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ON THE GREEN'S FUNCTION FOR SECOND ORDER ELLIPTIC DIFFERENTIAL EQUATIONS

BY

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Let D be a bounded domain in the Euclidean m-space R^m; let the partial differential operator L defined by

$$Lu = \sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

be elliptic there. Denote by $\tilde{D}$ the set

$$\tilde{D} = (\overline{D}\times\overline{D})\setminus(\partial D\times\partial D).$$

In this paper we derive a number of properties for the Green function of the equation $Lu = 0$ with respect to the domain D. These results are used for the construction of a fundamental solution (for the equation $Lu = 0$) in another paper [3] by the author.

LEMMA. *Suppose that the coefficients of the operator L ($m \geqslant 2$) satisfy the assumptions of Hopf theorem* (see [2], p. 4). *Furthermore, let $V(x, y)$ be a function continuous on $\tilde{D}$ for $x \neq y$ and such that the integral*

$$I_V(x) = \int_D V(x, y)\Theta(y)\,dy$$

is of class C^2 on D, continuous on $\overline{D}$ and satisfies the inequalities

$$I_V(x) \geqslant 0 \ \textit{for}\ x\in\partial D, \qquad L[I_V(x)] \leqslant 0 \ \textit{for}\ x\in D$$

for all non-negative $\Theta(x)$ in $C_0^1(D)$ (of class C^1 on D and with compact support). Under the above conditions we have

$$V(x, y) \geqslant 0 \quad \textit{for} \quad x, y\in D, x\neq y.$$

Proof. Suppose the contrary, i.e. $V(x_0, y_0) < 0$, where $x_0, y_0 \in D$, $x_0 \neq y_0$. Then $V(x_0, y) < 0$ in a neighbourhood $\Omega(y_0)$ of y_0, $x_0 \notin \Omega(y_0)$.

Let $\Theta(y)$ be a non-negative $C^1(D)$ function vanishing outside the neighbourhood $\Omega(y_0)$ and positive in its interior. Then we have

$$(1)\qquad I_V(x_0) < 0.$$

On the other hand, $L[I_V(x)] \leqslant 0$ on D and $I_V(x) \geqslant 0$ on ∂D. Applying Hopf's theorem we get $I_V(x) \geqslant 0$ on D, which contradicts (1).

A function $G = G_D(x, y)$ will be called (see [1], p. 167) a *Green function for the equation $Lu = 0$ with respect to the domain D* if

(i) $G_D(x, y)$ is continuous on $\tilde{D}$ for $x \neq y$;

(ii) $G_D(x, y) = 0$ for $x \in \partial D,\ y \in D$;

(iii) the interval

$$I_G(x) = \int_D G_D(x, y)\Theta(y)\,dy$$

is of class C^2 on D, continuous on $\bar{D}$ and satisfies the identity

$$L[I_G(x)] = -\Theta(x)$$

for all $\Theta(x)$ in $C_0^1(D)$.

Under suitable conditions, the above definition is equivalent to the definition of the Green's function which has been used in [2].

The following theorem is an easy corollary of the Lemma:

THEOREM 1. *Under the hypotheses of the Lemma there is at most one Green's function with respect to the domain D, and the inequality $G_D(x, y) \geqslant 0$ holds for any two points x, y in D, $x \neq y$, provided $G_D(x, y)$ exists.*

THEOREM 2. *Under the hypotheses of the Lemma, if $D \subset D_1$, then*

$$G_D(x, y) \leqslant G_{D_1}(x, y) \quad \textit{for} \quad (x, y) \in \tilde{D}, \quad x \neq y,$$

provided $G_D(x, y)$ and $G_{D_1}(x, y)$ exist.

Theorem 2 can be obtained easily from the Lemma for $V(x, y)$ $G_{D_1}(x, y) - G_D(x, y)$ and from Theorem 1.

In the sequel we shall use the function

$$(2)\qquad \mathscr{A}(x, y) = \frac{\sum_{i=1}^{m}(x_i - y_i)^2\Big[\sum_{i=1}^{m} a_{ii}(x) + \sum_{i=1}^{m} b_i(x)(x_i - y_i) + (-m+2+a)^{-1}c(x)\sum_{i=1}^{m}(x_i - y_i)^2\Big]}{\sum_{i,j=1}^{m} a_{ij}(x)(x_i - y_i)(x_j - y_j)}$$

for $x \neq y$, $0 < a < 1$.

In the special case of Laplace equation we have $\mathscr{A}(x, y) = m$.

THEOREM 3. *Let the coefficients of the elliptic operator of* L $(m > 2)$ *be Hölder continuous (with exponent* λ, $0 < \lambda < 1$) *on* $D_0 \supset \overline{D}$, $c(x) \leqslant 0$ *on* D_0, *and* $\mathscr{A}(x, y) > m-1+\varepsilon$ *for* $x \in D_0$, $y \in D_0$, $x \neq y$, *where* ε *is a positive constant* $(1-\varepsilon < \lambda)$. *Then for every point* $y_0 \in D$ *and for any number* $\varrho > 0$ *such that* $K(y_0, 2\varrho) = \{y|\ |y-y_0| < 2\varrho\} \subseteq D$, *there exist two constants* $M > 0$ *and* α $(1-\varepsilon < \alpha < \lambda)$ *such that*

$$G_D(x, y) \leqslant U_0(x, y) + Mr^{-m+2+\alpha}$$

for $x \in \overline{D}$, $y \in K(y_0, \varrho)$, *where*

$$U_0(x, y) = \begin{cases} \dfrac{1}{(m-2)\omega_m\sqrt{A(y)}}\Big[\sum\limits_{i,j=1}^{m} A_{ij}(y)(x_i-y_i)(x_j-y_j)\Big]^{(-m+2)/2}\Big(1-\Big(\dfrac{r}{\varrho}\Big)^3\Big) & \text{for } r \leqslant \varrho, \\ 0 & \text{for } r > \varrho, \end{cases}$$

$r = \big[\sum\limits_{i=1}^{m}(x_i-y_i)^2\big]^{1/2}$, $[A_{ij}(y)]$ *is the inverse of the matrix* $[a_{ij}(y)]$, ω_m *is the area of the unit sphere in* R^m *and* $A(y)$ *is the determinant of the matrix* $[A_{ij}(y)]$, *provided* $G_D(x, y)$ *exists. Constants* M *and* α *can be chosen independently of* D.

Proof. Consider the function

$$\Phi(x, y) = -U_0(x, y) - Mr^{-m+2+\alpha} + G_D(x, y).$$

The constants M, α, ϱ will be defined later. Let $I_\Phi(x)$ denote the integral

$$I_\Phi(x) = \int_D \Phi(x, y)\Theta(y)\,dy,$$

where $\Theta(x)$ is in the class $C^1(D)$. Then

$$I_\Phi(x) = -\int_D U_0(x, y)\Theta(y)\,dy - M\int_D r^{-m+2+\alpha}\Theta(y)\,dy + \int_D G_D(x, y)\Theta(y)\,dy. \tag{3}$$

$U_0(x, y)$ is the Levi's function (see [2], p. 13). The integral

$$\int_D U_0(x, y)\Theta(y)\,dy$$

is in the class $C^2(D)$ (see [2], p. 23, theorem 13. II). The integral $\int_D r^{-m+2+\alpha}\Theta(y)\,dy$ is in the class $C^2(D)$, too (see [2], p. 22). Hence, from the definition of the Green function $I_\Phi(x)$ is in the class $C^2(D)$ and

$$(4)\qquad L[I_\Phi(x)] = L\Big[-\int_D U_0(x,y)\Theta(y)\,dy\Big] - ML\Big[\int_D r^{-m+2-a}\Theta(y)\,dy\Big] + \\ + L\Big[\int_D G_D(x,y)\Theta(y)\,dy\Big].$$

From the Poisson theorem [(2], p. 25, formula (13.7)) we conclude that

$$(5)\qquad L\Big[-\int_D U_0(x,y)\Theta(y)\,dy\Big] = \Theta(x) - \int_D L_x[U_0(x,y)]\Theta(y)\,dy \quad \text{for} \quad x\in D,$$

where L_x denotes the operator L applied with respect to the variable x. Now we have

$$(6)\qquad L\Big[\int_D r^{-m+2+a}\Theta(y)\,dy\Big] = \int_D L_x[r^{-m+2+a}]\Theta(y)\,dy.$$

From (4), (5), (6) and the definition of the Green function we obtain

$$L[I_\Phi(x)] = -\int_D L_x[U_0(x,y)]\Theta(y)\,dy - M\int_D L_x(r^{-m+2+a})\Theta(y)\,dy,$$

or

$$L[I_\Phi(x)] = \int_D L_x[-U_0(x,y) - Mr^{-m+2+a})]\Theta(y)\,dy.$$

On the other hand, we have

$$L_x[-U_0(x,y)] = O(r^{-m+\lambda})$$

and

$$L_x(Mr^{-m+2+a}) = M\sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 r^{-m+2+a}}{\partial x_i\partial x_j} + M\sum_{i=1}^{m} b_i\frac{\partial r^{-m+2+a}}{\partial x_i} + Mcr^{-m+2+a}$$

$$= M(-m+2+a)r^{-m+2+a}\sum_{i,j=1}^{m} a_{ij}(x_i-y_i)(x_j-y_j)[(-m+a)+\mathscr{A}(x,y)],$$

where $\mathscr{A}(x,y)$ is given by formula (2). Note that we can fix $a = (1-\varepsilon+\lambda)/2$ independently of D. Let y_0 be an arbitrary point in D. Consider two spheres $K(y_0,\varrho)$ and $K(y_0,2\varrho)\subseteq D$. From $L_x[Mr^{-m+2+a}] = O(r^{-m+a})$ and $\mathscr{A}(x,y) > m-1+\varepsilon$ we infer that there exists a constant $M>0$ such that

$$L_x[-U_0(x,y) - Mr^{-m+2+a}] \geqslant 0$$

for x and y in $K(y_0,2\varrho)$, $x\neq y$. We can fix M independently of D. Let $\Theta(y)$ be a non-negative function of the class $C^1(D)$, vanishing outside

$K(y_0, \varrho)$ and positive in its interior. Then we have

$$I_\Phi(x) = \int_{K(y_0,\varrho)} \Phi(x, y)\Theta(y)\,dy$$

and

$$L[I_\Phi(x)] \geqslant 0 \tag{7}$$

in $K(y_0, 2\varrho)$. From the definition of $U_0(x, y)$ and from the hypotheses of Theorem 3 it follows that inequality (7) holds in D. Now, function (3) can be rewritten as follows:

$$I_\Phi(x) = \int_{K(y_0,\varrho)} -U_0(x, y)\Theta(y)\,dy - M \int_{K(y_0,\varrho)} r^{-m+2+a}\Theta(y)\,dy + \int_{K(y_0,\varrho)} G_D'(x, y)\Theta(y)\,dy. \tag{8}$$

The first and the third integrals on the right-hand side of (8) vanish on ∂D. The second integral is non-positive. Hence $I_\Phi(x) \leqslant 0$ on ∂D. From the Hopf theorem it follows that

$$I_\Phi(x) = \int_D \Phi(x, y)\Theta(y)\,dy \leqslant 0$$

on $\bar{D}$. Then $\Phi(x, y) \leqslant 0$ for $x \in \bar{D}$, $y \in K(y_0, \varrho)$. Thus the proof of Theorem 3 is complete.

REFERENCES

[1] K. Maurin, *Metody przestrzeni Hilberta*, Warszawa 1957.

[2] C. Miranda, *Equazioni alle derivate parziali di tipo ellittico*, Berlin-Göttingen-Heidelberg 1955.

[3] B. Szafirski, *The fundamental solution for second order elliptic differential equations*, Annales Polonici Mathematici (to appear).

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