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$L^1(a, b)$ with order convolution

by

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The linear order on an interval I of real numbers from a to b, where a or b may be infinite and I may or may not include one or the other of the end points (if it is necessary to indicate, for example, that the point b is in the interval I and the point a is not in the interval, then we shall write $I = (a, b])$, determines a convolution on $L^1(a, b)$ and with this product $L^1(a, b)$ is a Banach algebra. The convolution arises as follows. Since the interval is a linearly ordered set, it is a commutative semigroup with the product of $x, y \in I$ defined by $xy = \max\{x, y\}$. When I is provided with the usual interval topology, I is a locally compact topological semigroup. Let $M(a, b)$ denote the Banach space of all regular bounded Borel measures on I and let $C_0(a, b)$ denote the Banach space of all continuous functions on I which vanish at infinity (the phrase "vanish at infinity" is used in the general topological sense, that is, if $f \in C_0(a, b)$ and $\varepsilon > 0$, then there exists a compact subset K of I such that $|f(x)| < \varepsilon$ for all $x \notin K$). One can then define the convolution of two measures $\mu, \nu \in M(a, b)$ in the usual way by viewing $M(a, b)$, by means of the Riesz representation theorem, as the dual of $C_0(a, b)$. Then $\mu * \nu$ is the measure determined by the linear functional L on $C_0(a, b)$ where

$$L(\tau) = \int_I \int_I \tau(xy)\, d\mu(x)\, d\nu(y) = \int_I \tau(z)(\mu * \nu)(z) \tag{1}$$

for all $\tau \in C_0(a, b)$. With this product $M(a, b)$ is a Banach algebra. The structure of measure algebras of this form is discussed in [1] not only for an interval of real numbers but for an arbitrary totally ordered set X, with semigroup product defined as above, such that X is compact in the interval topology. As indicated in [1], many of the results for compact X are also valid for locally compact X. In particular, these results show that the maximal ideal space of $M(a, b)$ can be identified with the set of semicharacters on I, that is, all bounded complex-valued functions φ on I such that φ is not identically zero and $\varphi(xy) = \varphi(x)\varphi(y)$

for all $x, y \in I$. These semicharacters are of the following types:

(2) $$\varphi(x) = \langle x, \alpha\rangle = \begin{cases} 1, & x \leqslant \alpha, \\ 0, & x > \alpha, \end{cases} \quad \text{or} \quad \varphi(x) = \langle x, \alpha) = \begin{cases} 1, & x < \alpha, \\ 0, & x \geqslant \alpha, \end{cases}$$

where $\alpha \in I$ and $\alpha \neq a$ in the second case. The corresponding homomorphism of $M(a, b)$ into the complex numbers is given by $\int_I \varphi(x)\, d\mu(x)$ for all $\mu \in M(a, b)$ (see [1] for the details). In this paper we show that $L^1(a, b)$ is a subalgebra of $M(a, b)$ and study the structure of this algebra. The convolution on $L^1(a, b)$ will be called *order convolution*. The maximal ideal space of $L^1(a, b)$ is obtained from an identification of the semicharacters $\langle x, \alpha\rangle$ and $\langle x, \alpha)$, hence it corresponds to an interval of real numbers. The Gelfand topology is the usual interval topology and the Gelfand transform has the form of the indefinite integral.

$L^1(a, b)$ as a subalgebra of $M(a, b)$

In the interest of brevity we shall write simply L^1 and M for $L^1(a, b)$ and $M(a, b)$ throughout the remainder of the paper. Every function $f \in L^1$ determines a measure $\mu_f \in M$ where

$$\mu_f(E) = \int_E f(x)\, dx.$$

The measure μ_f is absolutely continuous with respect to Lebesgue measure and, according to the Radon-Nikodym theorem, every measure which is absolutely continuous with respect to Lebesgue measure is of this form. Since functions in L^1 which are equal almost everywhere are identified, different functions determine different measures. Moreover, we have $\|\mu_f\| = \|f\|$ and we can thus view L^1 as a linear subspace of M. Our first result is that this linear subspace is in fact a subalgebra.

It will be convenient to consider the semicharacters as functions of two variables and for $(x, y) \in I \times I$ we write $\langle x, y\rangle$ and $\langle x, y)$ for the functions defined by (2) above. It is clear that for any function f on I we have

(3) $$f(xy) = \langle x, y\rangle f(y) + \langle y, x\rangle f(x) \quad \text{almost everywhere on } I \times I.$$

THEOREM. *If $f, g \in L^1$, then $f * g \in L^1$ and*

(4) $$(f * g)(x) = f(x) \int_a^x g(y)\, dy + g(x) \int_a^x f(y)\, dy \quad \text{a.e.}$$

Proof. We will show first that the function defined on the right-hand side of (4) belongs to L^1.

We have

$$\int_a^b \left| f(x) \int_a^x g(y)\, dy + g(x) \int_a^x f(y)\, dy \right| dx$$
$$\leqslant \int_a^b \int_a^b |f(x)|\, |g(y)| \langle y, x\rangle\, dy dx + \int_a^b \int_a^b |f(y)|\, |g(x)| \langle y, x\rangle\, dy dx$$
$$= \int_a^b \int_a^b |f(x)|\, |g(y)| \langle y, x\rangle\, dy dx + \int_a^b \int_a^b |f(x)|\, |g(y)| \langle x, y\rangle\, dy dx$$
$$= \int_a^b \int_a^b |f(x)|\, |g(y)| (\langle x, y\rangle + \langle y, x\rangle)\, dx dy$$
$$= \int_a^b \int_a^b |f(x)|\, |g(y)|\, dx dy = \|f\|\, \|g\|.$$

Thus

$$f(x) \int_a^x g(y)\, dy + g(x) \int_a^x f(y)\, dy \in L^1.$$

To determine the convolution product of f and g, let τ be an arbitrary function in $C_0(a, b)$. Then, according to (1) and (3)

$$\int_I \tau(z)\, d(\mu_f * \mu_g)(z) = \int_a^b \int_a^b \tau(xy) f(x) g(y)\, dx dy$$
$$= \int_a^b \int_a^b [\langle x, y\rangle \tau(y) + \langle y, x\rangle \tau(x)] f(x) g(y)\, dx dy$$
$$= \int_a^b \tau(y) g(y) \Big[\int_a^b \langle x, y\rangle f(x)\, dx\Big] dy + \int_a^b \tau(x) f(x) \Big[\int_a^b \langle y, x\rangle g(y)\, dy\Big] dx$$
$$= \int_a^b \tau(y) \Big[g(y) \int_a^y f(x)\, dx\Big] dy + \int_a^b \tau(y) \Big[f(y) \int_a^y g(x)\, dx\Big] dy$$
$$= \int_a^b \tau(y) \Big[g(y) \int_a^y f(x)\, dx + f(y) \int_a^y g(x)\, dx\Big] dy.$$

Thus the measure $\mu_f * \mu_g$ agrees with the measure determined by the function

$$f(x) \int_a^x g(y)\, dy + g(x) \int_a^x f(y)\, dy.$$

From the above theorem we have that L^1 is a subalgebra of M and from the results of [1] M is semisimple. Thus we have the following result:

COROLLARY. *L^1 with order convolution is a commutative semisimple Banach algebra.*

If G is a locally compact group, then the group algebra $L^1(G)$ is not only a subalgebra of the measure algebra $M(G)$ but $L^1(G)$ is also an ideal in $M(G)$. This is definitely not the case for order convolution since the characteristic function of the subinterval $[c, d]$, where c and d are distinct interior points of the interval, when convolved with the unit point mass at d gives the mass $(d-c)$ at the point d. Of course the set of discrete measures in M, l_1, is a subalgebra of M. It is easy to check that $l_1 \oplus L^1$, that is those measures in M of the form $\mu_d + \mu_f$ where μ_d is discrete and $f \in L^1$, is also a subalgebra of M.

THEOREM. *If I is finite, then L^1 is generated by the function f_0, where $f_0(x) = 1$ for all $x \in I$.*

Proof. We base the proof on the observation that if f and g are differentiable functions with continuous derivative and if $f(a) = g(a) = 0$- then $f' * g' = (fg)'$. This follows immediately from (4) and the funda, mental theorem of calculus. By induction we have $(f_1' * \ldots * f_n') = (f_1 \ldots f_n)'$. Since f_0 is the derivative of $(x-a)$, the convolution product of n factors of f_0 is given by $(f_0 * \ldots * f_0)(x) = \big((x-a)^n\big)' = n(x-a)^{n-1}$. Thus linear combinations of powers of f_0 are dense in L^1.

The Gelfand representation

The Gelfand theory for commutative Banach algebras provides a representation of these algebras as algebras of continuous functions on a locally compact space. We show that this representation for L^1 with order convolution is given by the indefinite integral.

Since L^1 is a subalgebra of M, each homomorphism from M into the complex numbers will be a homomorphism of L^1 into the complex numbers. Thus each semicharacter will provide a homomorphism of L^1 into the complex numbers, however, since $\langle x, a\rangle = \langle x, a)$ almost everywhere, distinct semicharacters will not give distinct homomorphisms. There is the possibility too that there are homomorphisms on L^1 which are not of this form. The following theorem shows that each homomorphism is given by a semicharacter:

THEOREM. *Every homomorphism h of L^1 onto the complex numbers is of the form*

$$h(f) = \int_a^b f(x)\langle x, \alpha\rangle dx = \int_a^\alpha f(x)\,dx \quad \text{for some } a < \alpha \leqslant b. \tag{5}$$

Proof. Since h is a non-zero linear functional on L^1, there is a non-trivial function $c \in L^\infty$ such that

$$h(f) = \int_a^b f(x)c(x)\,dx.$$

By using the fact that h is multiplicative, we have

$$\begin{aligned} h(f*g) &= \int_a^b \Big(f(x)\int_a^x g(y)\,dy + g(x)\int_a^x f(y)\,dy\Big)c(x)\,dx \\ &= \int_a^b\int_a^b f(x)g(y)\big(\langle y, x\rangle c(x) + \langle x, y\rangle c(y)\big)\,dy\,dx \\ &= \int_a^b\int_a^b f(x)g(y)c(xy)\,dy\,dx. \end{aligned}$$

On the other hand,

$$h(f)h(g) = \int_a^b\int_a^b f(x)g(y)c(x)c(y)\,dy\,dx.$$

Since the linear span of $\{f(x)g(y)\colon f, g \in L^1\}$ is dense in $L^1(I\times I)$, we conclude that $c(xy) = c(x)c(y)$ almost everywhere on $I\times I$. The following lemma completes the proof:

LEMMA. *Let c be a measurable function on I such that $c(xy) = c(x)c(y)$ almost everywhere on $I\times I$. If c is not equal to zero almost everywhere on I, then there is an $a < \alpha \leqslant b$ such that $c(x) = \langle x, \alpha\rangle$ almost everywhere on I.*

Proof. Let $S_\lambda = \{x \in I\colon x \geqslant \lambda \text{ and } c(x) \neq 0\}$ and let $\alpha = \inf\{\lambda : S_\lambda$ has measure zero$\}$. Since $c(x)$ is non-trivial, $\alpha > a$. If $c(x)$ were to differ from $\langle x, \alpha\rangle$ on a set of positive measure, then there would be a set A of positive measure contained in $[a, \alpha)$ such that $c(x) \neq 1$ for $x \in A$. The set $D = \{(x, y)\colon y \leqslant x, y \in A\}$ would have positive measure in $I\times I$. Since $c(y) \neq 1$ for $(x, y) \in D$ and since $c(x)c(y) = c(xy) = c(x)$ for almost all $(x, y) \in D$, the set $D' = \{(x, y) \in D\colon c(x) \neq 0\}$ has measure zero. Hence the sets $D_y' = \{x \in I\colon (x, y) \in D'\} = \{x \in I\colon y \leqslant x, c(x) \neq 0\} = S_y$ have measure zero for almost all y. But this is impossible since S_y has positive measure for all $y \in A$ and A has positive measure. This completes the proof.

We can thus identify the carrier space of L^1 with the interval $(a, b]$ and the Gelfand transform of $f \in L^1$ is the indefinite integral of f. The following theorem shows that this identification is in fact topological:

THEOREM. *The Gelfand topology on $(a, b]$ coincides with the interval topology.*

Proof. The Gelfand topology, τ, is the weakest topology for which the functions

$$\hat{f}(\alpha) = \int_a^\alpha f(x)\,dx$$

are continuous. Since these functions are continuous with respect to the interval topology, τ', τ is weaker than τ'. The functions $\hat{f}(\alpha)$ clearly separate the points of $(a, b]$, vanish at infinity and do not all vanish at a particular point in $(a, b]$. Thus the weak topology, τ, induced on $(a, b]$ by these functions coincides with τ' ([2], p. 12).

Remark. The semicharacters $\langle x, \alpha\rangle$, $\alpha \in (a, b]$, form a semigroup under pointwise multiplication. Indeed, $\langle x, \alpha\rangle\langle x, \alpha'\rangle = \langle x, \alpha''\rangle$ where $\alpha'' = \min\{\alpha, \alpha'\}$. By simply reversing the interval of semicharacters we have for intervals of the form $[a, b)$ a situation similar to that for the group algebra of the real numbers. That is, the maximal ideal space of the algebra L^1 of the semigroup $[a, b)$ can be identified with the semigroup $[a, b)$.

Some additional properties of L^1

It follows from the above results that L^1 has no identity. It is clear that the adjunction of an identity to L^1 is equivalent to the adjunction of the unit mass at the point a to the algebra L^1. However, there are approximate identities in L^1.

THEOREM. *Given $f \in L^1$ and $\varepsilon > 0$, there exists $t \in I$ such that if u is any non-negative function in L^1 which vanishes to the right of t and*

$$\int_a^b u(x)\,dx = 1,$$

then $\|f - u * f\| \leqslant \varepsilon$.

Proof. Choose $t > a$ such that

$$\int_a^t |f(x)|\,dx < \varepsilon/3.$$

If u satisfies the conditions of the theorem, then for $x > t$, $(u*f)(x) = f(x)$. Thus

$$\begin{aligned}\|u*f-f\| &= \int_a^t \Big| u(x)\int_a^x f(y)\,dy + f(x)\int_a^x u(y)\,dy - f(x)\Big|\,dx \\ &\leqslant \int_a^t \Big[u(x)\int_a^x |f(y)|\,dy + |f(x)|\int_a^x u(y)\,dy + |f(x)|\Big]\,dx \\ &\leqslant \int_a^t u(x)\Big[\int_a^x |f(y)|\,dy\Big]\,dx + \int_a^t |f(x)|\,dx + \int_a^t |f(x)|\,dx \\ &\leqslant \int_a^t (\varepsilon/3)\,u(x)\,dx + \varepsilon/3 + \varepsilon/3 \leqslant \varepsilon.\end{aligned}$$

If we set $f^*(x) = \overline{f(x)}$, then the map $f \to f^*$ defines an involution in L^1 and it is immediate that L^1 is self-adjoint.

If C is a closed set in $(a, b]$ and $\alpha_0 \notin C$ and if f has a continuous derivative $f' \in L^1$, vanishes at a and on C but does not vanish at α_0, then $\hat{f}'(\alpha)$ is zero on C and $\hat{f}'(\alpha_0) \neq 0$. Thus we have

THEOREM. *L^1 is regular and self-adjoint.*

THEOREM. *Every proper closed ideal in L^1 is contained in a regular maximal ideal.*

Proof. The theorem follows from the fact that the set of functions $f \in L^1$ such that

$$\hat{f}(\alpha) = \int_a^\alpha f(x)\,dx$$

has compact support is dense in L^1 ([2], p. 85).

Finally we give an analogue of the Herglotz-Bochner theorem which is similar to that obtained in [1]. Using the terminology of [2] we call a functional P on L^1 *positive* if $P(f*f^*) \geqslant 0$ for all $f \in L^1$. Since L^1 contains approximate identities, a positive functional P on L^1 is extendable if and only if it is continuous ([2], p. 126). A function $p \in L^\infty$ is called *positive definite* if and only if the corresponding linear functional

$$P(f) = \int_I f(x)\overline{p(x)}\,dx$$

is positive. If $p \in L^\infty$, then, since $P(f) = \overline{P(f^*)}$ for extendable positive functionals, it is easy to see that $p(x) = \overline{p(x)}$ a. e., whenever p is positive definite.

THEOREM. *A function $p \in L^\infty$ is positive definite if and only if there exists a finite positive Baire measure μ_p on $I' = (a, b]$ such that*

$$p(x) = \int_{I'} \langle x, \alpha\rangle\,d\mu_p(\alpha)$$

almost everywhere.

Proof. If $p \in L^\infty$ is positive definite and P is the corresponding linear functional, then there exists a finite positive Baire measure μ_p on I' ([2], p. 97) such that

$$P(f) = \int_I f(x)p(x)\,dx = \int_{I'} \hat{f}(\alpha)\,d\mu_p(\alpha)$$

$$= \int_{I'}\int_I f(x)\langle x, \alpha\rangle\,dx\,d\mu_p(\alpha) = \int_I f(x)\Big[\int_{I'}\langle x, \alpha\rangle\,d\mu_p(\alpha)\Big]\,dx.$$

Hence

$$p(x) = \int_{I'}\langle x, \alpha\rangle\,d\mu_p(\alpha)$$

almost everywhere.

Conversely, if

$$p(x) = \int_{I'}\langle x, \alpha\rangle\,d\mu_p(\alpha)$$

for some finite positive Baire measure μ_p on I', then $p \in L^\infty$, $p(x) = \overline{p(x)}$ and

$$\int_I (f*f^*)(x)p(x)\,dx = \int_I\int_{I'}(f*f^*)(x)\langle x, \alpha\rangle\,d\mu_p(\alpha)\,dx$$

$$= \int_{I'}\int_I (f*f^*)(x)\langle x, \alpha\rangle\,dx\,d\mu_p(\alpha) = \int_{I'}|\hat{f}(\alpha)|^2\,d\mu_p(\alpha) \geqslant 0.$$

Hence p is positive definite.

COROLLARY. *A function $p \in L^\infty$ is positive definite if and only if p is positive, monotone non-increasing and left-continuous.*

Proof. The above theorem shows that positive definite functions are positive, monotone decreasing and left-continuous. On the other hand, such a function determines, in the usual way, a finite positive Baire measure such that

$$p(x) = \mu_p[x, b] = \int_{I'}\langle x, \alpha\rangle\,d\mu_p(\alpha).$$

References

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Commutators of singular integrals

by

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Introduction. Calderón and Zygmund considered in [2] singular integral operators, K, of type C_β^∞, $\beta > 1$, and proved results involving commutators of singular integral operators and the operator, Λ. It is the purpose of this paper to prove similar results for $K \in C_\beta^\infty$, $0 < \beta \leqslant 1$, and for the operator Λ^α, $\alpha < \beta$, defined so that $\widehat{\Lambda^\alpha f} = |x|^\alpha \hat{f}$, where $\hat{f}$ denotes the Fourier transform of f.

Notation. $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$ will denote points of E^n. $C_0^\infty(E^n)$ denotes the class of functions $f \in C^\infty(E^n)$ with compact support.

"a.e." designates the phrase "almost everywhere with respect to Lebesgue measure".

$$x \circ y = \sum_{i=1}^n x_i y_i; \qquad \Sigma = \{x \in E^n\colon |x| = 1\}.$$

$$\|f\|_p = \Big(\int_{E^n}|f(x)|^p\,dx\Big)^{1/p}, \qquad \hat{f}(x) = \int_{E^n} f(y)\,e^{2\pi i x\circ y}\,dy.$$

$\gamma = (\gamma_1, \ldots, \gamma_n)$ will denote a point in E^n with each γ_i representing a non-negative integer.

Finally,

$$(\partial/\partial x)^\gamma f(x) = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}, \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} f(x).$$

Assume $f(x) \in C^\infty(E^n)$ and that every derivative, $(\partial/\partial x)^\gamma f$, satisfies $(\partial/\partial x)^\gamma f = O(|x|^{-n})$. For such f we define

$$(S_{j,\varepsilon}^\alpha f)(x) = \int_{|y|>\varepsilon} f(x-y)\frac{y_j}{|y|^{n+1+\alpha}}\,dy, \qquad 0 < \alpha < 1.$$

REMARK 1. $\lim_{\varepsilon\to 0}(S_{j,\varepsilon}^\alpha f)(x)$ *exists point-wise for every x, and in $L^p(E^n)$, for every p $(1 < p < \infty)$.*