$$P = \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{(t-s)} \Big\{ \int_{E_n} \left|\Omega\left(x-(t-s)^{-1/a}y\right) - \Omega(x)\right| dx \Big\} dt$$

$$\leqslant C|y| \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{(t-s)^{1+1/a}} dt \leqslant C$$

using (3.3).

A similar inequality is obtained for Q using (3.4).

$$R = \int_{2^a\varrho^a(y,s)}^{\infty} t^{n/a} \left| \frac{1}{(t-s)^{n/a+1}} - \frac{1}{t^{n/a+1}} \right| \Big\{ \int_{E_n} |\Omega(x)|\, dx \Big\} dt$$

$$\leqslant C \int_{2^a\varrho^a(y,s)}^{\infty} t^{n/a} \frac{|s|}{|t+s|^{n/a+2}} dt \leqslant \left(\frac{1}{2}\right)^{n/a+2} C|s| \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{t^2} dt \leqslant BC,$$

and the condition (1.1) is finally proved.

References

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), p. 85-139.

[2] — *On singular integrals*, Amer. Jour. Math. 78 (1956), p. 289-309.

[3] — *Singular integral operators and differential equations*, ibidem 79 (1957), p. 901-921.

[4] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), p. 93-140.

[5] B. F. Jones, *On a class of singular integrals*, Amer. Jour. Math. 86 (1964), p. 441-462.

[6] S. G. Mihlin, *Singular integral equations*, Uspehi Matematicheskih Nauk (N. S.) 3, no. 3 (25) (1948), p. 29-112.

[7] A. Zygmund, *Trigonometric series*, Cambridge 1959.

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On the convergence structure of Mikusiński operators

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1. Introduction. Let $\mathscr{C}$ denote the complex algebra, the elements of which are continuous complex-valued functions of a non-negative real variable with the operation of multiplication defined by finite convolution; the operations of addition and scalar multiplication defined in the usual manner. $\mathscr{C}$ has no zero divisors, hence the quotient field may be constructed. This field, which we will denote by $\mathscr{M}$, is called the *field of operators*, and is the foundation of the operational calculus developed by Mikusiński [4].

Mikusiński [4] (Part Two, Chapter I, p. 144) states a definition of convergence of sequences of operators. Urbanik [6] has shown that there is no topology satisfying the first axiom of countability in which convergence of sequences is convergence in the sense defined by Mikusiński. The definition of convergence as given by Mikusiński is generalized to nets and filters and is referred to as *M-convergence*. We show that M-convergence defines a Limitierung, τ_M, on the field of operators which is the direct limit of Limitierungen on subspaces of $\mathscr{M}$. It is shown that the Limitierung, τ_M, is not topological. Thus there is no topology on $\mathscr{M}$ for which convergence of nets and filters is precisely M-convergence.

Some properties of the limit space $(\mathscr{M}, \tau_M)$ are investigated and the notion of a linear limit space is defined. The topology defined by Norris in [5] is shown to be the direct limit of Limitierungen on certain subspaces of the field of operators

2. Preliminaries. If the complex algebra $\mathscr{C}$ is provided with the topology of compact convergence it is a routine matter to verify that $\mathscr{C}$ is a topological complex algebra. The collection

$$\mathfrak{B}(f) = \{B(a, \varepsilon, f) \colon a \geqslant 0, \varepsilon > 0\},$$

where

$$B(a, \varepsilon, f) = \{g \in \mathscr{C} \colon \max_{0 \leqslant t \leqslant a} |f(t) - g(t)| < \varepsilon\},$$

is a fundamental system of neighborhoods of the element $f \in \mathscr{C}$ with respect to the topology of compact convergence.

THEOREM 1. *Let* $\mathscr{C}_1 = \{a+f\colon a \in C\}$, *where* C *denotes the complex field. If the operations of addition, scalar multiplication, and product are defined as in* $\mathscr{M}$, *then* $\mathscr{C}_1$ *has the following properties*:

(i) $\mathscr{C}_1$ *is an integral domain.*

(ii) $\mathscr{C}_1$ *is a complex algebra.*

(iii) $\mathscr{C}_1$ *is isomorphic to the direct sum of the complex algebras* $\mathscr{C}$ *and* C.

(iv) $\mathscr{M}$ *is the quotient field of* $\mathscr{C}_1$.

(v) $a+f$ *is a unit of* $\mathscr{C}_1$ *iff* $a \neq 0$.

Proof. The proofs of statements (i)-(iv) are straightforward and will not be duplicated here. The multiplicative identity of $\mathscr{C}_1$, one will note, is the element 1. $a+f$ is a unit in $\mathscr{C}_1$ iff there exists $\beta \in C$ and $g \in \mathscr{C}$ such that:

(i) $a\beta = 1$, and

(ii) $ag+\beta f+fg = 0$.

$a\beta = 1$ iff $a \neq 0$. Then $\beta = 1/a$ and condition (i) is satisfied. Condition (ii) becomes: $g(a+f) = -f/a$. Thus, formally

$$g(t) = \frac{-f(t)/a}{a+f(t)} = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{1}{a}f(t)\right]^{n+1}.$$

By induction one can verify

$$(1)\qquad |f^n(t)| \leqslant \frac{K_T^n t^{n-1}}{(n-1)!} \quad \text{for} \quad n = 1, 2, \ldots,$$

where $K_T = \max\limits_{0\leqslant t\leqslant T}\{|f(t)|\}$.

Application of (1) yields

$$\left|\sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{1}{a}f(t)\right]^{n+1}\right| \leqslant \frac{K_T}{|a|}\exp(tK_T/|a|).$$

Thus the series

$$\frac{1}{a}\sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{1}{a}f(t)\right]^{n+1}$$

is uniformly convergent on every finite interval and the function g defined by

$$g(t) = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{1}{a}f(t)\right]^{n+1}$$

is an element of $\mathscr{C}$.

By direct computation it may be shown that the function g satisfies condition (ii) and the theorem is proved.

We provide the direct sum, $C \oplus \mathscr{C}$, with the product topology. Thus the isomorphism of Theorem 1 (iii) induces a topology on $\mathscr{C}_1$ with the properties:

(i) $\mathscr{C}_1$ is a topological C-algebra.

(ii) $\mathscr{C}_1$ is a Hausdorff space.

Definition 1. Let $\mathscr{C}^* = C \cup \mathscr{C} - \{0\}$ be provided with the relation $|$ defined by $c \mid d$ iff there exists $b \in \mathscr{C}^*$ such that $bc = d$.

The relation $|$ defines a preorder on $\mathscr{C}^*$ and directs $\mathscr{C}^*$.

Definition 2. We shall denote by $\mathscr{C}_p$ the image of $\mathscr{C}_1$ under the mapping $\varphi_p\colon \mathscr{C}_1 \to \mathscr{M}$ defined by:

$$\varphi_p(a+f) = \frac{a}{p} + \frac{f}{p}, \qquad p \in \mathscr{C}^*.$$

φ_p is a C-module homomorphism and if, for $x \in \mathscr{C}_1$, $y \in \mathscr{C}_1$, we define the product $x\varphi_p(y) = \varphi_p(xy)$, it follows that $\mathscr{C}_p$ is a $\mathscr{C}_1$-module.

$\mathscr{C}_p$, however, is not a C-algebra. For if $p \in \mathscr{C} - \{0\}$ then $1/p \in \mathscr{C}_p$. If $\mathscr{C}_p$ were a C-algebra then $(1/p)(1/p) = 1/p^2 \in \mathscr{C}_p$, hence $1/p \in \mathscr{C}_1$. But $p \in \mathscr{C}$ is not a unit in $\mathscr{C}_1$, thus $1/p \notin \mathscr{C}_1$.

$\mathscr{C}_p$, provided with the topology T_p induced by the mapping φ_p is a topological $\mathscr{C}_1$-module. With respect to the topology T_p on $\mathscr{C}_p$, φ_p is a homeomorphism.

LEMMA 1. *If* $p \mid q$, *then* $\mathscr{C}_p \subset \mathscr{C}_q$.

Proof. If $x \in \mathscr{C}_p$, then $x = u/p$ for some $u \in \mathscr{C}_1$. Since $p \mid q$ there exists $k \in \mathscr{C}^*$ such that $kp = q$. Thus $kpx = qx = u$ and $x = u/q$. Hence $x \in \mathscr{C}_q$.

LEMMA 2. *If* $p \in \mathscr{C}^*$, $q \in \mathscr{C}^*$ *and* $p \mid q$, *then the topology* T_p *is finer than* $T_q \mid \mathscr{C}_p$ (*the* T_q *topology relative to* $\mathscr{C}_p$).

Proof. If $p \mid q$, then there exists $r \in \mathscr{C}^*$ such that $rp = q$. The injection map $j_{p,q}\colon \mathscr{C}_p \to \mathscr{C}_q$ can be expressed $j_{p,q} = \varphi_q r \varphi_p^{-1}$ where φ_p and φ_q are the homeomorphisms defined above and r is the continuous mapping $r\colon \mathscr{C}_1 \to \mathscr{C}_1$ defined by $r(x) = rx$. Since $T_q \mid \mathscr{C}_p$ is the least fine topology such that the injection map $j_{p,q}$ is continuous, it follows that T_p is finer than $T_q \mid \mathscr{C}_p$.

Definition 3. Let X be an arbitrary set. Let $\boldsymbol{B}$ be the family of filter bases on X preordered by the relation $\leqslant$ defined by: $\mathfrak{B} \leqslant \mathfrak{G}$ if for each $B \in \mathfrak{B}$ there exists $G \in \mathfrak{G}$ such that $G \subset B$. A *convergence structure on* X is a relation $\tau\colon \boldsymbol{B}(X) \to X$ which satisfies

(i) $\{\{x\}\}\tau x$ for each $x \in X$ where $\{\{x\}\}$ denotes the filter base of which the only element is $\{x\}$.

(ii) $\mathfrak{B}\tau x$ and $\mathfrak{B} \leqslant \mathfrak{J}$ implies $\mathfrak{J}\tau x$.

If, in addition to properties (i) and (ii), τ satisfies

(iii) $\mathfrak{B}\tau x$ and $\mathfrak{J}\tau x$ implies $(\mathfrak{J} \cap \mathfrak{B})\tau x$, then τ is called a *Limitierung* on X and the pair (X, τ) is termed a *limit space*.

Convergence structures were defined by Kent in [2] and limit spaces were formulated by Fischer in [1].

Definition 4. If τ and σ are two convergence structures on a set X, then τ is finer than σ iff $\mathfrak{I}\tau x$ implies $\mathfrak{I}\sigma x$.

A topology T on a space X uniquely defines a Limitierung τ on X in the following manner: $\mathfrak{I}\tau x$ iff $\mathfrak{N}(x) \leqslant \mathfrak{I}$ where $\mathfrak{N}(x)$ is fundamental T-neighborhood system of the point x.

Proposition 1. *Let S and T be two topologies on a set X, and let σ and τ be the Limitierungen defined by S and T respectively. Then S is finer than T iff σ is finer than τ.*

Definition 5. A function f from a limit space (X, τ) to a limit space (Y, σ) is *continuous* iff for each $x \in X$, $\mathfrak{B}\tau x$ implies $f(\mathfrak{B})\sigma f(x)$.

3. M-convergence

Definition 6. A net $(b_\lambda)_{\lambda\in\Lambda}$, $b_\lambda \in \mathscr{M}$ for each λ, *M-converges* to b iff there exists $p \in \mathscr{C}^*$ such that (pb_λ) is a net in $\mathscr{C}_1$ which converges to pb.

In the special case where (b_λ) is a sequence, the above definition is equivalent to the definition given by Mikusiński.

Definition 7. A filter base $\mathfrak{I}$ *M-converges* to a point x iff there exists a $p \in \mathscr{C}^*$ such that $x \in \mathscr{C}_p$ and $\mathfrak{N}_p(x) \leqslant \mathfrak{I}$ (where $\mathfrak{N}_p(x)$ is a fundamental T_p-neighborhood system of x in $\mathscr{C}_p$).

Theorem 2. *A net (b_λ) M-converges to $b \in \mathscr{M}$ iff the base $\mathfrak{B}$ of the net filter of (b_λ) M-converges to b.*

Proof. (i) If (b_λ) M-converges to b, then there exists $p \in \mathscr{C}^*$ such that (b_λ) converges to b in $\mathscr{C}_p$. $b \in \mathscr{C}_p$, and if $N_p(b) \in \mathfrak{N}_p(b)$ there exists $\omega \in \Lambda$ such that $\lambda \geqslant \omega$ implies $B_\lambda \subset N_p(b)$ where $B_\lambda = \{b_\lambda : \lambda \geqslant \omega\}$. Thus $\mathfrak{N}_p(b) \leqslant \mathfrak{B}$ ($\mathfrak{B} = \{B_\lambda : \lambda \in \Lambda\}$) and $\mathfrak{B}$ M-converges to b.

(ii) If $\mathfrak{B}$ M-converges to b, then there exists $p \in \mathscr{C}^*$ such that $b \in \mathscr{C}_p$ and $\mathfrak{N}_p(b) \leqslant \mathfrak{B}$. If $N_p(b) \in \mathfrak{N}_p(b)$ there exists $B_\lambda \in \mathfrak{B}$ such that $B_\lambda \subset N_p(b)$. Thus for $\omega \geqslant \lambda$, $b_\omega \in N_p(b)$, hence (b_λ) M-converges to b.

Corollary 1. *A filter base $\mathfrak{I}$ M-converges to $b \in \mathscr{M}$ iff every net with base of the net filter finer than $\mathfrak{I}$ M-converges to b.*

Proof. (i) Suppose $\mathfrak{I}$ M-converges to b, $\mathfrak{B}$ is the base of the net filter of (b_λ), and $\mathfrak{I} \leqslant \mathfrak{B}$. $\mathfrak{I}$ M-converges to b implies that there exists $p \in \mathscr{C}^*$ such that $b \in \mathscr{C}_p$ and $\mathfrak{N}_p(b) \leqslant \mathfrak{I}$. Thus $\mathfrak{N}_p(b) \leqslant \mathfrak{B}$ and by Theorem 2, (b_λ) M-converges to b.

(ii) Define a net $(b_{a,F})$ where $b_{a,F} = a$ if $a \in F$ and $F \in \mathfrak{I}$. Let $\leqslant$ be defined by: $(a', F') \leqslant (a, F)$ iff $F \subset F'$. Clearly $\leqslant$ directs the index set.

The base $\mathfrak{B}$ of the net filter of the net $(b_{a,F})$ is precisely $\mathfrak{I}$. If $(b_{a,F})$ M-converges to b, then by Theorem 2, $\mathfrak{I}$ M-converges to b.

Definition 8. Let $\tau_M : \boldsymbol{B}(\mathscr{M}) \to\!\!\to \mathscr{M}$ be defined by $\mathfrak{I}\tau_M x$ iff $\mathfrak{I}$ M-converges to x for each $x \in \mathscr{M}$.

Theorem 3. *$(\mathscr{M}, \tau_M)$ is a limit space.*

Proof. (i) For each $x \in \mathscr{M}$ there exists $p \in \mathscr{C}^*$ such that $x \in \mathscr{C}_p$. Clearly $\mathfrak{N}_p(x) \leqslant \{\{x\}\}$.

(ii) Suppose $\mathfrak{I}\tau_M x$ and $\mathfrak{I} \leqslant \mathfrak{G}$. Then for some $p \in \mathscr{C}^*$, $\mathfrak{N}_p(x) \leqslant \mathfrak{I}$ hence $\mathfrak{N}_p(x) \leqslant \mathfrak{G}$. Thus $\mathfrak{G}\tau_M x$.

(iii) $\mathfrak{I}\tau_M x$ and $\mathfrak{G}\tau_M x$ iff there exist $p \in \mathscr{C}^*$, $q \in \mathscr{C}^*$ such that $x \in \mathscr{C}_p \cap \mathscr{C}_q$, $\mathfrak{N}_p(x) \leqslant \mathfrak{I}$ and $\mathfrak{N}_q(x) \leqslant \mathfrak{G}$. There exists $r \in \mathscr{C}^*$ such that $p \mid r$ and $q \mid r$, hence $\mathscr{C}_p \subset \mathscr{C}_r$ and $\mathscr{C}_q \subset \mathscr{C}_r$. By Lemma 2, T_p is finer than $T_r \mid \mathscr{C}_p$ and T_q is finer than $T_r \mid \mathscr{C}_q$, thus in $\mathscr{C}_r$, $\mathfrak{N}_r(x) \leqslant \mathfrak{N}_p(x) \cap \mathfrak{N}_q(x)$. Therefore $x \in \mathscr{C}_r$ and $\mathfrak{N}_r(x) \leqslant \mathfrak{I} \cap \mathfrak{G}$, hence $\mathfrak{I} \cap \mathfrak{G}\tau_M x$.

Definition 9. A Limitierung τ on a set X defines a *pretopology* on X iff for each $x \in X$, $\mathfrak{N}\tau x$ where $\mathfrak{N} = \bigcap_{\mathfrak{I}\tau x} \mathfrak{I}$. $\mathfrak{N}$ is called the *base of the neighborhood filter of the point x* in the pretopological space (X, τ).

Every topological space is a pretopological space, but not conversely.

Theorem 4. *The Limitierung τ_M does not define a topology or pretopology on $\mathscr{M}$.*

Proof. Consider the sequence defined by

$$f_n(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant \dfrac{n-1}{n}, \\ nt-(n-1) & \text{for} \quad \dfrac{n-1}{n} \leqslant t \leqslant 1, \\ -nt+(n+1) & \text{for} \quad 1 \leqslant t \leqslant \dfrac{n+1}{n}, \\ 0 & \text{for} \quad \dfrac{n+1}{n} \leqslant t. \end{cases}$$

Clearly f_n does not converge to 0 in $\mathscr{C}_1$, but for any $f \in \mathscr{C}$, $|(ff_n)(t)| \leqslant k_a/n$ for $0 \leqslant t \leqslant a$ (where $k_a = \max\limits_{0 \leqslant t \leqslant a} |f(t)|$). Hence ff_n converges to 0 in $\mathscr{C}_1$.

Suppose τ_M is pretopological (i.e., τ_M defines a pretopology on $\mathscr{M}$). Let $\mathfrak{N}(0)$ be the base of the neighborhood filter of 0. Then $\mathfrak{N}(0) \leqslant \mathfrak{N}_p(0)$ for all $p \in \mathscr{C}^*$ and there exists $q \in \mathscr{C}^*$ such that $\mathfrak{N}_q(0) \leqslant \mathfrak{N}(0)$ since $\mathfrak{N}_q(0)\tau_M 0$. For $r = fq$, $f \in \mathscr{C} - \{0\}$, the sequence (f_n/q) M-converges to 0 in $\mathscr{C}_r$. Thus $\mathfrak{N}_r(0) \leqslant \mathfrak{I}$, $\mathfrak{I}$ filter base of f_n/q, hence $\mathfrak{N}_q(0) \leqslant \mathfrak{I}$. However, the sequence f^n/q does not converge to 0 in $\mathscr{C}_q$ since (f_n) does not converge to 0 in $\mathscr{C}$. Thus $\mathfrak{N}_q(0) \leqslant \mathfrak{I}$ and a contradiction is established. Hence τ_M does not define a pretopology on $\mathscr{M}$.

Corollary 2. *If $p \in \mathscr{C} \subset \mathscr{C}^*$, $q \in \mathscr{C} \subset \mathscr{C}^*$ and $p \mid q$, then the topology T_p is strictly finer than the relative topology $T_q \mid C_p$.*

Proof. For the sequence (f_n) constructed in Theorem 4, (f_n/p) converges to 0 in $\mathscr{C}_q$ since $q/p \in \mathscr{C}$. But (f_n/p) does not converge to 0 in $\mathscr{C}_p$. Lemma 2 has established that T_p is finer than $T_q \mid \mathscr{C}_p$, hence T_p is strictly finer than $T_q \mid \mathscr{C}_p$.

4. Direct limits of Limitierungen

Theorem 5. *Let $\mathscr{E} = \{(E_\lambda, \tau_\lambda): \lambda \in \Lambda\}$ be a collection of limit spaces. Suppose the index set Λ is directed by $\leqslant$ which is defined by $\lambda \leqslant \lambda'$ provided that $E_\lambda \subset E_{\lambda'}$, and the injection map $j_{\lambda,\lambda'}: E_\lambda \dashrightarrow E_{\lambda'}$, is continuous. Then if $E = \bigcup_{\lambda \in \Lambda} E_\lambda$ there exists a unique finest Limitierung τ such that for each $\lambda \in \Lambda$, the injection map $j_\lambda: E_\lambda \dashrightarrow E$ is continuous.*

Proof. We define τ as follows: for each $x \in E$ $\mathfrak{J}\tau x$ iff for some $\lambda \in \Lambda$

(i) $x \in E_\lambda$,

(ii) there exists $F \in \mathfrak{J}$ such that $F \subset E_\lambda$,

(iii) $\mathfrak{J}_{E_\lambda}\tau_\lambda x$ where $\mathfrak{J}_{E_\lambda} = \{F \cap E_\lambda : F \in \mathfrak{J}\} \in \boldsymbol{B}(E_\lambda)$.

Then

(1) $\{\{x\}\}\tau x$ since for some $\lambda \in \Lambda$, $x \in E_\lambda$, and $\{\{x\}\}_{E_\lambda}\tau_\lambda x$.

(2) If $\mathfrak{J}\tau x$ and $\mathfrak{J} \leqslant \mathfrak{G}$ then there exists $\lambda \in \Lambda$ such that $\mathfrak{J}_{E_\lambda}\tau_\lambda x$. $\mathfrak{J} \leqslant \mathfrak{G}$ implies $\mathfrak{J}_{E_\lambda} \leqslant \mathfrak{G}_{E_\lambda}$ thus $\mathfrak{G}_{E_\lambda}\tau_\lambda x$, hence $\mathfrak{G}\tau x$.

(3) Suppose $\mathfrak{J}\tau x$ and $\mathfrak{G}\tau x$. Then there exist λ and λ' in Λ such that $\mathfrak{J}_{E_\lambda}\tau_\lambda x$ and $\mathfrak{G}_{E_{\lambda'}}\tau_{\lambda'} x$. Λ is directed hence there exists $\lambda'' \in \Lambda$ such that $\lambda \leqslant \lambda''$ and $\lambda' \leqslant \lambda''$. This implies $E_\lambda \subset E_{\lambda''}$, $E_{\lambda'} \subset E_{\lambda''}$, and the injection maps $j_{\lambda,\lambda''}: E_\lambda \dashrightarrow E_{\lambda''}$, $j_{\lambda',\lambda''}: E_{\lambda'} \dashrightarrow E_{\lambda''}$ are continuous. Thus $\mathfrak{J}_{E_{\lambda''}}\tau_{\lambda''}x$ and $\mathfrak{G}_{E_{\lambda''}}\tau_{\lambda''}x$. $(E_{\lambda''}, \tau_{\lambda''})$ is a limit space, hence $(\mathfrak{J}_{E_{\lambda''}} \cap \mathfrak{G}_{E_{\lambda''}})\tau_{\lambda''}x$ and $(\mathfrak{J} \cap \mathfrak{G})\tau x$.

The fact that for each $\lambda \in \Lambda$ the injection map $j_\lambda: E_\lambda \dashrightarrow E$ is continuous follows directly from the definition of τ.

Finally, suppose σ is another Limitierung on E such that the injection map $j_\lambda: E_\lambda \dashrightarrow E$ is continuous for each $\lambda \in \Lambda$. Let $x \in E$, then if $\mathfrak{J}\tau x$ there exists $\lambda \in \Lambda$ such that $\mathfrak{J}_{E_\lambda}\tau_\lambda x$. Since $j_\lambda: E_\lambda \dashrightarrow E$ is (τ_λ, σ) continuous, $\{j_\lambda(\mathfrak{J}_{E_\lambda})\} \leqslant \mathfrak{J}$ hence $\mathfrak{J}\sigma x$. Thus τ is finer than σ.

Theorem 6. *Let (E, τ) be defined as in Theorem 5, and let (X, σ) be a limit space. If f is a mapping $f: E \dashrightarrow X$, then f is continuous iff $fj_\lambda: E_\lambda \dashrightarrow X$ is continuous for each $\lambda \in \Lambda$.*

Proof. (i) If f is continuous then $fj_\lambda: E_\lambda \dashrightarrow X$ is continuous for each $\lambda \in \Lambda$ since each j_λ is continuous.

(ii) Suppose $\mathfrak{J}\tau x$, then there exists $\lambda \in \Lambda$ such that: $x \in E_\lambda$, there exists $F \in \mathfrak{J}$ such that $F \subset E_\lambda$, and $\mathfrak{J}_{E_\lambda}\tau_\lambda x$. Since $j_\lambda: E_\lambda \dashrightarrow X$ is continuous, $(fj_\lambda)(\mathfrak{J}_{E_\lambda})\sigma fj_\lambda x$. But $(fj_\lambda)(\mathfrak{J}_{E_\lambda}) \leqslant f(\mathfrak{J})$, thus $f(\mathfrak{J})\sigma f(x)$ and $f: E \dashrightarrow X$ is continuous.

Definition 10. The unique finest Limitierung τ on the set E described in Theorem 5 is the *direct limit* of the Limitierungen τ_λ.

Theorem 7. *τ_M is the direct limit of the Limitierungen τ_p defined by the topologies T_p on $\mathscr{C}_p$, $p \in \mathscr{C}^*$.*

This theorem is a consequence of Lemma 1, Lemma 2 and Proposition 1 and from the fact that $\mathscr{M} = \bigcup_{p \in \mathscr{C}^*} \mathscr{C}_p$. The definition of τ_M is precisely that of τ in Theorem 5.

If the limit spaces $(E_\lambda, \tau_\lambda)$ are linear topological spaces, as is the case with $(\mathscr{C}_p, T_p)$ then E is a linear space. In this case (E, τ) has been termed the "réunion pseudotopologique" by Marinescu [3] (Chapter I, Section 5).

5. Properties of $(\mathscr{M}, \tau_M)$

Definition 11. A limit space (X, τ) is a *Hausdorff limit space* iff $\tau: \boldsymbol{B}(X) \dashrightarrow X$ is a function.

Theorem 8. *The direct limit of Hausdorff limit spaces is a Hausdorff limit space.*

Proof. Let (E, τ) be the direct limit of the Hausdorff limit spaces $(E_\lambda, \tau_\lambda)$, $\lambda \in \Lambda$. Suppose $\mathfrak{J}\tau x$ and $\mathfrak{J}\tau y$. Then there exist $\lambda \in \Lambda$, $\lambda' \in \Lambda$ such that $\mathfrak{J}_{E_\lambda}\tau_\lambda x$ and $\mathfrak{J}_{E_{\lambda'}}\tau_{\lambda'} x$. Since Λ is directed, there exists λ'' such that $E_\lambda \cup E_{\lambda'} \subset E_{\lambda''}$ and the injection maps $j_{\lambda,\lambda''}$ and $j_{\lambda',\lambda''}$ are continuous. Thus $j_{\lambda,\lambda''}(\mathfrak{J}_{E_\lambda}) \leqslant \mathfrak{J}_{E_{\lambda''}}$ and $j_{\lambda',\lambda''}(\mathfrak{J}_{E_{\lambda'}}) \leqslant \mathfrak{J}_{E_{\lambda''}}$ hence $\mathfrak{J}_{E_{\lambda''}}\tau_{\lambda''}x$ and $\mathfrak{J}_{E_{\lambda''}}\tau_{\lambda''}y$. Since $(E_{\lambda''}, \tau_{\lambda''})$ is a Hausdorff limit space, $x = y$.

Corollary 3. *$(\mathscr{M}, \tau_M)$ is a Hausdorff limit space.*

Definition 12. Given the limit space (X_1, τ_1) and (Y_2, τ_2), the product space $(X_1 \times X_2, \tau_1 \times \tau_2)$ is defined by: For $\mathfrak{J} \in \boldsymbol{B}(X_1 \times X_2)$, $\mathfrak{J}\tau_1 \times \tau_2 x$ iff $\mathrm{pr}_\lambda(\mathfrak{J})\tau_\lambda x$ for $\lambda = 1, 2$ (where pr_λ is the projection map $\mathrm{pr}_\lambda: X \dashrightarrow X_\lambda$, $\lambda = 1, 2$).

The proof that $(X_1 \times X_2, \tau_1 \times \tau_2)$ is a limit space may be found in [1]. The Limitierung $\tau_1 \times \tau_2$ is the least fine Limitierung such that the projection maps are continuous.

Definition 13. A complex linear space E equipped with a Limitierung τ such that the mappings: $+: (E \times E, \tau \times \tau) \dashrightarrow (E, \tau)$ defined by $+(x, y) = x + y$; and $\varrho: (\boldsymbol{C} \times E, \tau_{\boldsymbol{C}} \times \tau) \dashrightarrow (E, \tau)$ defined by $\varrho(a, x) = ax$ (where $\tau_{\boldsymbol{C}}$ is the Limitierung defined by the usual topology on $\boldsymbol{C}$), are continuous, is termed a *linear limit space*.

Theorem 9. *$(\mathscr{M}, \tau_M)$ is a linear limit space in which products are jointly continuous.*

Proof. (i) Let $\mathfrak{J}(\tau_M \times \tau_M)(x, y)$, then $\mathrm{pr}_1(\mathfrak{J})\tau_M x$ and $\mathrm{pr}_2(\mathfrak{J})\tau_M y$.

There exist $p \in \mathscr{C}^*$ and $q \in \mathscr{C}^*$ such that $\mathfrak{N}_p(x) \leqslant \{\mathrm{pr}_1(\mathfrak{I})\}_{\mathscr{C}_p}$ and $\mathfrak{N}_q(y) \leqslant \{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_q}$. If $r \in \mathscr{C}^*$ such that $p \mid r$ and $q \mid r$, then $\mathfrak{N}_r(x) \leqslant \{\mathrm{pr}_1(\mathfrak{I})\}_{\mathscr{C}_r}$ and $\mathfrak{N}_r(y) \leqslant \{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_r}$. But $\mathscr{C}_r$ is a topological C-algebra thus $\{\mathrm{pr}_1(\mathfrak{I})+\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_r}$ converges to $x+y$ in $\mathscr{C}_r$, that is $\mathfrak{N}_r(x+y) \leqslant \{\mathrm{pr}_1(\mathfrak{I})+\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_r}$. Hence $\{\mathrm{pr}_1(\mathfrak{I})+\mathrm{pr}_2(\mathfrak{I})\}\tau_M(x+y)$ and $+$ is continuous.

(ii) Let $\mathfrak{I}\tau_C \times \tau_M(a, x)$. Then $\mathrm{pr}_1(\mathfrak{I})\tau_C a$ and $\mathrm{pr}_2(\mathfrak{I})\tau_M x$. There exists $p \in \mathscr{C}^*$ such that $\mathfrak{N}_p(x) \leqslant \{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_p}$. Since $\mathscr{C}_p$ is a topological C-algebra $\mathfrak{N}_p(ax) \leqslant \{\mathrm{pr}_1(\mathfrak{I})\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_p}$. Hence $\{\mathrm{pr}_1(\mathfrak{I})\mathrm{pr}_2(\mathfrak{I})\}\tau_M ax$ and scalar multiplication is continuous.

(iii) Let $\mathfrak{I}\tau_M \times \tau_M(x, y)$. Then $\{\mathrm{pr}_1(\mathfrak{I})\}\tau_M x$ and $\{\mathrm{pr}_2(\mathfrak{I})\}\tau_M y$. There exist $p \in \mathscr{C}^*$ and $q \in \mathscr{C}^*$ such that $\mathfrak{N}_p(x) \leqslant \{\mathrm{pr}_1(\mathfrak{I})\}_{\mathscr{C}_p}$ and $\mathfrak{N}_q(y) \leqslant \{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_q}$. For $r = pq$, $r \in \mathscr{C}^*$, $\mathscr{C}_p \cup \mathscr{C}_q \subset \mathscr{C}_r$, $xy \in \mathscr{C}_r$, $\mathfrak{N}_r(x) \leqslant \{\mathrm{pr}_1(\mathfrak{I})\}_{\mathscr{C}_r}$ and $\mathfrak{N}_r(y) \leqslant \{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_r}$. But products in $\mathscr{C}_r$ are continuous whenever defined. In particular if $x \in \mathscr{C}_p \subset \mathscr{C}_r$ then

$$xy = \left(\frac{a}{p}+\frac{f}{p}\right)\left(\frac{\beta}{q}+\frac{g}{q}\right) = (a+f)\left(\frac{\beta}{r}+\frac{g}{r}\right)$$

for $a+f \in \mathscr{C}_1$ and $\beta+g \in \mathscr{C}_1$. This operation is continuous since $\mathscr{C}_r$ is a topological $\mathscr{C}_1$-module. Thus $\mathfrak{N}_r(xy) \leqslant \{\{\mathrm{pr}_1(\mathfrak{I})\}_{\mathscr{C}_r}\{\mathrm{pr}_2(\mathfrak{I})\}_{\mathscr{C}_r}\}_{\mathscr{C}_r}$ and the joint continuity of multiplication in $(\mathscr{M}, \tau_M)$ follows directly.

6. On the topology defined by D. O. Norris. The purpose of this section is to characterize the results on convergence obtained by Norris in [5] in the environment of a limit space. It is necessary, first, to list some definitions and a theorem which appear in [5]. These definitions and the theorem will appear in a form consistent with the previous developments of this paper.

Definition 14. (i) $\mathscr{C}'_c$ shall denote the class of continuous, complex-valued functions of a real variable with the property:

$$f(t+c) = 0 \quad \text{for} \quad t \leqslant -c.$$

(ii) $\mathscr{C}'$ shall denote the set:

$$\mathscr{C}' = \bigcup \mathscr{C}'_c \quad (c \text{ a real number}).$$

(iii) Products in $\mathscr{C}'$ are defined by:

$$(fg)(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du;$$

but if $f \in \mathscr{C}'_{c_1}$ and $g \in \mathscr{C}'_{c_2}$, $c_1 \leqslant c_2$,

$$(fg)(t) = \int_{-c_1}^{t+c_2} f(u)g(t-u)\,du.$$

We note that the quotient field of $\mathscr{C}'$ is precisely $\mathscr{M}$ and the natural imbedding of $\mathscr{C}'$ in $\mathscr{M}$ identifies $\mathscr{C}'_0$ with that subset of $\mathscr{C}$ which consists of (continuous) functions f with the property: $f(0) = 0$.

Definition 15. (i) Let $D = \{f \in \mathscr{C}' : f \text{ is infinitely differentiable}\}$.

(ii) Let S be the subalgebra of $\mathscr{M}$ defined by: $S = \{a \in \mathscr{M} : aD \subset D\}$.

$\mathscr{C}'$ is provided with the topology of compact convergence. D is equipped with the topology induced from $\mathscr{C}'$. A topology on S, referred to as the S-topology, is defined as the coarsest topology such that each of the functions $d: S \to D$ defined by $d(a) = da$ is continuous.

Definition 16 (Restricted Mikusiński Convergence in S). A filter base $\mathfrak{I}$ N-converges to x in S iff there exists $p \in \mathscr{C}^*$, p a unit in S, such that the filter base $p\mathfrak{I}$ converges to px in some space $\mathscr{C}'_c$, $c \geqslant 0$.

Theorem 10 (Norris). *If a filter base $\mathfrak{I}$ N-converges to x in S, then $\mathfrak{I}$ converges to x in the S-topology.*

Lemma 3. *Let* $N = \left\{p \in \mathscr{C}^* : \frac{1}{p}D \subset D\right\}$ *and let* $\mathscr{C}_N = \bigcup_{p \in N} \mathscr{C}_p$. *Then* $\mathscr{C}_N$ *is a* $\mathscr{C}_1$*-algebra.*

Proof. We first note that $\mathscr{C}^*$ is a multiplicative cancellation semi-group. This follows from the fact that:

(i) C is closed under multiplication and $\mathscr{C}$ is closed under multiplication and scalar multiplication;

(ii) Associativity and the cancellation property follow from corresponding properties in $\mathscr{C}_1$;

(iii) The (unique) unit element of $\mathscr{C}^*$ is $1 \in \mathscr{C}^*$.

N is a multiplicative sub-semigroup of $\mathscr{C}^*$ since (i) N is a non-empty subset of $\mathscr{C}^*$; (ii) if $p \in N$ and $q \in N$, then

$$\frac{1}{pq}D = \frac{1}{p}\left(\frac{1}{q}D\right) \subset \frac{1}{p}D \subset D$$

hence N is closed under multiplication.

N is directed by the pre-order $\mid$ on $\mathscr{C}^*$ restricted to N. Each $\mathscr{C}_p$ is a $\mathscr{C}_1$-module $(p \in N)$ hence $\mathscr{C}_N$ is a $\mathscr{C}_1$-module. Since N is a multiplicative sub-semigroup of $\mathscr{C}^*$, $\mathscr{C}_N$ is a $\mathscr{C}_1$-algebra.

Lemma 4. $\mathscr{C}_N \subset S$.

Proof. Let $x \in \mathscr{C}_N$. Then for some $p \in N$, $x = (a+f)/p$, thus

$$xD = \frac{a+f}{p}(D) = (a+f)\left(\frac{1}{p}D\right) \subset (a+f)D \subset D.$$

Hence $x \in S$.

Definition 17. Let $(\mathscr{C}_N, \tau_N)$ denote the direct limit of the limit spaces $\{(\mathscr{C}_p, \tau_p): p \in N\}$.

The proof of the existence of the limit space $(\mathscr{C}_N, \tau_N)$ parallels the proof of Theorem 5.

THEOREM 11. *$\mathfrak{I}\tau_N x$ iff $\mathfrak{I}$ N-converges to x.*

Proof. $\mathfrak{I}\tau_N x$ implies that there exists $p \in N$ such that $x \in \mathscr{C}_p$ and $\mathfrak{N}_p(x) \leqslant \mathfrak{I}$. $p \in N$ implies p is a unit in S; $\mathfrak{N}_p(x) \leqslant \mathfrak{I}$ iff $p\,\mathfrak{I}$ converges to px in $\mathscr{C}$.

The operator $e^{cs}\colon \mathscr{C}' \to \mathscr{C}'$ is defined by:

$$e^{cs}\{f(t)\} = \begin{cases} 0 & \text{for} \quad t < -c, \\ f(t+c) & \text{for} \quad -c \leqslant t. \end{cases}$$

(A derivation of the operator e^{cs} may be found in [4], Part III, Chap. 2). From the definition of e^{cs} one notes $e^{cs}D \subset D$, hence $e^{cs} \in S$. If $\mathfrak{I}$ N-converges to x, then there exists $p \in \mathscr{C}^*$, p a unit in S, such that $p\,\mathfrak{I}$ converges to px in $\mathscr{C}'_c$ for some $c \geqslant 0$. If $p\mathfrak{I}$ converges to px in $\mathscr{C}'_c$ then $e^{-cs}p\mathfrak{I}$ converges to $e^{-cs}px$ in $\mathscr{C}'_0$, hence in $\mathscr{C}_1$. Since $e^{-cs}p \in N$, the proof is complete.

Two questions arise in connection with this section:

(a) It was established that $\mathscr{C}_N \subset S$. It is suspected, but as yet unverified, that $\mathscr{C}_N = S$.

(b) Let T_N denote the topology induced on $\mathscr{C}_N$ by the S-topology. Let σ denote the Limitierung induced on $\mathscr{C}_N$ by the topology T_N. Theorem 10 and Theorem 11 show that $\sigma \leqslant \tau_N$. Is T_N the finest topology on $\mathscr{C}_N$ with this property?

Bibliography

[1] H. R. Fischer, *Limesräume*, Math. Annalen 137 (1959), p. 269-303.

[2] D. C. Kent, *Convergence functions and their related topologies*, Fundamenta Mathematicae 54 (1964), p. 125-133.

[3] G. Marinescu, *Espaces vectoriels pseudotopologiques et théorie des distributions*, Berlin 1963.

[4] J. G. Mikusiński, *Operational calculus*, 5th ed., New York 1959.

[5] D. O. Norris, *A topology for Mikusiński operators*, Studia Mathematica 24 (1964), p. 245-255.

[6] K. Urbanik, *Sur la structure non-topologique du corps des opérateurs*, ibidem 14 (1953-4), p. 243-246.

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Strong differentials in L^p

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CHAPTER I

This chapter contains the statement of the main results. Chapters II-VI contain their proofs. Chapter VII contains some additional remarks.

In what follows the function $f(x) = f(x_1, x_2, \ldots, x_n)$ is defined in the n-dimensional unit cube: $0 \leqslant x_j \leqslant 1, j = 1, \ldots, n$, and is of the class L^p there, $1 \leqslant p < \infty$. We assume once for all that $n \geqslant 2$.

Definition 1. *The function f has at a point x a k-th differential in L^p* — for brevity, a (k, p) *differential* — if there is a polynomial $P(t) = P(t_1, \ldots, t_n)$ of degree k or less such that

$$(1.1) \qquad \left(\frac{1}{|Q|}\int_Q |f(x+t)-P(t)|^p dt\right)^{1/p} = o(h^k), \quad h \to 0,$$

where Q is an n-dimensional cube containing the origin and of edge h.

The purpose of this paper is to investigate the connections between this differential and certain other notions of differential. In [3] a connection between this and what may be thought of as the partial (k, p) differential is discussed. The main theorem of [3] is:

THEOREM A. *If f has a (k, p) differential at each point of a set E, then for any integer m satisfying $1 \leqslant m < n$ the function f has a (k, p) differential almost everywhere in E with respect to the variable $x' = (x_1, x_2, \ldots, x_m)$.*

Actually what we shall need here is the following result, also proved in [3], of which Theorem A is a simple consequence.

THEOREM A'. *Let $x' = (x_1, \ldots, x_m)$, $x'' = (x_{m+1}, \ldots, x_n)$ and let $f(x) = f(x_1 \ldots, x_n) = f(x', x'')$ be non-negative and integrable over the unit cube Q^0. Let a be any positive number and let Q and I denote respectively arbitrary n-dimensional and m-dimensional cubes with edge h. If at each point $x = (x', x'')$ of a set $E \subset Q^0$ we have*

$$\int_Q f(\xi)\,d\xi = o(h^{n+a}), \quad h \to 0,$$