ON THE REGULARITY WITH RESPECT TO TIME OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Consider in E_3 a fixed bounded domain Ω and put $\Omega_T = \Omega \times [0, T]$, where T is a fixed positive constant. Denote by D_T the class of 3-dimensional C^∞ -vectors $\varphi = \varphi(x, t)$ defined on Ω_T which are divergence free and have compact supports in $\Omega \times [0, T]$. Denote further by (φ, ψ) the ordinary $L_2(\Omega)$ scalar product of two vectors φ, ψ , that is put

$$(\varphi, \psi) = \sum_i \int_{\Omega} \varphi_i \psi_i dx,$$

and let $\|\varphi(t)\|^2 = (\varphi, \varphi)$. The letter t retained in the notation of the norm reminds of its dependence upon t . Denote $\sum_i (\text{grad} \varphi_i, \text{grad} \psi_i)$ by (φ_x, ψ_x) .

Thus

$$\|\varphi_x(t)\|^2 = \int_{\Omega} \varphi_x^2 dx$$

is the Dirichlet integral of the vector φ . Finally, let $|\varphi|$ denote the length of the vector φ , i.e. $|\varphi| = (\sum_i \varphi_i^2)^{1/2}$.

A 3-dimensional vector $u = u(x, t)$, defined on Ω_T , is called a *weak solution* (in the sense of E. Hopf) of the non-stationary Navier-Stokes equations subject to the boundary condition $u|_S = 0$ (S denotes the boundary of Ω) and to the initial condition $u(x, 0) = u_0(x)$ if $\|u(t)\|$ is finite for each $t \in [0, T]$, u is in the closure $\bar{D}_T$ of D_T under the norm

$$\|\varphi\|_{1,T}^2 = \int_0^T (\|\varphi(t)\|^2 + \|\varphi_x(t)\|^2) dt$$

and the equation

$$(1) \quad \int_0^T \{(u, \varphi_t) - \nu(u_x, \varphi_x) + (u, u\varphi_x) + (f, \varphi)\} dt = -(u_0, \varphi_0)$$

is satisfied for each $\varphi \in D_T$. In the last formula φ_t denotes $\partial\varphi/\partial t$, ν is a positive constant (kinematic coefficient of viscosity), $(u, u\varphi_x) = \sum_i (u, u_i \text{grad} \varphi_i)$,

$f = f(x, t)$ denotes a given vector (exterior force) which belongs to $L_2(\Omega_T)$ and, finally, $\varphi_0 = \varphi(x, 0)$. The vector u_0 is assumed to belong to $L_2(\Omega)$.

As it is well known, we have for u the a priori estimate

$$(2) \quad \|u(t)\|^2 + 2\nu \int_0^t \|u_x(t)\|^2 dt \leq C,$$

valid for all $t \in [0, T]$, with a constant C depending only on L_2 -norms of f and u_0 (cf., for example, [1]). Moreover, u , when considered as a function of t with values in $L_2(\Omega)$, depends continuously on t in the weak topology of $L_2(\Omega)$. In fact, much more may be deduced about the character of this dependence. It is the aim of this paper to prove the following

THEOREM 1. *If u is a weak solution, as defined above, in Ω_T of the nonstationary Navier-Stokes equations, and, moreover, if the boundary S of Ω is sufficiently regular, say, of class C^2 , then u is Hölder-continuous in the mean, i.e. we have*

$$(3) \quad \int_0^{T-\tau} \|\Delta_\tau u(t)\|^2 dt \leq C\tau^{2\kappa}$$

for sufficiently small $\tau > 0$, where $\Delta_\tau u = u(x, t+\tau) - u(x, t)$. The exponent κ is $\leq 2/7$ and the constant C depends only on Ω , T and on the constant of the estimate (2).

(We shall constantly use, when denoting any constant, the same letter C although its numerical value may be different in different cases).

Remark. Our theorem gives a generalization of a result of Lions [2], p. 218; who proved that the Fourier transform $\hat{u}(x, \sigma)$, with respect to t , of u satisfies the condition

$$\int_{-\infty}^{\infty} (1 + |\sigma|^{2\gamma}) \|\hat{u}(\sigma)\|^2 d\sigma < \infty$$

for $\gamma < 1/4$, from which (3) results with $\kappa < 1/4$. For the 2-dimensional case, which is also contained in the result of Lions, see below, Theorem 2.

We begin with two lemmas. Denote by $W_2^1(\Omega)$ and $W_2^2(\Omega)$, as usually, the Sobolev's spaces and by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, the standard norms in these spaces, i.e. $\|u\|_1^2 = \|u\|^2 + \|u_x\|^2$, $\|u\|_2^2 = \|u_x\|_1^2 + \|u\|^2$. Finally, let $\overset{\circ}{W}_2^1(\Omega)$ denote the closure in the norm $\|\cdot\|_1$ of $C_0^\infty(\Omega)$ (infinitely often differentiable with compact supports in Ω) functions.

LEMMA 1. *If a vector $w = w(x)$ defined on Ω is in the closure under the $\|\cdot\|$ - norm of the class of vectors which are regular, divergence free and have compact supports in Ω , then w may be represented in the form $w = \text{rot } v$,*

where v is divergence free, belongs to $W_2^1(\Omega)$ and its tangential component v_σ on S vanish. We have also

$$(4_1) \quad \|v\|_1 \leq C \|w\|$$

with C depending only on Ω .

If, moreover, $w \in W_2^1(\Omega)$, then $v \in W_2^2(\Omega)$ and

$$(4_2) \quad \|v\|_2 \leq C \|w\|_1.$$

The boundary S of Ω is assumed to be of class C^2 .

Actually, it suffices to suppose that only the normal component w_n of w vanishes along S . For the proof of Lemma 1 see [3].

In the sequel, the standard procedure of mollifying of functions will be used. Let a fixed function $j(x) \in C_0^\infty(E_3)$ satisfy the conditions: $j(x) \geq 0$, $j(x) = 0$ for $|x| \geq 1$, $\int_{E_3} j(x) dx = 1$. Let $j_\varepsilon(x) = \varepsilon^{-3} j(x/\varepsilon)$ and put

$$(5) \quad f_\varepsilon(x) = \int j_\varepsilon(z) f(x-z) dz$$

for any locally integrable function $f(x)$, the integration being extended over the whole E_3 . Besides the estimate

$$(6) \quad \|f_\varepsilon\| \leq \|f\|$$

valid for any $f \in L_2(E_3)$ we shall make use also of two other properties of f expressed in Lemma 2.

LEMMA 2. Let $w \in \overset{\circ}{W}_2^1(\Omega)$ and let Ω be a bounded domain with a boundary S of class C^1 . Then, for sufficiently small ε , we have the estimates

$$(7) \quad \|w_\varepsilon - w\| \leq C\varepsilon \|w_x\|,$$

$$(8) \quad \|w_{\varepsilon,x}\| \leq C\varepsilon^{-1} \|w\|,$$

with w_ε resulting from w due to (5) and C independent of w and ε . $w_{\varepsilon,x}$ in formula (8) denotes $\text{grad } w_\varepsilon$.

Proof. Assuming ε to be positive and sufficiently small, denote by ω_ε the boundary strip of Ω of the width ε , i.e. $\omega_\varepsilon = \{x: x \in \Omega, \text{dist}(x, S) \leq \varepsilon\}$. Take a function $\zeta(x) \in C_0^1(\Omega)$, $0 \leq \zeta \leq 1$, vanishing in ω_ε and equal to 1 in $\Omega - \omega_{3\varepsilon}$. Denote by $\tilde{w}$ a function which is equal to ζw in Ω and zero outside of Ω . We can write

$$\tilde{w}_\varepsilon - \tilde{w} = - \int j_\varepsilon(z) dz \int_0^1 z \cdot \tilde{w}_x(x - \lambda z) d\lambda,$$

in accordance with (5). Hence, changing the order of integration and applying the Schwarz inequality, we get

$$|\tilde{w}_\varepsilon - \tilde{w}|^2 \leq \int_0^1 d\lambda \int j_\varepsilon(z) |z \cdot \tilde{w}_x(x - \lambda z)|^2 dz$$

which gives us, when we note that $j(z) = 0$ for $|z| \geq \varepsilon$, the estimate

$$\|\tilde{w}_\varepsilon - \tilde{w}\| \leq \varepsilon \|\tilde{w}_x\|.$$

To get from the last inequality the desired inequality (7) it suffices to use the two obvious inequalities

$$\begin{aligned} \|w_\varepsilon - w\|^2 &\leq \|\tilde{w}_\varepsilon - \tilde{w}\|^2 + \int_{\omega_{4\varepsilon}} |w_\varepsilon - w|^2 dx, \\ \|\tilde{w}_x\|^2 &\leq \|w_x\|^2 + \int_{\omega'} |\tilde{w}_x|^2 dx, \quad \omega' = \omega_{3\varepsilon}, \end{aligned}$$

and then to apply to the integrals which have appeared in the last two formulae the estimates

$$(9) \quad \int_{\omega_{4\varepsilon}} |w_\varepsilon - w|^2 dx \leq 6 \int_{\omega_{5\varepsilon}} w^2 dx \leq C\varepsilon^2 \int_{\omega_{5\varepsilon}} w_x^2 dx,$$

$$(10) \quad \int_{\omega'} \tilde{w}_x^2 dx \leq C \|w_x\|^2,$$

where C is independent of w and ε . To derive (9) and (10) we extend w outside of Ω putting there $w = 0$. Note that this new w does not cease to belong to $\overset{\circ}{W}_2^1(\Omega)$. Further, it is easy to see that we have, for any $f \in \overset{\circ}{W}_2^1(\Omega)$ and for δ sufficiently small, the inequality

$$(11) \quad \int_{\omega_\delta} f^2 dx \leq C\varepsilon^2 \int_{\omega_\delta} f_x^2 dx$$

with C independent of f and δ . Inequality (9) is an immediate consequence of (11). Suppose now that ζ has been chosen in such a way that $|\zeta_x| \leq C\varepsilon^{-1}$ throughout ω' with C depending only on S , which is always possible. Due to (11) and to the condition imposed on ζ we have

$$(12) \quad \int_{\omega'} \tilde{w}_x^2 dx \leq 2 \int_{\omega'} (|\zeta_x \cdot w|^2 + w_x^2) dx \leq C \|w_x\|^2,$$

and this is the desired inequality (10). This completes the proof of (7).

To prove (8) let us note that, on account of (5), we can write

$$w_{\varepsilon,x}(x) = \int j_{\varepsilon,z}(z) w(x - z) dz$$

with w denoting, as before, the extension of the original w by putting 0 outside of Ω and $j_{\varepsilon,z} = \text{grad} j_{\varepsilon}(z)$. Hence, by applying Minkowski's inequality, we get

$$\|w_{\varepsilon,x}\| \leq \|w\| \int |j_{\varepsilon,z}(z)| dz \leq C\varepsilon^{-1} \|w\|$$

with C independent of w and ε , and this is already the wanted inequality (8).

Lemma 2 is a special case of a more general theorem of the first author (see [4]).

Proof of Theorem 1. By Lemma 1, u may be represented, for each $t \in [0, T]$, in the form $u = \text{rot} v$ with $v = v(x, t)$ having zero tangential components v_{σ} along S , $v_{\sigma}|_S = 0$, and for this v , due to (2) and (4₁), we have in $[0, T]$ a uniform estimate

$$(13) \quad \|v(t)\|_1 \leq C.$$

Choose now a function $\zeta(x)$ having all the properties formulated in the proof of Lemma 2, including the estimate $|\zeta_x| \leq C\varepsilon^{-1}$. Moreover, let ζ take constant values along the surfaces S_{ϱ} parallel to S (a surface S_{ϱ} is called *parallel* to S if every point of it lies at the same distance ϱ from S). A construction of such a ζ is, for sufficiently small ε , guaranteed by the required regularity of S .

Let $\tilde{u} = \tilde{u}(x, t)$ be a vector defined as follows:

$$\tilde{u}(x, t) = \begin{cases} \zeta_0(t) \text{rot} \{\zeta(x) v(x, t)\} & \text{for } (x, t) \in \Omega \times [0, T], \\ 0 & \text{for } (x, t) \in \Omega \times ([-\tau, 0] \cup [T, T+\tau]). \end{cases}$$

Here $\zeta_0(t)$ denotes a $C^\infty([0, T])$ function, $0 \leq \zeta_0 \leq 1$, $\zeta_0(t)$ is equal to 1 for $3\tau \leq t \leq T-3\tau$ and vanishes for $0 \leq t \leq 2\tau$ and $T-2\tau \leq t \leq T$. Take $\tilde{u}_\varepsilon$ resulting by application of (5) to $\tilde{u}$ and put

$$(14) \quad U(x, t) = -\Delta_{-\tau} \int_0^\tau \tilde{u}_\varepsilon(x, t+\eta) d\eta$$

for $\tau > 0$, where Δ has the same meaning as in formula (3). The function U thus defined is, as a function of x , of class C^∞ and is divergence free, since the mollifying operation (5) preserves this property. Moreover, U vanishes in the neighbourhood of $t = T$. However, for lack of the required regularity with respect to t , U does not fall into the class D_T . Now the proof of the theorem will run as follows: we replace φ by U in (1), extract the desired term

$$H^2 = H^2(\tau) = \int_0^{T-\tau} \|\Delta_\tau u(t)\|^2 dt$$

and estimate all the remaining terms. From the inequality obtained in this way, we deduce the desired estimate (3). To justify this procedure

we have to show that the replacing φ by U in (1) is admissible and this will be done later.

Substitute $\varphi = U$ into (1). If we write

$$R = - \int_0^{T-\tau} (\Delta_\tau u, \Delta_\tau (\tilde{u}_\varepsilon - u)) dt,$$

we may transform the first term of the expression thus obtained as follows:

$$(15) \quad \int (u, \varphi_t) dt = \int (u, U_t) dt = - \int (\Delta_\tau u, \Delta_\tau \tilde{u}_\varepsilon) dt = -H^2 + R.$$

Here as well as in the sequel we omit the limits of integration in the integrals if these are taken over $(0, T)$. When performing the transformation of (15) we have used the evident identity $U_t = -\Delta_{-\tau} \Delta_\tau u$ and then we have applied the formula

$$\int h(t) \cdot \Delta_{-\tau} g(t) dt = \int \Delta_\tau h(t) \cdot g(t) dt$$

valid for any pair f, g , with g vanishing in $(-\tau, 0)$ and $(T-\tau, T)$. Now we apply to R the Schwarz' inequality and we get in this way the inequality

$$(16) \quad R^2 \leq 4H^2 \int \|\tilde{u}_\varepsilon(t) - u(t)\|^2 dt.$$

Similarly, the Schwarz inequality will be applied to the remaining terms of (1). This leads to the following inequalities:

$$(17) \quad \left| \int (u_x, U_x) dt \right| \leq 2 \int_0^\tau d\eta \int \|u_x(t)\| \|\tilde{u}_{\varepsilon,x}(t+\eta)\| dt,$$

$$(18) \quad \left| \int (u, u U_x) dt \right| \leq 2 \int_0^\tau d\eta \int \|u^2(t)\| \|\tilde{u}_{\varepsilon,x}(t+\eta)\| dt,$$

$$(19) \quad \left| \int (f, U) dt \right| \leq 2 \int_0^\tau d\eta \int \|f(t)\| \|\tilde{u}_\varepsilon(t+\eta)\| dt.$$

Note that the right-hand member of (1), according to the definition of U , vanishes, $(u_0, U_0) = 0$.

The subsequent estimates of the expressions which have appeared in (16)-(19) will be based on the following inequalities valid for those $t \in [0, T]$, where $\zeta_0(t) = 1$:

$$(20) \quad \|\tilde{u}(t)\| \leq C,$$

$$(21) \quad \|\tilde{u}(t) - u(t)\| \leq C\varepsilon^{1/2} \|u_x(t)\|,$$

$$(22) \quad \|\tilde{u}_{\varepsilon,x}(t)\| \leq C\varepsilon^{-1/2} \|u_x(t)\|.$$

For the remaining t of the interval $[0, T]$, where $\zeta_0(t) < 1$, inequality (20) remains unchanged, (21) should be replaced by $\|\tilde{u}(t) - u(t)\| \leq C$ and $C\varepsilon^{-1}$ should be added to the right-hand member of (22).

Assume for a moment that inequalities (20)-(22) are true and neglect by the the time being the presence of $\zeta_0(t)$ in $\tilde{u}$. Replace $\|\tilde{u}_\varepsilon - u\|^2$ in (16) by $\|\tilde{u}_\varepsilon - u_\varepsilon\|^2 + \|u_\varepsilon - u\|^2$. We have $\|\tilde{u}_\varepsilon - u_\varepsilon\|^2 \leq \|\tilde{u} - u\|^2$ and this, according to (21) and (11), is less than $C\varepsilon\|u_x(t)\|^2$, whereas $\|u_\varepsilon - u\|^2$ is less than $C\varepsilon^2\|u_x(t)\|^2$ due to (7) applied to u . Finally, making use of estimate (2), we get

$$(23) \quad R \leq C H \varepsilon^{1/2}.$$

Now the right-hand member of (17) may be estimated by using (22) and (2) which yields

$$(24) \quad \left| \int (u^x, U^x) dt \right| \leq C \varepsilon^{-1/2} \tau.$$

Due to (8) applied to $\tilde{u}$ and due to (2), (22) and (20) we have

$$\|\tilde{u}_{\varepsilon,x}(t)\|^2 \leq C \varepsilon^{-3/2} \|u_x(t)\| \|\tilde{u}(t)\| \leq C \varepsilon^{-3/2} \|u_x(t)\|$$

and hence we get

$$(25) \quad \left| \int \tilde{u}(u, u U_x) dt \right| \leq C \varepsilon^{-3/4} \int_0^\tau d\eta \int \|u^2(t)\| \|u_x(t+\eta)\|^{1/2} dt.$$

Let us now recall the estimate

$$(26) \quad \|f^2\| \leq C \|f\|^{1/2} \|f_x\|^{3/2}$$

valid for any $f \in \overset{\circ}{W}_2^1(\Omega)$ with C independent of f (and even of Ω ; the proof of the last inequality may be found, e.g., in [1]). Applying this inequality, with $f = u$, to (25) we get

$$(27) \quad \left| \int (u, u U_x) dt \right| \leq C \varepsilon^{-3/4} \tau.$$

Finally, due to (6), (20) and $f \in L_2(\Omega_T)$, the integral $\int (f, U) dt$ in formula (19) may be estimated by a constant times τ .

Thus equation (1) with φ replaced by U yields the inequality

$$H^2 \leq C(\varepsilon^{1/2} H + \tau \varepsilon^{-3/4} + \tau \varepsilon^{-1/2} + \tau) \leq C(\varepsilon^{1/2} H + \tau \varepsilon^{-3/4}),$$

and hence

$$H^2 \leq C(\varepsilon + \tau \varepsilon^{-3/4}).$$

Putting $\varepsilon = \tau \varepsilon^{-3/4}$ we get $H^2 \leq C \tau^{4/7}$ which is the desired estimate (3) with $\varkappa = 2/7$.

To complete the proof of the theorem we have, first of all, to demonstrate inequalities (20)-(22).

Let v be the vector appearing in the representation $u = \operatorname{rot} v$. Denote by $v_\sigma = v_\sigma(x, t)$ the component of $v(x, t)$ perpendicular to the direction of the vector ζ_x taken in the point x if $\zeta_x \neq 0$ there. Making use of the equation $\operatorname{rot}(\zeta v) = \zeta_x \times v + \zeta \operatorname{rot} v$ we may replace v by v_σ in the term $\zeta_x \times v$, hence, due to the supposed estimate $|\zeta_x| \leq C\varepsilon^{-1}$, we have

$$(28) \quad \|\tilde{u}(t) - u(t)\|^2 = \int_{\omega'} (\tilde{u} - u)^2 dx \leq C\varepsilon^{-2} \int_{\omega'} v_\sigma^2 dx + 2 \int_{\omega'} u^2 dx.$$

In view of inequality (11), of the relation $v_\sigma|_S = 0$ and of the supposed regularity of S , we have

$$(29) \quad \int_{\omega'} v_\sigma^2 dx \leq C\varepsilon^2 \int_{\omega'} (v^2 + v_x^2) dx$$

and hence $\|\tilde{u}(t) - u(t)\|^2 \leq C\|u(t)\|^2 \leq C$ due to the inequality $\|v(t)\|_1 \leq \|u(t)\| \leq C$ resulting from (4₁) and from (2). Inequality (20) follows now simply from the obvious inequality $\|\tilde{u}\|^2 \leq \|u\|^2 + \|\tilde{u} - u\|^2$.

In order to get (21), we shall use one of the Sobolev's inequalities, namely the inequality of the form

$$(30) \quad \int_{S'} f^2 ds \leq C\|f\|_1^2,$$

valid for any function $f \in W_2^1(\Omega)$ and any plane section S' of Ω , with C depending only on Ω . It is clear that this inequality will remain true also for S' being any surface S_ϱ parallel to S (with $\varrho \leq 3\varepsilon_0$). Applying (30) to $f = v$ and then to $f = v_x$ we get

$$\int_{S_\varrho} (v^2 + v_x^2) dx \leq C\|v\|_2^2$$

and hence, integrating both sides of the last inequality with respect to ϱ over $(0, 3\varepsilon)$ we are led to

$$\int_{\omega'} (v^2 + v_x^2) dx \leq C\varepsilon\|v\|_2^2.$$

The last member of this inequality is, due to the estimate (4₂) of Lemma 1 applied to $w = u$ and due to (11), less than $C\varepsilon\|u_x(t)\|^2$. Combining this with (28) and (29) we get (21).

(22) follows from the inequality $\|\tilde{u}_{\varepsilon,x}\|^2 \leq \|\tilde{u}_{\varepsilon,x} - u_{\varepsilon,x}\|^2 + \|u_{\varepsilon,x}\|^2$ by applying estimate (6) to its last member and the estimate

$$\|\tilde{u}_{\varepsilon,x}(t) - u_{\varepsilon,x}(t)\|^2 \leq C\varepsilon^{-2}\|\tilde{u}(t) - u(t)\|^2 \leq C\varepsilon^{-1}\|u_x(t)\|^2,$$

resulting from (8) and (21), to the member $\|\tilde{u}_{\varepsilon,x} - u_{\varepsilon,x}\|$.

A little modification of the reasoning leading to (20) shows that (20) is also true for $0 \leq t \leq 3\tau$, $T-3\tau \leq t \leq T$. Hence, due to the triangle inequality, for those t inequality (21) should be replaced by $\|\tilde{u}(t) - u(t)\| \leq C$. Finally, for the same t , $C\varepsilon^{-1}$ should be added to the right-hand member of (22) which is clear from the procedure leading to (22) and from the preceding estimate.

We have now to take into consideration the influence of $\xi_0(t)$, when estimating H^2 . This leads to additional integrals $\int_{2\tau}^{3\tau} dt$ and $\int_{T-3\tau}^{T-2\tau} dt$ which, as it is simply seen, give rise to the additional terms in estimates (23), (24), (27) and (19) and these additional terms are resp. $CH\tau^{1/2}$, $C\varepsilon^{-1}\tau^{3/2}$, $C\varepsilon^{-1}\tau^{5/4}$ and $C\tau$. It follows now from the relation between ε and τ that these new terms leave unchanged the final estimate of H^2 .

It remains only to show that φ may be replaced by U in formula (1). We already know that all the terms of (1) are meaningful for $\varphi = U$. Therefore it suffices to show that U may be approximated in a norm by some elements U' of D_T and that in the limiting process $U' \rightarrow U$ (in this norm) all the terms of (1) behave continuously.

The required approximations U' may be constructed by using u' instead of u in the formulas defining U , u' denoting the result of mollifying the vector u with respect to the single variable t . We have $\|u' - u\|_{1,T} \rightarrow 0$ and hence $\|U' - U\|_C \rightarrow 0$, where the norm $\|\cdot\|_C$ is given by $\|\varphi_2^C\| = \int (\|\varphi\|^2 + \|\varphi_x\|^2 + \|\varphi_t\|^2) dt$. The continuity of all terms of (1) with respect to φ in the norm $\|\cdot\|_C$, except of the term $\int (u, u\varphi_x) dt$, is obvious. To show that also the term $\int (u, uU'_x) dt$ behaves continuously when $\|U' - U\|_C \rightarrow 0$ we have only to repeat the reasonings leading to the estimate (27). In this way we get

$$(31) \quad \left| \int (u, uU'_x - uU_x) dt \right| \leq C\varepsilon^{-3/4} \int_0^\tau d\eta \int \|u_x(t)\|^{3/2} \|u'_x(t+\eta) - u_x(t+\eta)\|^{1/2} \|u'(t+\eta) - u(t+\eta)\|^{1/2} dt$$

and hence, due to the estimate $\|u'(t)\| \leq \|u(t)\|$ and to the relation $\|u' - u\|_{1,T} \rightarrow 0$, the right-hand side of (31) tends to zero. This completes the proof of the theorem.

We shall now briefly discuss the case $n = 2$. The case under consideration is, in comparison to that one treated before, very simple. All the procedure of applying the operation of mollifying and of truncating by using the function $\zeta(x)$ appears now to be completely superfluous, since the term (u, uU_x) may be estimated directly by using the inequality $\|u^2\| \leq C\|u\| \|u_x\|$, which is the 2-dimensional analogue of inequality

(26). Applying the last inequality to (18), where now $\tilde{u}_\varepsilon$ is replaced simply by $\zeta_0 u$, we get instead of (27) an estimate with $C\tau$ on the right-hand side. Also $C\tau$ will replace the right-hand side of (24). The first term of (1) will be simply H^2 plus some term less than $C\tau$, the appearance of the last being caused by the presence of $\zeta_0(t)$. Thus we get $H^2 \leq C\tau$ and hence

THEOREM 2. *Under the same assumptions on u as in Theorem 1, this time Ω being any bounded domain of E_2 , inequality (3) holds true for $\kappa \leq 1/2$.*

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