varphi(t_i) = \xi_i, \quad i = 0, \ldots, n;$$

$$m_i = \frac{1}{\delta_i}\int_{t_{i-1}}^{t_i} x(t)\,dt, \quad i = 1, \ldots, n;$$

$\xi_i = \frac{1}{2}(m_{i+1} + m_i)$ *for* $i = 1, \ldots, n-1$; $\quad \xi_0 = m_1$ *and* $\xi_n = m_n$.

Then, for $1 \leqslant p < \infty$,

$$\|\varphi - H_n(x)\|_p \leqslant \Big(2\int_0^{1-\delta} |\Delta_\delta x(t)|^p\,dt\Big)^{1/p} \leqslant 2^{1/p}\omega_1^{(p)}\Big(\frac{1}{n}; x\Big).$$

Proof. Notice that

$$m_i - \xi_i = \tfrac{1}{2}(m_i - m_{i+1}) \quad \text{if} \quad i = 1, \ldots, n-1;$$

and

$$m_i - \xi_{i-1} = \tfrac{1}{2}(m_i - m_{i-1}) \quad \text{if} \quad i = 2, \ldots, n;$$

and that for $i = 1, \ldots, n$ we have

$$\int_{t_{i-1}}^{t_i} |H_n(x;t) - \varphi(t)|^p\,dt = \int_{t_{i-1}}^{t_i} |m_i - \varphi(t)|^p\,dt$$
$$= \int_{t_{i-1}}^{t_i} \left|\frac{t - t_{i-1}}{\delta_i}(m_i - \xi_i) + \frac{t_i - t}{\delta_i}(m_i - \xi_{i-1})\right|^p dt$$
$$\leqslant \delta_i \max(|m_i - \xi_i|^p, |m_i - \xi_{i-1}|^p) \leqslant \delta_i(|m_i - \xi_i|^p + |m_i - \xi_{i-1}|^p).$$

This gives

$$\|H_n(x) - \varphi\|_p^p \leqslant \sum_{i=1}^{n}\delta_i|m_i - \xi_i|^p + \sum_{i=1}^{n}\delta_i|m_i - \xi_{i-1}|^p \tag{35}$$
$$= \sum_{i=1}^{n-1}\delta_i|\tfrac{1}{2}(m_i - m_{i+1})|^p + \sum_{i=2}^{n}\delta_i|\tfrac{1}{2}(m_i - m_{i-1})|^p$$
$$= 2^{-p}\sum_{i=1}^{n-1}(\delta_i + \delta_{i+1})|m_i - m_{i+1}|^p$$
$$= 2^{-p}\sum_{i=1}^{n-1}(\delta_i + \delta_{i+1})^{p+1}|[t_{i-1}, t_i, t_{i+1}; X]|^p,$$

where

$$X(t) = \int_0^t x(s)\,ds.$$

Write

$$A = \sum_{i=1}^{n-1}(\delta_i + \delta_{i+1})^{p+1}|[t_{i-1}, t_i, t_{i+1}; X]|^p,$$

$$A_1 = \sum_{i=1}^{2k-1}(\delta_i + \delta_{i+1})^{p+1}|[t_{i-1}, t_i, t_{i+1}; X]|^p,$$

$$A_2 = (\delta_{2k} + \delta_{2k+1})^{p+1}|[t_{2k-1}, t_{2k}, t_{2k+1}; X]|^p,$$

$$A_3 = \sum_{i=2k+1}^{n-1}(\delta_i + \delta_{i+1})^{p+1}|[t_{i-1}, t_i, t_{i+1}; X]|^p.$$

We notice that there are three possibilities:

$$A = \begin{cases} A_1 & \text{if} \quad k = 2^m, \\ A_1 + A_2 & \text{if} \quad k = 2^m - 1, \\ A_1 + A_2 + A_3 & \text{if} \quad k < 2^m - 1. \end{cases} \tag{36}$$

Without any restrictions on k, i.e. for $1 \leqslant k \leqslant 2^m$, we have

$$A_1 = \sum_{i=1}^{2k-1}(2\delta)^{p+1}(2\delta)^{-p}\left|\frac{1}{\delta}\int_{t_{i-1}}^{t_i}\Delta_\delta x(t)\,dt\right|^p \tag{37}$$
$$\leqslant 2\sum_{i=1}^{2k-1}\int_{t_{i-1}}^{t_i}|\Delta_\delta x(t)|^p\,dt = 2\int_0^{t_{2k-1}}|\Delta_\delta x(t)|^p\,dt.$$

If k is such that $1 \leqslant k \leqslant 2^m - 1$, then the well-known technic of divided differences (see Popoviciu [13], p. 7) gives

$$A_2 = (3\delta)^{p+1}|[t_{2k-1}, t_{2k}, t_{2k+1}; X]|^p \tag{38}$$
$$= (3\delta)^{p+1}\left|\frac{2}{3}[t_{2k-1}, t_{2k-1}+\delta, t_{2k-1}+2\delta; X] + \frac{1}{3}[t_{2k-1}+\delta, t_{2k-1}+2\delta, t_{2k-1}+3\delta; X]\right|^p$$
$$= (3\delta)^{p+1}\left|\frac{2}{3}\,\frac{1}{2\delta^2}\Delta_\delta^2 X(t_{2k-1}) + \frac{1}{3}\,\frac{1}{2\delta^2}\Delta_\delta^2 X(t_{2k-1}+\delta)\right|^p$$
$$\leqslant (2\delta)^{-p}(3\delta)^{p+1}\left[\frac{2}{3}\left|\frac{1}{\delta}\Delta_\delta^2 X(t_{2k-1})\right|^p + \frac{1}{3}\left|\frac{1}{\delta}\Delta_\delta^2 X(t_{2k-1}+\delta)\right|^p\right]$$
$$\leqslant 3\Big(\frac{3}{2}\Big)^p\left[\frac{2}{3}\int_{t_{2k-1}}^{t_{2k}}|\Delta_\delta x(t)|^p\,dt + \frac{1}{3}\int_{t_{2k}}^{t_{2k}+\delta}|\Delta_\delta x(t)|^p\,dt\right].$$

To estimate A_3 we need the following identities and inequalities (cf. [13]).

Let $2k+1 \leqslant i \leqslant n-1$. Then

$$[t_{i-1}, t_i, t_{i+1}; X] = [t_{i-1}, t_{i-1}+2\delta, t_{i-1}+4\delta; X]$$
$$= \frac{1}{4}[t_{i-1}, t_{i-1}+\delta, t_{i-1}+2\delta; X]+$$
$$+\frac{1}{2}[t_{i-1}+\delta, t_{i-1}+2\delta, t_{i-1}+3\delta; X]+$$
$$+\frac{1}{4}[t_{i-1}+2\delta, t_{i-1}+3\delta, t_{i-1}+4\delta; X],$$

whence

$$|[t_{i-1}, t_i, t_{i+1}; X]|$$
$$\leqslant \frac{1}{4}\frac{1}{2\delta^2}|\Delta_\delta^2 X(t_{i-1})| + \frac{1}{2}\frac{1}{2\delta^2}|\Delta_\delta^2 X(t_{i-1}+\delta)| + \frac{1}{4}\frac{1}{2\delta^2}|\Delta_\delta^2 X(t_{i-1}+2\delta)|.$$

This gives

(39) $$|[t_{i-1}, t_i, t_{i+1}; X]|^p \leqslant \frac{1}{4}(2\delta)^{-p}\frac{1}{\delta}\int_{t_{i-1}}^{t_{i-1}+\delta}|\Delta_\delta x(t)|^p dt+$$
$$+\frac{1}{2}(2\delta)^{-p}\frac{1}{\delta}\int_{t_{i-1}+\delta}^{t_i}|\Delta_\delta x(t)|^p dt + \frac{1}{4}(2\delta)^{-p}\frac{1}{\delta}\int_{t_i}^{t_i+\delta}|\Delta_\delta x(t)|^p dt$$
$$= \frac{1}{4}2^{-p}\delta^{-(p+1)}\int_{t_{i-1}}^{t_i}|\Delta_\delta x(t)|^p dt + \frac{1}{4}2^{-p}\delta^{-(p+1)}\int_{t_{i-1}+\delta}^{t_i+\delta}|\Delta_\delta x(t)|^p dt.$$

Applying (39) to the definition of A_3 we get

(40) $$A_3 \leqslant 2^p \sum_{i=2k+1}^{n-1}\Big(\int_{t_{i-1}}^{t_i}|\Delta_\delta x(t)|^p dt + \int_{t_{i-1}+\delta}^{t_i+\delta}|\Delta_\delta x(t)|^p dt\Big)$$
$$= 2^p\int_{t_{2k}}^{t_{n-1}}|\Delta_\delta x(t)|^p dt + 2^p\int_{t_{2k}+\delta}^{t_{n-1}+\delta}|\Delta_\delta x(t)|^p dt$$
$$\leqslant 2^{p+1}\int_{t_{2k}}^{1-\delta}|\Delta_\delta x(t)|^p dt - 2^p\int_{t_{2k}}^{t_{2k}+\delta}|\Delta_\delta x(t)|^p dt.$$

Now, in the case $k = 2^m$, according to (36), Lemma 11 is a consequence of (35) and (37). If $k = 2^m-1$, again by (36), (35), (37) and (38) we obtain

$$\|H_n(x)-\varphi\|_p^p \leqslant 2^{-p}(A_1+A_2)$$
$$\leqslant 2^{1-p}\int_0^{t_{2k-1}}|\Delta_\delta x(t)|^p dt + 2\Big(\frac{3}{4}\Big)^p\int_{t_{2k-1}}^{t_{2k}}|\Delta_\delta x(t)|^p dt + \Big(\frac{3}{4}\Big)^p\int_{t_{2k}}^{t_{2k}+\delta}|\Delta_\delta x(t)|^p dt$$
$$\leqslant 2\Big(\frac{3}{4}\Big)^p\int_0^{t_{2k}+\delta}|\Delta_\delta x(t)|^p dt = 2\Big(\frac{3}{4}\Big)^p\int_0^{1-\delta}|\Delta_\delta x(t)|^p dt$$
$$\leqslant 2\int_0^{1-\delta}|\Delta_\delta x(t)|^p dt.$$

Finally, if $1 \leqslant k \leqslant 2^m-2$, then according to (36) we obtain from (35), (37), (38) and (40) the following inequalities

$$\|H_n(x)-\varphi\|_p^p \leqslant 2^{-p}(A_1+A_2+A_3)$$
$$\leqslant 2^{1-p}\int_0^{t_{2k-1}}|\Delta_\delta x(t)|^p dt + 2\Big(\frac{3}{4}\Big)^p\int_{t_{2k-1}}^{t_{2k}}|\Delta_\delta x(t)|^p dt + \Big(\frac{3}{4}\Big)^p\int_{t_{2k}}^{t_{2k}+\delta}|\Delta_\delta x(t)|^p dt+$$
$$+2\int_{t_{2k}}^{1-\delta}|\Delta_\delta x(t)|^p dt - \int_{t_{2k}}^{t_{2k}+\delta}|\Delta_\delta x(t)|^p dt \leqslant 2\int_0^{1-\delta}|\Delta_\delta x(t)|^p dt;$$

this completes the proof.

THEOREM 8. *Let $n \geqslant 1$. Then for $x \in L_p\langle 0,1\rangle$ if $1 \leqslant p < \infty$, and for $x \in C\langle 0,1\rangle$ if $p = \infty$ we have*

(41) $$E_n^{(p)}(x) \leqslant \|x - S_n(x)\|_p \leqslant 4E_n^{(p)}(x),$$

and

(42) $$E_n^{(p)}(x) \leqslant 8\omega_1^{(p)}\Big(\frac{1}{n}; x\Big).$$

Proof. It is clear that it sufficient to prove this theorem for finite p.

The left-hand side inequality in (41) is a consequence of the definition of $E_n^{(p)}(x)$. The right-hand side inequality follows from Theorem 1′ of [1], (compare Theorem 6 of [1]) (²).

(²) In the proof of Theorem 6 of [1] there are three misprints: $E_{2^m+k}(x)$ should be replaced by $\sigma_{2^m+k}(x)$.

The proof of (42) goes as follows. If φ denotes the polynomial defined in Lemma 11, then the definition of $E_n^{(p)}(x)$, Theorem 7 and Lemma 11 give for $n \geqslant 2$

$$E_n^{(p)}(x) \leqslant \|x-\varphi\|_p \leqslant \|x-H_n(x)\|_p+\|H_n(x)-\varphi\|_p$$
$$\leqslant 6\omega_1^{(p)}\left(\frac{1}{n}; x\right)+2^{1/p}\omega_1^{(p)}\left(\frac{1}{n}; x\right) \leqslant 8\omega_1^{(p)}\left(\frac{1}{n}; x\right).$$

If $n=1$ we infer from Theorem 7 that

$$E_1^{(p)}(x) \leqslant \|x-H_1(x)\|_p \leqslant 6\omega_1^{(p)}(1; x),$$

and this completes the proof.

THEOREM 9. *Let* $0<\alpha\leqslant 2$, $x\in C\langle 0,1\rangle$ *and let* $\omega_2(\delta; x)\leqslant M\delta^\alpha$ *for* $0<2\delta\leqslant 1$. *Then,*

$$\|x-S_n(x)\| \leqslant \frac{4M}{2^\alpha-1}\frac{1}{(n+1)^\alpha}, \quad n\geqslant 1. \tag{43}$$

Proof. Notice that $\|\sigma_n(x)\|\leqslant\|x\|$, hence it follows that

$$E_n(x) \leqslant \|x-\sigma_n(x)\| \leqslant 2E_n(x). \tag{44}$$

Theorem 1′ of [1] and (44) imply

$$\|x-\sigma_n(x)\| \leqslant 2\|x-S_n(x)\|, \quad \|x-S_n(x)\| \leqslant 4\|x-\sigma_n(x)\|. \tag{45}$$

We know also that (see Theorem 3 of [5])

$$x(t) = x(0)+\sum_{n=1}^{\infty} c_n\varphi_n(t), \quad t\in\langle 0,1\rangle; \tag{46}$$

where

$$c_1 = x(1)-x(0), \quad c_{2^m+k} = -2^{\frac{1}{2}m}\Delta^2_{1/2^{m+1}}x\left(\frac{k-1}{2^m}\right). \tag{47}$$

Modifying slightly the argument of Theorem 4 of [5] we get from (45), (46) and (47) inequality (43).

Remark. It follows from the results of [1] that for $0<\alpha<1$ the order of approximation in (43) is the best one. If $\alpha=2$, then for $x(t)=t^2$ we have

$$x(t_{2k-1})-\sigma_{2^m+k-1}(t_{2k-1}) = \sigma_{2^m+k}(t_{2k-1})-\sigma_{2^m+k-1}(t_{2k-1})$$
$$= c_{2^m+k}\varphi_{2^m+k}(t_{2k-1}) = -\tfrac{1}{2}\Delta^2_{1/2^{m+1}}x(t_{2k-2}) = -\frac{1}{(2^{m+1})^2},$$

whence

$$\|x-S_n(x)\| \geqslant \frac{1}{8}\frac{1}{(n+1)^2}, \quad n\geqslant 1.$$

THEOREM 10. *Let* $x\in L_p\langle 0,1\rangle$ *if* $1\leqslant p<\infty$ *and let* $x\in C\langle 0,1\rangle$ *if* $p=\infty$. *Then, for* $1\leqslant p\leqslant\infty$ *we have*

$$\omega_1^{(p)}\left(\frac{1}{n}; x\right) \leqslant 48\frac{1}{n+1}\sum_{i=0}^{n} E_i^{(p)}(x), \quad n\geqslant 1, \tag{48}$$

and for $1\leqslant p<\infty$ *we have*

$$\omega_2^{(p)}\left(\frac{1}{n}; x\right) \leqslant \frac{5\cdot 2^7}{n^{1+1/p}}\sum_{i=1}^{n} i^{1/p}E_i^{(p)}(x), \quad n\geqslant 2. \tag{49}$$

Proof. It is clear that the proof of (48) can be restricted to finite p. Let us denote by ψ_n the $L_p\langle 0,1\rangle$ best approximating polynomial of the n-th degree and corresponding to the function $x\in L_p\langle 0,1\rangle$. Then, for $k\geqslant 1$, we have

$$x = \psi_0+(\psi_1-\psi_0)+\sum_{m=0}^{k-1}(\psi_{2^{m+1}}-\psi_{2^m})+(x-\psi_{2^k}).$$

Let $h\in(0,2^{-k}\rangle$. Then by Minkowski's inequality and by (21) of Theorem 5 we obtain

$$\left(\int_0^{1-h}|\Delta_h x(t)|^p\,dt\right)^{1/p}$$
$$\leqslant\left(\int_0^{1-h}|\Delta_h(\psi_1(t)-\psi_0(t))|^p\,dt\right)^{1/p}+\sum_{m=0}^{k-1}\left(\int_0^{1-h}|\Delta_h(\psi_{2^{m+1}}(t)-\psi_{2^m}(t))|^p\,dt\right)^{1/p}+$$
$$+\left(\int_0^{1-h}|\Delta_h(x(t)-\psi_{2^k}(t))|^p\,dt\right)^{1/p}$$
$$\leqslant 4\min(\tfrac{1}{2},2^{-k})\|\psi_1-\psi_0\|_p+4\sum_{m=0}^{k-1}\min(2^{m+1-k},\tfrac{1}{2})\|\psi_{2^{m+1}}-\psi_{2^m}\|_p+2E_{2^k}^{(p)}(x).$$

Now, if $k=0$ we have

$$\left(\int_0^{1-h}|\Delta_h x(t)|^p\,dt\right)^{1/p} \leqslant 2\|\psi_1-\psi_0\|_p+2E_1^{(p)}(x) \leqslant 2E_0^{(p)}(x)+4E_1^{(p)}(x),$$

whence

$$\omega_1^{(p)}(2^{-k}, x) \leqslant 8\frac{1}{2^{k+1}}\sum_{i=0}^{2^k}E_i^{(p)}(x).$$

If $k \geqslant 1$, then

$$\Big(\int_0^{1-h} |\Delta_h x(t)|^p\,dt\Big)^{1/p} \leqslant \frac{4}{2^k}\big(E_0^{(p)}(x)+E_1^{(p)}(x)\big)+\frac{8}{2^k}\sum_{m=0}^{k-1} 2^m\big(E_{2^m+1}^{(p)}(x)+E_{2^m}^{(p)}(x)\big)+2E_{2^k}^{(p)}(x)$$

$$\leqslant \frac{8}{2^{k+1}}E_0^{(p)}(x)+\frac{48}{2^{k+1}}\sum_{m=0}^{k} 2^{m-1}E_{2^m}^{(p)}(x) \leqslant \frac{48}{2^{k+1}}\sum_{i=0}^{2^k}E_i^{(p)}(x).$$

Thus, we infer for all $k \geqslant 0$ that

$$\omega_1^{(p)}\Big(\frac{1}{2^k};x\Big) \leqslant 48\,\frac{1}{2^{k+1}}\sum_{i=0}^{2^k}E_i^{(p)}(x).$$

For given n, $n \geqslant 1$, let k be such that $2^k \leqslant n < 2^{k+1}$, then the last inequality gives (48).

For the proof of (49) we assume that $0 < h \leqslant 2^{-k}$, $k \geqslant 1$. Theorem 5, (22), gives

$$\Big(\int_0^{1-2h} |\Delta_h^2 x(t)|^p\,dt\Big)^{1/p} \leqslant \sum_{m=0}^{k-1}\Big(\int_0^{1-2h} \big|\Delta_h^2\big(\psi_{2^{m+1}}(t)-\psi_{2^m}(t)\big)\big|^p dt\Big)^{1/p}+4E_{2^k}^{(p)}(x)$$

$$\leqslant \sum_{m=0}^{k-1} 8\min\big[(h2^{m+1})^{1+1/p},\tfrac{1}{2}\big]\|\psi_{2^{m+1}}-\psi_{2^m}\|_p+4E_{2^k}^{(p)}(x)$$

$$\leqslant \sum_{m=0}^{k-1} 8\,(2^{m+1-k})^{1+1/p}\big[E_{2^m+1}^{(p)}(x)+E_{2^m}^{(p)}(x)\big]+4E_{2^k}^{(p)}(x)$$

$$\leqslant 5\cdot 2^7\cdot 2^{-(k+1)(1+1/p)}\sum_{i=1}^{2^k} i^{1/p}E_i^{(p)}(x),$$

hence (49) follows immediately.

Remarks. Theorem 10 shows that for finite p the condition $E_n^{(p)} = O(1/n)$ implies $\omega_2^{(p)}(\delta;x) = O(\delta)$. If $p = \infty$, then this is not true. An example of such function is constructed as follows (compare with [4], Theorem 4):

$$x(t) = \sum_{m=1}^{\infty}\frac{1}{2^{m/2}}\sum_{k=1}^{2^m}\varphi_{2^m+k}(t), \qquad t \in \langle 0,1\rangle.$$

One checks easily that

$$\Delta^2_{1/2^{n+1}}x(\tfrac{1}{2}) = -\frac{n}{2^n},$$

hence it follows that

$$\omega_2(2^{-(n+1)};x) \geqslant n\cdot 2^{-n}.$$

However, for $p = \infty$ and $x \in C\langle 0,1\rangle$ the converse is true. Namely, if $\omega_2(\delta;x) = O(\delta)$, then $E_n(x) = O(1/n)$ (cf. [5], Theorem 4). We do not know whether a similar result holds for finite p.

The following result generalizes Corollary of [1], p. 156:

THEOREM 11. *Let $x \in L_p\langle 0,1\rangle$ if $1 \leqslant p < \infty$ and let $x \in C\langle 0,1\rangle$ if $p = \infty$. If a_{n-1} denotes the n-th Fourier-Franklin coefficient of x and if $0 < \alpha < 1$, then the following conditions are equivalent:*

(i) $E_n^{(p)}(x) = O(n^{-\alpha})$,

(ii) $\|x-S_n(x)\|_p = O(n^{-\alpha})$,

(iii) $2^{m(1/2-1/p)}\Big(\sum_{2^m+1}^{2^{m+1}}|a_n|^p\Big)^{1/p} = O(2^{-\alpha m})$,

(iv) $\omega_1^{(p)}(\delta;x) = O(\delta^\alpha)$.

The theorem remains true replacing O by o.

Proof. The equivalence of (i) and (ii) is due to Theorem 8, (41). Corollary to Theorem 6 shows that (ii) is equivalent to (iii). Finally, (42) and (48) give the equivalence of (iv) and (i).

The next theorem generalizes the results proved in [3].

THEOREM 12. *Let $0 < \alpha < 1$, $1 \leqslant p < \infty$. Then there is a linear isomorphism between the spaces $\langle L_p^{(\alpha)}, \|\;\|_p^{(\alpha)}\rangle$, $\langle L_p^{(\alpha,0)}, \|\;\|_p^{(\alpha)}\rangle$ and $\langle m_p, \|\;\|\rangle$, $\langle m_p^{(0)}, \|\;\|\rangle$, respectively. In both cases the isomorphism is given by the Fourier-Franklin series*

$$x = \sum_{n=1}^{\infty}a_n f_n^{(\alpha,p)}, \qquad a_n = \|f_n\|_p^{(\alpha)}(x,f_n),$$

where $f_n^{(\alpha,p)} = f_n/\|f_n\|_p^{(\alpha)}$. Moreover, $\{f_n^{(\alpha,p)}\}$ is a Schauder basis for the space $\langle L_p^{(\alpha,0)}, \|\;\|_p^{(\alpha)}\rangle$.

This theorem is a simple consequence of Lemma 10 and of Theorem 11.

7. Fourier-Franklin coefficients. Absolute convergence.

THEOREM 13. *Let $\alpha > 0$, $x \in C\langle 0,1\rangle$ if $p = \infty$ and let $x \in L_p\langle 0,1\rangle$ if $1 \leqslant p < \infty$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\,\omega_1^{(p)}\Big(\frac{1}{n};x\Big) < \infty \tag{50}$$

if and only if

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}E_n^{(p)}(x) < \infty. \tag{51}$$

Proof. According to Theorem 8 condition (50) implies (51). The converse is a consequence of Theorem 10. Indeed,

$$\sum_{n=1}^{\infty}\frac{1}{n^{a}}\,\omega_1^{(p)}\Big(\frac{1}{n};x\Big)\leqslant 48\sum_{n=1}^{\infty}\frac{1}{n^{a}(n+1)}\sum_{i=0}^{n}E_i^{(p)}(x)$$

$$\leqslant E_0^{(p)}(x)\,48\sum_{n=1}^{\infty}\frac{1}{n^{1+a}}+48\sum_{i=1}^{\infty}E_i^{(p)}(x)\sum_{n=i}^{\infty}\frac{1}{n^{1+a}}=O\Big(\sum_{n=1}^{\infty}\frac{1}{i^{a}}E_i^{(p)}(x)\Big).$$

The following two theorems generalize Theorems 8 and 9 of [1].

THEOREM 14. *Let $x\in L_p\langle 0,1\rangle$ if $1\leqslant p<\infty$ and $x\in C\langle 0,1\rangle$ if $p=\infty$. If $a_n=(x,f_n)$ and $a\geqslant 0$, then*

$$\sum_{n=1}^{\infty}\frac{E_n^{(p)}(x)}{n^{1-a}}<\infty$$

implies

$$\sum_{m=0}^{\infty}2^{m(1/2+a)}\Big(2^{-m}\sum_{2^m+1}^{2^{m+1}}|a_n|^p\Big)^{1/p}<\infty.$$

Proof. According to (34) and (41) we have

$$2^{m(1/2+a)}\Big(2^{-m}\sum_{2^m+1}^{2^{m+1}}|a_n|\Big)^{1/p}$$

$$\leqslant 2^5\cdot 3^{1/2}\cdot 2^{am}\|S_{2^{m+1}}(x)-S_{2^m}(x)\|_p\leqslant 2^8\cdot 3^{1/2}\cdot 2^{am}E_{2^m}^{(p)}(x),$$

hence the theorem follows.

COROLLARY. *If $x\in L_1\langle 0,1\rangle$ and*

$$\sum_{n=1}^{\infty}\frac{E_n^{(1)}(x)}{n^{1/2}}<\infty,$$

then

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

THEOREM 15. *Let $1\leqslant p\leqslant\infty$ and let*

$$\sum_{m=0}^{\infty}2^{m/2}\Big(2^{-m}\sum_{2^m+1}^{2^{m+1}}|a_n|^p\Big)^{1/p}<\infty.$$

Then the series $\sum_{n=1}^{\infty}|a_nf_n(t)|$ converges a.e. and in the norm $\|\ \|_p$.

This result follows immediately from (34).

The idea of the proof of the next theorem is due to P. L. Uljanov [17], p. 362.

THEOREM 16. *Let x be a function of bounded variation on $\langle 0,1\rangle$ and let $a_n=(x,f_n)$. Then,*

$$\sum_{2^m+1}^{2^{m+1}}|a_n|\leqslant 2^{11}\cdot 3\cdot 3^{1/2}\cdot 2^{-m/2}\operatorname*{var}_{\langle 0,1\rangle}x,\qquad m\geqslant 0.$$

Proof. Since x is of bounded variation, it follows that for $0<\delta\leqslant 1$ (cf. Uljanov [17], p. 358)

$$\omega_1^{(1)}(\delta;x)\leqslant 3\,\delta\operatorname*{var}_{\langle 0,1\rangle}x.$$

This, (34) and Theorem 8 give

$$2^{-m/2}\sum_{2^m+1}^{2^{m+1}}|a_n|\leqslant 2^5\cdot 3^{1/2}\|S_{2^{m+1}}(x)-S_{2^m}(x)\|_1\leqslant 2^8\cdot 3^{1/2}E_{2^m}^{(1)}(x)$$

$$\leqslant 2^{11}\cdot 3^{1/2}\,\omega_1^{(1)}\Big(\frac{1}{2^m};x\Big)\leqslant 2^{11}\cdot 3\cdot 3^{1/2}\cdot 2^{-m}\operatorname*{var}_{\langle 0,1\rangle}x,$$

and this completes the proof.

COROLLARY. *If $t\in\langle 0,1\rangle$ and $m\geqslant 0$, then*

$$\sum_{2^m+1}^{2^{m+1}}\Big|\int_0^t f_n(s)\,ds\Big|\leqslant 2^{11}\cdot 3\cdot 3^{1/2}2^{-m/2}.$$

A similar inequality holds for the Schauder functions (see [4], p. 142).

THEOREM 17. *Let x be of bounded variation on $\langle 0,1\rangle$ and let $a_n=(x,f_n)$. Then,*

$$\sum_{n=1}^{\infty}n^{a}|a_n|<\infty\quad\text{for}\quad a<\tfrac{1}{2},$$

and

$$\sum_{n=1}^{\infty}|a_n|^{\beta}<\infty\quad\text{for}\quad \beta>\tfrac{2}{3}.$$

Proof. The first part follows immediately from Theorem 16.

The second part can be proved in the same way as a corresponding result for the Haar system (cf. Uljanov [17], p. 373).

THEOREM 18. *Let $0<\varepsilon<\frac{4}{3}$ and let $1\geqslant a>\varepsilon/2(2-\varepsilon)$. Then for each x such that $\omega_1(\delta;x)=O(\delta^{a})$ we have*

$$\sum_{n=1}^{\infty}|a_n|^{2-\varepsilon}<\infty,$$

where $a_n=(x,f_n)$.

To prove this it is sufficient to apply Corollary from [1], p. 155.

THEOREM 19 (Carleman's singularity). *Let $0 < \varepsilon < \frac{4}{3}$. Then there exists x such that $\omega_1(\delta; x) = O(\delta^\alpha)$ with $\alpha = \varepsilon/2(2-\varepsilon)$ and such that*

$$\sum_{n=1}^{\infty} |a_n|^{2-\varepsilon} = \infty, \qquad a_n = (x, f_n).$$

Proof. Define

$$x(t) = \sum_{n=1}^{\infty} a_n f_n(t) \quad \text{with} \quad a_n = \frac{1}{n^{1/2+\alpha}}.$$

Then $|a_n|^{2-\varepsilon} = 1/n$. Now, Corollary ([1], p. 156) gives $\omega_1(\delta; x) = O(\delta^\alpha)$.

8. Integrated Franklin system as a basis. If we integrate the Haar functions and add to this new system the constant function 1 we obtain the Schauder basis of the Banach space $\langle C\langle 0,1\rangle, \| \,\|\rangle$. The same turns out to be true for the Franklin functions.

THEOREM 20. *The set of functions*

$$\Big\{1, \int_0^t f_n(s)\,ds,\ n = 0, 1, \ldots\Big\} \tag{52}$$

is a Schauder basis for the space $\langle C\langle 0,1\rangle, \| \,\|\rangle$. Moreover, for each $x \in C\langle 0,1\rangle$ the unique expansion with respect to the system (52) *is given by the formula*

$$x(t) = x(0) + \sum_{n=0}^{\infty} \Big[\int_0^1 f_n(s)\,dx(s)\Big] \int_0^t f_n(s)\,ds, \tag{53}$$

where the series converges uniformly in $\langle 0,1\rangle$.

Proof. The uniqueness is a consequence of the relation (3) of [1].

Since $\{f_n\}$ is a Schauder basis for the space $\langle C\langle 0,1\rangle, \| \,\|\rangle$, it follows that (53) is satisfied for a dense subset of $C\langle 0,1\rangle$ for x's which are continuously differentiable in $\langle 0,1\rangle$.

Let

$$D_n(x; t) = x(0) + \sum_{i=0}^{n} \Big[\int_0^1 f_i(s)\,dx(s)\Big] \int_0^t f_i(s)\,ds. \tag{54}$$

Notice that $\{D_n\}$ is a sequence of bounded linear operators in $\langle C\langle 0,1\rangle, \| \,\|\rangle$ such that $D_n(x) - x \to 0$ in a dense set. To complete the proof of our theorem it is sufficient to show that the norms $\|D_n\|$ of these operators form a bounded sequence.

The definition (54) gives

$$\begin{aligned} D_n(x; t) &= x(0) + x(1) \sum_{i=0}^{n} \int_0^t f_i(s)\,ds\, f_i(1) - x(0) \sum_{i=0}^{n} \int_0^t f_i(s)\,ds\, f_i(0) - \\ &\qquad - \sum_{i=1}^{n} \int_0^1 x(s) f_i'(s)\,ds \int_0^t f_i(s)\,ds \\ &= x(0) + x(1) S_n(\chi_t; 1) - x(0) S_n(\chi_t; 0) - \int_0^t x(s) G_n(s, t)\,ds, \end{aligned}$$

where $\chi_t(u)$ is 1 for $0 \leqslant u \leqslant t$ and 0 for $t < u \leqslant 1$, and

$$G_n(s, t) = \sum_{i=1}^{n} f_i'(s) \int_0^t f_i(u)\,du.$$

This and Theorem 1' of [1] imply

$$\|D_n(x)\| \leqslant 7\,\|x\| + \|x\| \sup_{0 \leqslant t \leqslant 1} \int_0^1 |G_n(s, t)|\,ds,$$

hence it follows that

$$\|D_n\| \leqslant 7 + \sup_{0 \leqslant t \leqslant 1} \int_0^1 |G_n(s, t)|\,ds. \tag{55}$$

Now, we notice that

$$\begin{aligned} G_n(s, t) &= \frac{d}{ds} \int_0^t K_n(s, u)\,du = \int_0^t \frac{d}{ds} K_n(s, u)\,du \\ &= -\frac{d}{ds} \int_t^1 K_n(s, u)\,du = -\int_t^1 \frac{d}{ds} K_n(s, u)\,du. \end{aligned}$$

Let $\{t_0, \ldots, t_n\}$ be the partition corresponding to n and let $\delta_i = t_i - t_{i-1}$. If $t_{k-1} \leqslant t \leqslant t_k$ and $i \leqslant k$, then Theorem 3 gives

$$\begin{aligned} \int_{t_{i-1}}^{t_i} |G_n(s, t)|\,ds &= \int_{t_{i-1}}^{t_i} \Big|\int_t^1 \frac{d}{ds} K_n(s, u)\,du\Big|\,ds \\ &= \frac{1}{\delta_i} \int_{t_{i-1}}^{t_i} \Big|\int_t^1 [K_n(t_i, u) - K_n(t_{i-1}, u)]\,du\Big|\,ds \\ &\leqslant \int_t^1 [|K_n(t_i, u)| + |K_n(t_{i-1}, u)|]\,du \\ &\leqslant \sum_{j=k}^{n} \int_{t_{j-1}}^{t_j} [|K_n(t_i, u)| + |K_n(t_{i-1}, u)|]\,du \end{aligned} \tag{56}$$

$$\leqslant \sum_{j=k}^{n} \delta_j\big[|K_n(t_i,t_j)|+|K_n(t_i,t_{j-1})|+|K_n(t_{i-1},t_j)|+$$

$$+|K_n(t_{i-1},t_{j-1})|\big] = O\Big(\sum_{j=k}^{n} \delta_j n e^{-a|j-i|}\Big)$$

$$= O\Big(e^{ai}\sum_{j=k}^{n} e^{-aj}\Big) = O(e^{-a(k-i)}).$$

If $i > k$, then quite similar argument shows that

$$\int_{t_{i-1}}^{t_i} |G_n(s,t)|\,ds = O(e^{-a(i-k)}). \tag{57}$$

Thus, (56) and (57) imply

$$\int_{t_{i-1}}^{t_i} |G_n(s,t)|\,ds = O(e^{-a|i-k|}) \quad \text{for} \quad t_{k-1} \leqslant t \leqslant t_k,$$

whence uniformly in k

$$\sup_{0\leqslant t\leqslant 1} \int_0^1 |G_n(s,t)|\,ds = O\Big(\sum_{i=1}^{n} e^{-a|k-i|}\Big) = O(1).$$

Combining this with (55) we complete the proof.

COROLLARIES. 1) *The set* (52) *is a Schauder basis in the Banach space* $C^{(1)}\langle 0,1\rangle$ *with respect to the usual norm.* 2) *If* $x \in C\langle 0,1\rangle$ *and* y *is of bounded variation on* $\langle 0,1\rangle$, *then the following Parseval's identity holds*:

$$\int_0^1 y(t)\,dx(t) = \sum_{n=0}^{\infty}\int_0^1 f_n(t)\,dx(t)\int_0^1 f_n(s)\,y(s)\,ds.$$

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