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SOME GENERAL, ALGEBRAIC REMARKS ON TENSOR CLASSIFICATION, THE GROUP $O(2,2)$ AND SECTIONAL CURVATURE IN 4-DIMENSIONAL MANIFOLDS OF NEUTRAL SIGNATURE

BY

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Dedicated to the memory of Witold Roter

Abstract. This paper presents a general discussion of the geometry of a manifold M of dimension 4 which admits a metric g of neutral signature $(+,+,-,-)$. The tangent space geometry at $m \in M$, the complete pointwise algebraic classification of second order symmetric and skew-symmetric tensors and the algebraic structure of the members of the orthogonal group $O(2,2)$ are given in detail. The sectional curvature function for (M,g) is also discussed and shown to be an essentially equivalent structure on M to the metric g in all but a few very special cases, and these special cases are briefly introduced. Some brief remarks on the Weyl conformal tensor, Weyl's conformal theorem and holonomy for (M,g) are also given.

1. Introduction and notation. There has been some recent interest in 4-dimensional manifolds of neutral signature, and the general idea of this paper is to describe the tangent space geometry of such manifolds, the classification techniques for their associated second order, symmetric and skew-symmetric tensors and orthogonal group, and also the sectional curvature functions on such manifolds. Inevitable comparisons will be made with the similar case of Lorentz signature used in general relativity theory. This introductory section will be used to give a résumé of notation and the tangent space geometry of a 4-dimensional manifold with metric g of neutral signature (this structure usually being denoted by (M,g)), whilst the next two sections will give details of the algebraic classification of symmetric and skew-symmetric second order tensors at $m \in M$. [Some remarks on such classifications have been given by Petrov [16] (and commented on briefly in [12]) but they are somewhat brief and incomplete, and it is believed that

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the work here is more structured, user-friendly and detailed and lists all the possible canonical forms for such tensors.] This will be followed by an algebraic discussion of the group $O(2,2)$. One should also include a study of the classification of the Weyl conformal tensor but this would significantly increase the length of this paper and thus it is given elsewhere [4]. Together with some other remarks it is very briefly commented on in the final section. This is followed by a discussion of the sectional curvature function associated with the Riemann tensor on (M,g), and it will be shown that this function, with a very special exception, uniquely determines the metric g on M from whence it came. Some remarks will be made on the Ricci-flat situation for sectional curvature. Finally, some brief concluding remarks will be given in the last section where the classification and conformal properties of the Weyl conformal tensor will be touched upon together with some comments on holonomy for the type of structures described in this paper.

To establish notation, M denotes a 4-dimensional, smooth manifold with smooth metric g of neutral signature $(+,+,-,-)$. The tangent space at $m \in M$ is denoted by T_mM and the vector space of 2-forms (usually referred to as *bivectors*) at m by $\Lambda_m M$. Due to the existence of the metric (and where no confusion could arise) the distinction between the tangent and cotangent spaces will sometimes be ignored, as will the index placing on tensors. The symbol $u.v$ denotes the inner product at m of $u,v \in T_mM$, $g(m)(u,v)$.

A non-zero member $u \in T_mM$ is called *spacelike* if $u.u > 0$, *timelike* if $u.u < 0$ and *null* if $u.u = 0$. The symbol $*$ denotes the usual Hodge duality (linear) operator on $\Lambda_m M$. Since g has neutral signature, one may choose a pseudo-orthonormal basis x,y,s,t at $m \in M$ with $x.x = y.y = -s.s = -t.t = 1$ and an associated *null* basis of (null) vectors l,n,L,N at m given by $\sqrt{2}\,l = x+t$, $\sqrt{2}\,n = x-t$, $\sqrt{2}\,L = y+s$ and $\sqrt{2}\,N = y-s$ so that $l.n = L.N = 1$ and all other such inner products are zero. The associated completeness relations are

$$g_{ab} = x_a x_b + y_a y_b - s_a s_b - t_a t_b = l_a n_b + n_a l_b + L_a N_b + N_a L_b.$$

A 2-dimensional subspace (2-space) V of T_mM is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two null 1-dimensional subspaces (*directions*) and which are necessarily non-orthogonal, *null* if V contains exactly one null direction, and *totally null* if each non-zero member of V is null. Thus a totally null 2-space consists, apart from the zero vector, of null vectors any two of which are orthogonal, and a null 2-space is such that it contains $u,v \in T_mM$ with $u.v = 0$ if and only if one of u and v spans the unique null direction. This list of types of 2-spaces is mutually exclusive and exhaustive. If V is spacelike (respectively, timelike, null or totally null) then so is its orthogonal complement. If V is spacelike or timelike then V and $V^\perp$

are complementary in T_mM; if V is null then V and $V^\perp$ intersect in their common (unique) null direction; and if V is totally null then $V = V^\perp$.

A bivector E at m with components E^{ab} $(= -E^{ba})$ necessarily has even matrix rank. If this rank is 2, E is called *simple*, and if 4, it is called *non-simple*. If E is simple, it may be written $E^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_mM$, and the 2-space spanned by u and v is uniquely determined by E and called the *blade* of E (and then, unless more precision is required, E or its blade is written $u \wedge v$). A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is spacelike (respectively, timelike, null or totally null).

For each $E \in \Lambda_m M$ its dual is defined by $\overset{*}{E}_{ab} = \frac{1}{2}\epsilon_{abcd}E^{cd}$ where $\epsilon_{abcd} \equiv \sqrt{\det g}\,\delta_{abcd}$ with δ denoting the usual alternating symbol and $\det g$ the determinant of g. Clearly $E \in \Lambda_m M$ is simple if and only if $\overset{*}{E}$ is, and their blades are then orthogonal complements of each other. For neutral signature $\overset{**}{E} = E$ and so the only eigenvalues of the linear map $*$ are ± 1. Now define the subspaces

$$\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\} \quad \text{and} \quad \overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$$

and also the subset $\widetilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$, of $\Lambda_m M$. Then $\overset{+}{S}_m \cap \overset{-}{S}_m = \{0\}$, and $E \in \Lambda_m M \setminus \widetilde{S}_m$ if and only if E and $\overset{*}{E}$ are independent members of $\Lambda_m M$. If E is totally null, it is in $\widetilde{S}_m$. If E and E' are independent and totally null and both are in $\overset{+}{S}_m$ or both are in $\overset{-}{S}_m$ then their blades intersect in only the zero vector, whereas if $E \in \overset{+}{S}_m$ and $E' \in \overset{-}{S}_m$, their blades intersect in a unique null direction. It follows that if $k \in T_mM$ is null, there are exactly two totally null 2-spaces containing k, one in $\overset{+}{S}_m$ and one in $\overset{-}{S}_m$. In fact if $E_1, E_2 \in \overset{+}{S}_m$ and E_3, $E_4 \in \overset{-}{S}_m$ then a unique null basis l, n, L, N arises from the four non-trivial intersections between them (up to appropriate scalings). Any $E \in \Lambda_m M$ may be written in exactly one way as $E = \overset{+}{E} + \overset{-}{E}$ with $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, and so one has $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Also if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, one has $[\overset{+}{E}, \overset{-}{E}] = 0$ where $[\ ]$ denotes matrix commutation. Then each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is a Lie algebra under $[\ ]$ Lie isomorphic to the Lie algebra $o(1,2)$, and so $\Lambda_m M$ is the Lie algebra product $\overset{+}{S}_m \oplus \overset{-}{S}_m$.

Following duality conventions on the bases (x, y, s, t) and (l, n, L, N) for T_mM, one may select as a basis for $\overset{+}{S}_m$ the set $\{x \wedge y + s \wedge t,\ x \wedge t + s \wedge y,\ x \wedge s + y \wedge t\}$ or the set $\{l \wedge n - L \wedge N,\ l \wedge N,\ n \wedge L\}$. (Similar ones are available for $\overset{-}{S}_m$.) More details on such matters may be found in [24, 23, 4].

2. Classification of symmetric tensors. Now let T be a (real) *symmetric* type $(0, 2)$ tensor at $m \in M$ with components T_{ab} and regard it, in component form, as a linear map $f : T_mM \to T_mM$ given for $u \in T_mM$ by $u^a \to T^a{}_b u^b$. One may classify such a tensor by finding all possible Jordan forms/Segre types for f. Some quite general remarks were made on such a classification in [16] and the idea here is to present a simpler and more complete alternative to this including all possible canonical forms.

First note that f always admits an invariant 2-space (and also, for convenience, the word "complex" will always mean "non-real"). So suppose that W is an invariant 2-space of f. Then, as a consequence of the symmetry of T, the orthogonal complement $W^\perp$ of W is also invariant under f.

One can easily show that if an invariant 2-space W is timelike, say $W = l \wedge n$ for *non-orthogonal* null vectors l and n, then either l *and* n are null eigenvectors of f with equal eigenvalues, or l *or* n is the only eigenvector in W, or there are two independent eigenvectors in W which are either both real, orthogonal and non-null and with distinct eigenvalues, or a complex conjugate pair of eigenvectors is admitted. In the last case one may choose l and n so that the complex eigenvectors are $l \pm in$.

Similarly, if W is spacelike, there are two real orthogonal eigenvectors in W which are either both spacelike or both timelike (Sylvester's law). If $W = l \wedge y$ is null, its orthogonal complement $W^\perp = l \wedge s$ is also invariant. Hence l always spans an eigendirection of f. Then either there are two independent real eigenvectors of f, or l is the only independent eigenvector. If $W = l \wedge L$ is totally null with l and L orthogonal null vectors then either there are one or two independent real null eigenvectors in W, or two independent (conjugate) complex null eigenvectors arise, and in this latter case one may choose l and L so that they are $l \pm iL$. If p, q are real or complex eigenvectors of T with eigenvalues α and β, respectively, then either $p.q = 0$ or $\alpha = \beta$.

The following lemma is known and will be needed (in the case $n = 3$) in what follows (and is useful in the $n = 4$ case in general relativity theory [5]).

Lemma 2.1. *Let M be an n-dimensional manifold* ($n \geq 3$) *with metric g of Lorentz signature* $(+, +, \ldots, +, -)$ *and let T be a* (*real*) *second order, symmetric tensor on M with associated linear map f at $m \in M$. The Jordan form for f may take one of the following Segre types:* $\{11\ldots1\}$, $\{z\bar{z}1\ldots1\}$, $\{21\ldots1\}$, *or* $\{31\ldots1\}$, *or a possible degeneracy of one of these types.* (*The notation* $\{z\bar{z}1\ldots1\}$ *means that one has diagonalisability over* $\mathbb{C}$ *with* $n-2$ *real eigenvalues and a conjugate pair of complex eigenvalues, whilst round brackets denote an eigenvalue degeneracy.*)

The next lemma can be stated in the general case but is again only needed in the $n = 4$ case. It collects together some simple results which are useful later.

LEMMA 2.2. *Let M be an n-dimensional manifold $(n \geq 4)$ with metric g of arbitrary signature and let T be a (real) symmetric tensor at $m \in M$ with associated linear map f, as above.*

(i) *Suppose f admits a conjugate pair of complex eigenvectors $x \pm iy$ with respective eigenvalues $a \pm ib$ with $x, y \in T_mM$ spanning an invariant 2-space W of f and $a, b \in \mathbb{R}$ with $b \neq 0$. Then W is totally null if and only if one, and hence both, of $x \pm iy$ are null, and W is timelike if and only if one, and hence both, of $x \pm iy$ are not null.*

(ii) *Any (real or complex) eigenvector of f corresponding to a non-simple elementary divisor is null (and so any (real or complex) non-null eigenvector is associated with a simple elementary divisor). Conversely, if f admits a real or complex, null eigenvector then either the corresponding eigenvalue is degenerate (that is, its eigenspace has dimension ≥ 2), or the associated elementary divisor is non-simple (cf. [16]).*

(iii) *If $x \pm iy$ and $p \pm iq$ are two conjugate pairs of complex eigenvectors of f with distinct complex eigenvalues $a \pm ib$ and $c \pm id$ (that is, $a+ib \neq c \pm id$ and $b \neq 0 \neq d$) then their associated invariant 2-spaces $x \wedge y$ and $p \wedge q$ are orthogonal and intersect only in $\{0\}$.*

(iv) *If $n/2 < m \leq n$ then there cannot exist m null, mutually orthogonal, real independent vectors in T_mM.*

Proof. (i) Since T is symmetric and $a + ib \neq a - ib$, it follows that $(x+iy).(x-iy) = 0$ ($\Leftrightarrow x.x+y.y = 0$). Then $x \pm iy$ null $\Rightarrow x.x-y.y = x.y = 0$, and so $x.x = y.y = x.y = 0$, and this implies that $x \wedge y$ is totally null. The converse is clear. If $x \pm iy$ is not null ($x.x + y.y = 0$ still holds) then either $x.x - y.y \neq 0$ or $x.y \neq 0$ (or both). Whatever the choices, $x \wedge y$ is timelike. The converse is clear.

(ii) For the first part, select a standard Jordan basis which includes the real or complex eigenvector k and another vector k' satisfying $f(k) = \alpha k$, $f(k') = \alpha k' + k$ ($\alpha \in \mathbb{C}$). Then since T is symmetric, one has $k.f(k') = k'.f(k)$ and so k is null. Now suppose a real or complex *null* eigenvector p is admitted with $f(p) = \alpha p$ for $\alpha \in \mathbb{C}$ such that α is not degenerate and is associated with a simple elementary divisor. Then $p.\bar{p} = 0$ (with a bar denoting conjugation) whether p is real or non-real, and if in addition f admits a complex eigenvector q independent of p and $\bar{p}$ then since α is not degenerate, one has $p.q = p.\bar{q} = 0$. If f admits another *real* eigenvector k with $f(k) = \beta k$ then $\alpha \neq \beta$ because either α is not real or it is not degenerate. Thus $p.k = 0$. It follows that p is orthogonal to each eigenvector of f and so the eigenvectors of f cannot form a basis, and hence f admits some non-simple elementary divisor whose eigenvalue is distinct from α. Then if u is any (real or complex) eigenvector of f with eigenvalue $\gamma \neq \alpha$ corresponding to any non-simple elementary divisor then $p.u = 0$, and for the associated members of a Jordan basis $u, r, s, \ldots$ one has $f(u) = \gamma u$, $f(r) = \gamma r + u$, $f(s) = \gamma s + r, \ldots$ On contracting these equations with p and using $p.f(r) = r.f(p)$, etc. one finds $p.r = p.s = \cdots = 0$. It follows that p is orthogonal to each member of a (full) Jordan basis, which is a contradiction.

(iii) If $x \pm iy$ and $p \pm iq$ are two complex conjugate pairs of eigenvectors with distinct eigenvalues $a \pm ib$ and $c \pm id$ ($a + ib \neq c \pm id$) then if $(x \wedge y) = (p \wedge q)$ an easy argument contradicts the fact that $a \pm ib$ and $c \pm id$ are distinct eigenvalues, whereas if $(x \wedge y) \cap (p \wedge q)$ is 1-dimensional it gives rise to a real eigenvector in $x \wedge y$ and $p \wedge q$ and hence a contradiction. It follows that $x \pm iy$ and $p \pm iq$ are independent over $\mathbb{C}$ and that $(x \wedge y) \cap (p \wedge q) = \{0\}$ so that the span of x, y, p, q is 4-dimensional.

(iv) Suppose $k_1, \ldots, k_m$ are m such null vectors and let $U = \text{span}\{k_1, \ldots, k_m\}$. Then $\dim U = m$ (so $\dim U^\perp = n - m$). But $k_1, \ldots, k_m$ are independent members of $U^\perp$ and one obtains the contradiction that $m \leq n - m$. ■

Lemma 2.3. *Let M be a 4-dimensional manifold with metric g of arbitrary signature and let T be a* (*real*) *symmetric tensor at $m \in M$ with associated linear map f as above. If f admits a pair of complex,* null *eigenvectors $x \pm iy$, with eigenvalues $a \pm ib$ ($b \neq 0$), then no other independent* real *eigenvector is admitted and no other complex eigenvector with a different eigenvalue is admitted* (*that is, $a \pm ib$ are the only eigenvalues of f*). *Thus either the eigenvalues $a \pm ib$ are degenerate, or the elementary divisors associated with $x \pm iy$ are non-simple of order* 2. *In the degenerate case when other eigenvectors are admitted, a basis of complex, non-null eigenvectors may be chosen.* [*It follows from this lemma and Lemma* 2.2(i) *that the* 2*-spaces $x \wedge y$ and $p \wedge q$ in Lemma* 2.2(iii) *are timelike.*]

Proof. If $x \pm iy$ is a complex conjugate pair of *null* eigenvectors, and k is a real eigenvector independent of $x \pm iy$, then they have different eigenvalues and so $k.x = k.y = 0$; hence $k \in (x \wedge y)^\perp = x \wedge y$ since $x \wedge y$ is totally null, and a contradiction follows. If $p \pm iq$ is any complex conjugate pair of eigenvectors with eigenvalues $c \pm id$ and $a + ib \neq c \pm id$ then $(x + iy).(p \pm iq) = 0$ and so $x.p = x.q = y.p = y.q = 0$. Hence $p, q \in x \wedge y$ and so $p \pm iq$ are complex linear combinations of $x \pm iy$, contradicting the independence of $x \pm iy$ and $p \pm iq$. The final part is clear. ■

Theorem 2.4. *Let M be a 4-dimensional manifold with metric g of signature $(+,+,-,-)$ and let T be a* (*real*) *second order symmetric tensor on M with associated linear map f at $m \in M$. The Jordan form for f takes one of the following Segre types:* $\{1111\}$, $\{z\bar{z}11\}$, $\{z\bar{z}w\bar{w}\}$, $\{211\}$, $\{2z\bar{z}\}$, $\{22\}$ *with eigenvalues complex,* $\{22\}$ *with eigenvalues real,* $\{31\}$, *or* $\{4\}$, *each with real eigenvalue*(*s*), *or a possible degeneracy of one of these types. Here,* $\{2z\bar{z}\}$ *means two complex* (*conjugate*) *eigenvalues each with simple elementary divisor, and a real eigenvalue with elementary divisor of order two.*

Proof. First suppose that f admits a real, non-null eigenvector k (hence with a simple elementary divisor from Lemma 2.2(ii)). Then the 3-dimensional subspace U of T_mM which is the orthogonal complement of k inherits a Lorentz metric, and the restriction of f to U satisfies the conditions of Lemma 2.1. Thus, incorporating k into the Jordan form for f, the Segre type of f is $\{1111\}$, $\{z\bar{z}11\}$, $\{211\}$ or $\{31\}$ or a possible degeneracy of one of these.

So suppose that the only eigenvectors of f are either complex, or real and null. If a complex eigenvector for f exists then, if it is null, Lemma 2.3 shows the Segre type is $\{22\}$ or $\{(zz)(\bar{z}\bar{z})\}$, each with complex conjugate eigenvalues. If a non-null, complex eigenvector exists, it gives rise to a timelike, invariant 2-space W of f (Lemma 2.2(i)), and $W^{\perp}$ is also invariant and timelike and $W \cap W^{\perp} = \{0\}$. The 2-space $W^{\perp}$ then admits either a single, independent, real, null eigenvector, or two independent real eigenvectors or a conjugate pair of complex eigenvectors (and the second of these then leads to the contradiction that a non-null real eigenvector exists). Thus, in this case, the Segre type is $\{2z\bar{z}\}$ or $\{z\bar{z}w\bar{w}\}$.

So suppose that all eigenvectors are real and null. If there is only one independent such eigenvector, the Segre type is $\{4\}$, whilst if there are exactly two such independent eigenvectors, they must be orthogonal (otherwise their eigenvalues would be equal and a non-null eigenvector would arise in their span). Thus the Segre type is either $\{(31)\}$ or $\{22\}$ ($\{31\}$ is forbidden by Lemma 2.2(ii)).

If the former holds, one has a Jordan basis l, k, r, s with $f(l) = \alpha l$, $f(r) = \alpha r + l$, $f(s) = \alpha s + r$ and $f(k) = \alpha k$ and with l and k null eigenvectors with $l.k = 0$ (so that $l \wedge k$ is totally null). The symmetry of T then gives $k.f(s) = s.f(k)$ and $l.f(s) = s.f(l)$. These give, respectively, $r.k = 0$ and $r.l = 0$, and so $r \in l \wedge k$, which contradicts the fact that l, k, r, s is a basis for T_mM.

Thus the Segre type is $\{22\}$ (with real eigenvalues) or its degeneracy. If three or more such eigenvectors exist, they are mutually orthogonal (to avoid non-null eigenvectors), and this contradicts Lemma 2.2(iv). This completes the proof. ∎

Possible canonical forms for each of these types may be found, and are given below in (2.2)–(2.11) in either a pseudo-orthonormal or null basis, by the following argument in which all Greek letters refer to real numbers and each Segre symbol is now understood to *include* all its intended degeneracies.

In the type $\{1111\}$ case one may always choose four independent, orthogonal, non-null, real eigenvectors as x, y, s, t with corresponding eigenvalues $\alpha, \beta, \gamma, \delta$, and the canonical form (2.2) is easily achieved.

In case $\{z\bar{z}11\}$ the complex eigenvectors must be associated with a timelike invariant 2-space (Lemmas 2.3 and 2.2(i)) and may be chosen as $l \pm in$ with respective eigenvalues $\gamma \pm i\delta$ ($\delta \neq 0$), and the real ones are in $(l \wedge n)^{\perp}$ and may be chosen as y and s with corresponding eigenvalues α and β. Then the canonical form (2.3) follows.

If the type is $\{z\bar{z}w\bar{w}\}$ with two distinct conjugate pairs of complex eigenvectors and associated, orthogonal, timelike invariant 2-spaces (which may be chosen as $l \wedge n$ and $L \wedge N$), then these eigenvectors may be taken as $l \pm in$ and $L \pm iN$ with respective eigenvalues $\alpha \pm i\beta$ and $\gamma \pm i\delta$ ($\beta \neq 0 \neq \delta$), and (2.4) follows. [In the event that the eigenvalues are equal, say $\alpha = \gamma$ and $\beta = \delta$, the $\alpha + i\beta$ eigenspace, spanned by $l + in$ and $L + iN$, contains exactly two complex *null* directions spanned by $(l \mp N) + i(n \pm L)$. A change of null basis then gives rise to a canonical form (2.5) for this Segre type, that is, $\{(zz)(\bar{z}\bar{z})\}$, with complex eigenvalues $\alpha + i\beta$ and $\alpha - i\beta$ and with corresponding eigenspaces spanned in this new basis by $l + iL$ and $n - iN$ (for $\alpha + i\beta$) and $l - iL$ and $n + iN$ (for $\alpha - i\beta$).]

For Segre type $\{211\}$ with real, null eigenvector l corresponding to the non-simple elementary divisor of order 2 with eigenvalue α, the other eigenvectors are orthogonal to l (see Lemma 2.1 and the proof of Theorem 2.4) and may be chosen as y and s with respective eigenvalues β and γ. Equation (2.6) then follows from this with $\lambda \neq 0$, and in fact a scaling of l may be used to set $\lambda = \pm 1$. [That there are no other independent eigenvectors resulting from (2.6) is easily checked.] Note that in this case, unlike the Lorentz case, if $\beta = \gamma$ then two other *null* eigenvectors L and N are admitted).

For type $\{2z\bar{z}\}$, one has a real, null eigenvector l corresponding to the non-simple elementary divisor of order 2 with (real) eigenvalue α and a conjugate pair of complex eigenvectors associated with a timelike 2-space orthogonal to l and which may be chosen as $L \wedge N$. Thus the complex eigenvectors are $L \pm iN$ with eigenvalues $\gamma \pm i\delta$ ($\delta \neq 0$). Then (2.7) follows with $\lambda \neq 0$, and in fact, a scaling of l may be used to set $\lambda = \pm 1$. [That no other independent (necessarily real) eigenvectors arise in (2.7) may again be easily checked.]

For type $\{22\}$ with complex eigenvalues, one has two independent, complex conjugate, *null* eigenvectors (associated with a totally null 2-space which may be chosen as $l \wedge L$) with elementary divisors of order 2 (see Lemma 2.2(i) and (ii)). Thus, after possibly adjusting the choice of l and L, the eigenvectors may be chosen as $l \pm iL$ with respective eigenvalues $\alpha \pm i\beta$ ($\beta \neq 0$), and the canonical form (2.8) follows (in which it is insisted that $\omega \neq 0$) but with an extra term of the form $\nu L_a L_b$, which may be removed by a change of null basis of the form $l \to l$, $L \to L$, $n \to n + bL$ and $N \to N - bl$ for some $b \in \mathbb{R}$. [In fact, the terms $\mu l_a l_b$ and $\omega(l_a L_b + L_a l_b)$ in (2.8) may be combined into a single term of the form $l_a L'_b + L'_a l_b$ with

$L' = (\mu/2)l + \omega L$.] It may be checked that no further independent eigenvectors arise in (2.8).

Now suppose that the Segre type is $\{22\}$ with real eigenvalues and hence (Lemma 2.2(ii)) the eigenvectors may be chosen as orthogonal real null vectors l and L, say. The orthogonality of l and L follows either directly from the proof of Theorem 2.4, or by noticing that otherwise the (necessarily timelike) orthogonal complement of their span would produce more eigenvectors and a contradiction. On extending l and L to a null basis l, n, L, N, one achieves the form (2.9).

Now if $\alpha \neq \beta$, a change of basis $l \to l$, $L \to L$, $n \to n + bL$ and $N \to N - bl$ ($b \in \mathbb{R}$) shows that ω may be set to zero in (2.9), and then one must take $\mu \neq 0 \neq \nu$ to avoid n and/or N becoming eigenvectors. Thus in this case then if any further eigenvectors are admitted then they cannot be complex and null (Lemma 2.3) or complex and non-null (since they would have to be orthogonal to l and L, and a contradiction follows). Thus any further independent eigenvector k is real and its eigenvalue must be either α or β (to avoid it being orthogonal to l and L, and hence being in the $l \wedge L$ 2-space). So, say, k has eigenvalue α and $k = al + bn + cL + dN$ with $a, b, c, d \in \mathbb{R}$ and $b^2 + d^2 \neq 0$. A substitution into (2.9) immediately gives the contradiction that $b = d = 0$.

If $\alpha = \beta$, the assumed Segre type forces $T - \alpha g$ to have rank two and hence $\mu\nu \neq \omega^2$. Then one can show that if, since $\alpha = \beta$, one uses the freedom to choose l and L within the 2-space $l \wedge L$ by the changes $l \to el + fL$ and $L \to e'L + f'l$ with $ee' - ff' \neq 0$, one may, if $\omega^2 > \mu\nu$, choose $e, f, e'f'$ so that $\mu = \nu = 0$ (and $\omega \neq 0$), whilst if $\omega^2 < \mu\nu$ one may similarly choose $\mu\nu > 0$ and $\omega = 0$. [The sign of the corresponding quantity $\omega^2 - \mu\nu$ is unchanged by such basis changes.] In either case an argument similar to the last one reveals that there are no further independent eigenvectors in (2.9).

If the Segre type is $\{31\}$ with real null eigenvector l corresponding to the elementary divisor of order 3 with eigenvalue α, suppose that the other eigenvector is spanned by $k \in T_m M$. Then (Lemma 2.2(ii)) if k is null then, to avoid extra independent eigenvectors, the invariant 2-space $l \wedge k$ must be either null or totally null. If it is null then k is non-null and a contradiction follows. If it is totally null then $l.k = 0$ and the previous argument shows that the Segre type is $\{22\}$ and gives another contradiction.

It follows that k is non-null and $l \wedge k$ is null. So in the usual tetrad choose k equal to say s with eigenvalue β to achieve the form (2.10) with the coefficients of $y_a y_b$ and $l_a y_b + y_a l_b$ equal to, say, A and B, respectively, and with an extra term of the form $\mu l_a l_b$ ($A, B, \mu \in \mathbb{R}$). Then if $B = 0$, y becomes a third eigenvector (so take $B \neq 0$), and if $A \neq \alpha$, $l + \frac{A-\alpha}{B} y$ is also an eigenvector (so take $A = \alpha$). Then, by a scaling of l (and also n), one may take $B = 1$. With this information, one can easily show (after a simplifying use of a completeness relation) that no other independent eigenvectors exist in (2.10) (and that a basis change which includes $y \to y + ml$ ($m \in \mathbb{R}$) can be used to set $\mu = 0$). Note here that the non-null eigenvector could have been chosen to be y; one simply switches y and s in (2.10) and changes the signs in the last two terms.

Finally, (2.11) is type $\{4\}$ with the single, independent, real eigenvector l. To see this, note that this Segre type has a single (necessarily real and null) eigendirection spanned by l (Lemma 2.2(ii)) and with real eigenvalue α. Then note that, temporarily, a Jordan basis containing l and $P \in T_mM$ may be chosen so that $f(l) = \alpha l$ and $f(P) = \alpha P + l$. Thus $l \wedge P$ is an invariant 2-space of f containing only one independent eigenvector l and is thus either null, timelike or totally null.

If $l \wedge P$ is null then $(l \wedge P)^{\perp} = l \wedge P'$ ($P' \in T_mM$, l, P, P' independent) is also null and must contain only the eigenvector l. Thus (Section 1) $f(P') = \alpha P' + dl$ ($0 \neq d \in \mathbb{R}$). After a scaling on P' one has $f(P') = \alpha P' + l$, and the fact that $P - P'$ is then an eigenvector of f independent of l gives a contradiction.

If $l \wedge P$ is timelike, $(l \wedge P)^{\perp}$ would give another independent eigenvector of f and another contradiction.

Thus $l \wedge P$ is totally null and one can replace P by N and, abandoning the Jordan basis, extend l, N to a null basis l, n, L, N with $f(l) = \alpha l$, $f(N) = \alpha N + l$ and $f(L) = aL + bN + cl$, with $a, b, c \in \mathbb{R}$, the latter after using the equality $L.f(l) = l.f(L)$ to remove a possible term in n. The equality $L.f(N) = N.f(L)$ then shows that $a = \alpha$. (Here one needs $b \neq 0$, otherwise either $c = 0$ and L becomes an independent eigenvector, or $c \neq 0$ in which case $N - c^{-1}L$ becomes an independent eigenvector; each giving a contradiction.) So in this basis, one finds

$$T_{ab} = \alpha g_{ab} + \mu l_a l_b + b N_a N_b + c(l_a N_b + N_a l_b) + (l_a L_b + L_a l_b). \tag{2.1}$$

Now perform a basis change $l \to l$, $N \to N + \lambda l$ and $L \to L$ ($\lambda \in \mathbb{R}$) to set $c = 0$, then change $l \to l$, $N \to N$ and $L \to L + \lambda' l$ ($\lambda' \in \mathbb{R}$) to set $\mu = 0$, and finally a scaling of N to set b equal to 1 or -1. One thus obtains (2.11) with $\nu = \pm 1$. The complete list is:

$$\alpha x_a x_b + \beta y_a y_b - \gamma s_a s_b - \delta t_a t_b, \tag{2.2}$$

$$\alpha y_a y_b - \beta s_a s_b + \gamma(l_a n_b + n_a l_b) + \delta(l_a l_b - n_a n_b), \tag{2.3}$$

$$\alpha(l_a n_b + n_a l_b) + \beta(l_a l_b - n_a n_b) + \gamma(L_a N_b + N_a L_b) + \delta(L_a L_b - N_a N_b), \tag{2.4}$$

$$\alpha g_{ab} + \beta(l_a N_b + N_a l_b - n_a L_b - L_a n_b), \tag{2.5}$$

$$\alpha(l_a n_b + n_a l_b) + \lambda l_a l_b + \beta y_a y_b - \gamma s_a s_b, \tag{2.6}$$

$$\alpha(l_a n_b + n_a l_b) + \lambda l_a l_b + \gamma(L_a N_b + N_a L_b) + \delta(L_a L_b - N_a N_b), \tag{2.7}$$

$$\alpha g_{ab} + \omega(l_a L_b + L_a l_b) + \mu l_a l_b + \beta(l_a N_b + N_a l_b - n_a L_b - L_a n_b), \tag{2.8}$$

$$\alpha(l_a n_b + n_a l_b) + \beta(L_a N_b + N_a L_b) + \mu l_a l_b + \nu L_a L_b + \omega(l_a L_b + L_a l_b), \tag{2.9}$$

$$\alpha g_{ab} + (l_a y_b + y_a l_b) + (\alpha - \beta) s_a s_b, \tag{2.10}$$

$$\alpha g_{ab} + (l_a L_b + L_a l_b + \nu N_a N_b). \tag{2.11}$$

Note that each permissable degeneracy of the Segre types listed in Theorem 1 is possible and implicitly contained in (2.2)–(2.11) above.

3. Classification of skew-symmetric tensors. Let F be a type $(0, 2)$ skew-symmetric tensor (bivector) at $m \in M$. Again F may be regarded as a linear map f on $T_m M$ given in components by $f : k^a \to F^a{}_b k^b$ for $k \in T_m M$ and $F^a{}_b = F_{cb} g^{ac}$. The tensor F may also be classified according to its Jordan–Segre type and this will be done in this section. Since F is skew-symmetric, any non-null eigenvector (real or complex) has a zero eigenvalue (and so any eigenvector with non-zero eigenvalue is null). It is also true, from the skew-symmetry of F, that if p and q are (real or non-real) eigenvectors of f with eigenvalues α and β, respectively, then either $\alpha = -\beta$ or $p.q = 0$. Simple bivectors (up to a scaling) are in a one-to-one correspondence with 2-spaces of $T_m M$, and their algebraic properties depend on the nature (timelike, etc.) of the 2-space $u \wedge v$ (see Section 1). Non-simple bivectors are a little more complicated but all their eigenvalues are non-zero and hence all their eigenvectors are null. If W is an invariant 2-space of f then so is $W^\perp$. If F is simple, its blade may be timelike, spacelike, null or totally null. For each of these cases, respectively, a pseudo-orthogonal basis x, y, s, t or a null basis l, n, L, N may be chosen so that, respectively, $F = l \wedge n$, $F = x \wedge y$ (or $s \wedge t$), $l \wedge y$ (or $l \wedge s$) and $l \wedge L$. The first three yield respective Segre types $\{11(11)\}$, $\{z\bar{z}(11)\}$, and $\{(31)\}$. In the totally null case the only eigenvectors are l and L, and $f(l) = 0$, $f(L) = 0$, $f(N) = l$ and $f(n) = -L$. It follows that each eigenvalue is zero and corresponds to a non-simple elementary divisor of order 2, and so the Segre type is $\{(22)\}$.

Now suppose that F is non-simple and admits a real eigenvector k and a complex conjugate pair of eigenvectors $p \pm iq$ with respective eigenvalues c and $a \pm ib$ ($a, b, c \in \mathbb{R}$ and $b \neq 0$), which are necessarily non-zero since F is non-simple. Thus $p \pm iq$ and k are null and so $p.p = q.q$, $p.q = 0 = k.k$. It follows that $p \wedge q$ is either spacelike or totally null, and since $a \pm ib \neq -c$, one has $k.(p \pm iq) = 0$. If $p \wedge q$ is spacelike, so is $(p \wedge q)^\perp$, and $k \in (p \wedge q)^\perp$ is a contradiction since k is null. If $p \wedge q$ is totally null then $k \in p \wedge q$, which is also a contradiction. Thus *the eigenvalues of* (a *non-simple*) F *are either all real or all complex* (and from the above, this fails if F is simple).

So suppose all eigenvalues of F are complex with $p \pm iq$ being a pair of complex eigenvectors with non-zero eigenvalues, $a \pm ib$ ($a, b \in \mathbb{R}$, $b \neq 0$). Again, $p \wedge q$ is spacelike or totally null.

Supposing it is spacelike, choose an orthonormal basis x, y, s, t such that $p = x$ and $q = y$. Then it easily follows that $a = 0$ and so $f(x) = -by$ and $f(y) = bx$. From this, using $-t.f(x) = x.f(t)$, etc., one easily finds that $f(t) = \gamma s$ and $f(s) = -\gamma t$ ($0 \neq \gamma \in \mathbb{R}$), and so $t \pm is$ are complex eigenvectors with corresponding eigenvalues $\mp i\gamma$ (and $s \wedge t = (p \wedge q)^{\perp}$). Thus the Segre type is $\{z\bar{z}w\bar{w}\}$ and

$$F = b(x \wedge y) - \gamma(s \wedge t) \tag{3.1}$$

(and if $b = \pm\gamma$, the degeneracy $\{(zz)(\bar{z}\bar{z})\}$ is achieved). [If $b \neq \pm\gamma$ then all eigenvectors are such that their real and imaginary parts span spacelike 2-spaces. In the event that $b = \pm\gamma$, however, complex eigenvectors are easily shown to arise whose real and imaginary parts span totally null 2-spaces. *This degenerate situation is the only way that complex eigenvectors with spacelike and totally null spans can arise at the same time.* To see this suppose that eigenvectors $p \pm iq$ with spacelike span and eigenvalues $\pm ib$ (as above) and eigenvectors $l \pm iL$ with totally null span and eigenvalues $c \pm id$ ($c, d \in \mathbb{R}$, $d \neq 0 \neq b$) arise. Then the equation ${F^a}_a = 0$ shows that $c = 0$, and if $b \neq \pm d$ then $p + iq$ is orthogonal to $l \pm iL$ and gives the contradiction $p \wedge q = l \wedge L$.]

Now suppose $p \wedge q$ is totally null and, say, $l \pm iL$ is an eigenvector of F with eigenvalue $a \pm ib$. Then $f(l) = al - bL$, $f(L) = bl + aL$, and in the usual basis l, n, L, N one has $f(N) = \rho l + bn - aN$ and $f(n) = -\rho L - an - bN$. Now F may be written in terms of an obvious bivector basis as

$$F = \alpha(l \wedge n) + \beta(l \wedge L) + \gamma(l \wedge N) + \delta(n \wedge L) + \mu(n \wedge N) + \nu(L \wedge N), \tag{3.2}$$

and so, in this case, $\alpha = \nu = a$, $\beta = \rho$, $\gamma = \delta = b$ and $\mu = 0$. Thus

$$F = a(l \wedge n + L \wedge N) + b(l \wedge N + n \wedge L) + \rho(l \wedge L). \tag{3.3}$$

If $a \neq 0$ then a change of null basis given by $l' = l$, $L' = L$, $n' = n + \kappa L$, $N' = N - \kappa l$ with $\kappa = \rho/(2a)$ will (after dropping primes for convenience) set $\rho = 0$ and thus remove the term in $l \wedge L$ in (3.3). If this is the case, it easily follows that $n \pm iN$ are also eigenvectors of f with corresponding eigenvalues $-a \pm ib$, and F is diagonalisable over $\mathbb{C}$ and of Segre type $\{z\bar{z}w\bar{w}\}$ with no degeneracies possible. If $a = \rho = 0$ in (3.3), a degeneracy of this type arises since $l \wedge N + n \wedge L = x \wedge y + s \wedge t$ (see the previous case).

Continuing, if $a = 0 \neq \rho$ suppose $k = p + iq$ is a complex eigenvector independent of $l + iL$ and with a totally null span $p \wedge q$ (see the above remarks on the degenerate case) and, since ${F^a}_a = 0$, with eigenvalue id ($0 \neq d \in \mathbb{R}$). Then $p.p = q.q = 0$ and $p.q = 0$. If the spaces $l \wedge L$ and $p \wedge q$ intersect in a 1-dimensional space, this would give rise to a real eigenvector and a contradiction, and if $l \wedge L = p \wedge q$ then $p + iq$ is a linear combination of the known eigenvectors $l \pm iL$ and again a contradiction occurs. Thus $l \wedge L$ and $p \wedge q$ intersect only in the trivial vector and so l, L, p, q are independent. If $d \neq \pm b$ then $(p + iq).(l \pm iL) = 0$, which implies that $p, q \in l \wedge L$ and a contradiction follows. If, say, $b = d$ (the case $b = -d$ is similar) then $(p + iq).(l + iL) = 0$, which is $p.l - q.L = 0 = q.l + p.L$. Now p cannot be orthogonal to both l and L (since it would then lie in $l \wedge L$), and so one may use the freedom to scale $l + iL$ by an arbitrary non-zero complex number to arrange $p.l = 0 \neq p.L$ and then $q.L = 0 \neq q.l$. (This will change the null basis but the form (3.3) still holds since the new $l \pm iL$ are again eigenvectors with eigenvalues $\pm ib$.) Then since $p.p = q.q = p.q = 0$, one finds, using the independence of l, L, p and q and after scalings of p and q, that $p = N + cl$ and $q = n - cL$. Thus $p + iq = (N + in) + c(l - iL)$.

On collecting these results together one has $f(l) = -bL$, $f(L) = bl$, $f(p) = -bq$, $f(q) = bp$, $p = N + cl$, $q = n - cL$, and the earlier results give $f(N) = \rho l + bn$ and $f(n) = -\rho L - bN$. Then $f(p) = -bq$ gives $\rho l + bn - cbL = -bn + bcL$ and the contradiction that $\rho = b = 0$. Thus no further independent eigenvectors are admitted and the Segre type can be checked to be $\{22\}$ with complex eigenvalues.

So suppose that all eigenvectors of f are real (and necessarily null). Then there exists a null $l \in T_mM$ with $f(l) = \lambda l$ and $0 \neq \lambda \in \mathbb{R}$ (since F is non-simple). Thus, basing a null tetrad on l, it follows that, in (3.2), $\alpha = \lambda$ and $\mu = \delta = 0$ and so, again since F is non-simple, $\nu \neq 0$. Suppose $\beta \neq 0 \neq \gamma$ and define

$$\kappa = \frac{\nu - \lambda}{\gamma} \quad \text{and} \quad \kappa' = \frac{-\nu - \lambda}{\beta}.$$

Then $\kappa = \kappa' = 0$, being equivalent to $\nu = \lambda = 0$, is impossible. If $\kappa \neq 0 \neq \kappa'$ (so that $\lambda \neq \pm\nu$), it is easily checked that $L' \equiv l + \kappa L$ and $N' \equiv l + \kappa' N$ are also real, null eigenvectors for f with eigenvalues ν and $-\nu$ respectively and $L'.N' = \kappa\kappa'$. Since $F^a{}_a = 0$, it follows that another independent (real, null) eigenvector exists, say n', with eigenvalue $-\lambda$ and that f is diagonalisable over $\mathbb{R}$. Thus, with the new basis $l' = l, L', N', n'$, one has $f(l') = \lambda l'$, $f(L') = \nu L'$, $f(N') = -\nu N'$ and $f(n') = -\lambda n'$ (and since $\lambda \neq \pm\nu$, $n'.L' = n'.N' = 0$, and $l'.n' \neq 0$ as otherwise n' would be orthogonal to each member of the basis l', n', L', N'). So f is diagonalisable over $\mathbb{R}$ with Segre type $\{1111\}$ and no degeneracies and, after any necessary scaling of n' and N' and dropping primes, is of the form

$$F = \lambda(l \wedge n) + \nu(L \wedge N), \tag{3.4}$$

which is (3.2) with the β and γ terms removed (and $\mu = \delta = 0$). Still with the case $\beta \neq 0 \neq \gamma$ in (3.2), suppose $\kappa = 0 \neq \kappa'$ (equivalently, $\lambda = \nu$).

Then in (3.2), where already one has $\alpha = \lambda$, $\mu = \delta = 0$, a change of null basis $l'' \equiv l$, $L'' \equiv L$, $n'' \equiv n - \frac{\beta}{2\lambda}L$, $N'' \equiv N + \frac{\beta}{2\lambda}l$ will remove the β term from (3.2). (Similar comments apply if $\kappa' = 0 \neq \kappa$ to the γ term.) So in this new basis (dropping primes) and with $\gamma \neq 0 \neq \lambda$ one has

$$F = \lambda(l \wedge n + L \wedge N) + \gamma l \wedge N, \tag{3.5}$$

and then $f(l) = \lambda l$ and $f(N) = -\lambda N$. It is then easily seen that $f(L) = \lambda L + \gamma l$ and $f(n) = -\lambda n - \gamma N$, and it can be checked that l and N are the only eigenvectors. Then the Segre type is $\{22\}$ with real eigenvalues. Similar comments hold in the procedure from $\kappa' = 0 \neq \kappa$ and treatment of the β term. Thus one may always remove at least one of the terms β and γ, whilst if both are zero the situation is as in (3.4), and if in addition $\lambda = \pm\nu$ then the type is $\{(11)(11)\}$. Thus one has the following result (and it is trivially remarked that different choices of basis members will restore the symmetry between bivectors in $\overset{+}{S}_m$ and $\overset{-}{S}_m$).

THEOREM 3.1. *Let F be a bivector on a 4-dimensional manifold with metric of neutral signature. If F is* simple, *then the Segre types for F are $\{11(11)\}$ (blade of F timelike), $\{z\bar{z}(11)\}$ (spacelike), $\{(31)\}$ (null) and $\{(22)\}$ (totally null). If F is* non-simple, *its eigenvalues are either all real or all complex and (adopting the convention that any coefficient not explicitly equated to zero is non-zero) the possible Segre types are $\{z\bar{z}w\bar{w}\}$ [(3.1) or (3.3) with $\rho = 0$], $\{(zz)(\bar{z}\bar{z})\}$ [(3.3) with $a = \rho = 0$ or (3.1) with $b = \pm\gamma$], $\{22\}$ (over $\mathbb{C}$) [(3.3) with $a = 0$], $\{1111\}$ [(3.4)], $\{(11)(11)\}$ [(3.4) with $\lambda = \pm\nu$] and $\{22\}$ (over $\mathbb{R}$) [(3.5)]. A non-simple bivector is in $\widetilde{S}_m$ if and only if it has an eigenvalue degeneracy. (Note that by taking appropriate combinations of bivectors, any bivector may be written as a sum of two simple ones.)*

4. The algebraic structure of the orthogonal group for neutral signature. Choose an orthonormal basis e_a for T_mM so that, with $\eta \equiv g(m)$, one has $\eta(e_a, e_b) = \eta_{ab}$ where η_{ab} denotes the matrix $\mathrm{diag}(1, 1, -1, -1)$, and let $\mathcal{L}$ denote the group of all linear maps $f : T_mM \to T_mM$ satisfying $\eta(f(x), f(y)) = \eta(x, y)$ for all $x, y \in T_mM$. Then $\mathcal{L}$ is a 6-dimensional Lie group under the usual rules of composition, and each such f is a bijective map. For any given $f \in \mathcal{L}$ let A be its representative matrix in this basis. Then, using a superscript T to denote a matrix transpose, one finds the condition $A\eta A^T = \eta$. With the above basis fixed one may identify members of $\mathcal{L}$ with matrices satisfying this condition, and this will be done. The following two lemmas contain basic facts which can be collected together easily following similar ones which apply to the 4-dimensional Lorentz group [5] (see also [18, 20, 19]).

LEMMA 4.1.

(a) $A \in \mathcal{L} \Rightarrow \det A = \pm 1$ *and* $A^T \in \mathcal{L}$.

(b) *If* $\alpha \in \mathbb{C}$ *is an eigenvalue of* A *then* $\alpha \neq 0$ *and* α^{-1} *is an eigenvalue of* A *and* A^{-1} (*and also of* A^T), *and if any corresponding* (*real or complex*) *eigenvector* v *is non-null then* $\alpha = \pm 1$ *and so* α *is real. Thus either* v *is real, or its real and imaginary parts are eigenvectors of* A *with equal eigenvalue* α.

(c) *If* u *and* v *are eigenvectors of* $A \in \mathcal{L}$ *with respective eigenvalues* α *and* β *then either* $\alpha = \beta^{-1}$ *or* $u.v = 0$. *If* $\alpha \in \mathbb{C}\backslash\mathbb{R}$ *then* (*from* (b)) u *is null and its real and imaginary parts span either an invariant totally null* 2*-space for* A ($\Leftrightarrow u.\overline{u} = 0$) *or an invariant spacelike* 2*-space* ($\Leftrightarrow u.\overline{u} \neq 0$). *In this latter case* $|\alpha| = 1$.

LEMMA 4.2. *Each* $f \in \mathcal{L}$ *admits an invariant* 2*-space* V *and its orthogonal complement* $V^{\perp}$ *is invariant. If* V *is spacelike then the restriction of* f *to* V *gives rise either to a pair of* (*real*) *orthogonal eigenvectors whose eigenvalues are each* ± 1, *or to a conjugate pair of complex null eigenvectors with eigenvalues of unit modulus. If* V *is timelike and spanned by null vectors* l *and* n *with* $l.n = 1$ *then either* $f(l) = \alpha l$ *and* $f(n) = \alpha^{-1} n$ ($\alpha \in \mathbb{R}$), *or* $f(l) = \lambda n$ *and* $f(n) = \lambda^{-1} l$, *in which case* $l \pm \lambda n$ *are a pair of spacelike and timelike real eigenvectors with eigenvalues* ± 1. *If* V *is null then its unique null direction is a real eigendirection of* f *and there may* (*or may not*) *be another* (*necessarily non-null*) *one in* V. *If* V *is totally null then it is easily checked that either there may be exactly one or exactly two independent real null eigenvectors in* V, *or a conjugate pair of complex null eigenvectors may arise whose real and imaginary parts span* V. *In the latter case* f *cannot admit any other independent* real *eigenvector in* $T_m M$ *since then if* $k \in T_m M$ *is such a vector, Lemma* 4.1(c) *shows that* k *is orthogonal to each member of* V *and hence lies in* V *and is thus not an independent eigenvector. If two totally null invariant* 2*-spaces arise, one in each of* $\overset{+}{S}_m$ *and* $\overset{-}{S}_m$, *their intersection leads to a real null eigendirection for* f (*Section* 1).

LEMMA 4.3. *If* V *is an invariant* 2*-space of* $f \in \mathcal{L}$ *which is spacelike* (*respectively timelike*) *then* $V^{\perp}$ *is invariant and spacelike* (*respectively timelike*) *and* V *and* $V^{\perp}$ *are complementary subspaces in* $T_m M$. *In either case* f *is diagonalisable over* $\mathbb{C}$ *or* $\mathbb{R}$ *with Segre type* $\{z\overline{z}w\overline{w}\}$, $\{11z\overline{z}\}$, $\{1111\}$ *or some degeneracy of these types. Further, if* f *has four independent* (*real or complex*) *eigenvectors then either* f *admits a pair of totally null invariant* 2*-spaces lying both in* $\overset{+}{S}_m$ *or both in* $\overset{-}{S}_m$, *or it admits a spacelike or timelike invariant* 2*-space.*

Proof. The first part follows from Lemma 4.2. For the last part, if f admits two conjugate pairs of complex eigenvectors then either an orthogonal pair of spacelike invariant 2-spaces are admitted or a pair of totally null invariant 2-spaces, both lying in $\overset{+}{S}_m$ or both in $\overset{-}{S}_m$, are admitted (in order that they be complementary—Section 1). If f admits a single conjugate pair of complex eigenvectors and two independent real ones, it follows from the last part of Lemma 4.2 that they arise from a spacelike invariant 2-space and its orthogonal complement. Next, suppose f admits four independent real eigenvectors l, k, p, q where one may be chosen null, say l. Then either an invariant timelike 2-space exists (containing l), or $l.k = l.p = l.q = 0$, which contradicts the independence of l, k, p, q. Finally, suppose l, k, p, q are real and none may be chosen null. Then the span of any two of them is invariant and cannot be null (since this would lead to a null eigenvector from Lemma 4.2) or totally null, and so must be spacelike or timelike. ■

At this point it is useful to note the following results. Let $V \equiv l \wedge N$ denote an invariant, totally null 2-space (in the notation of Section 1 using a null tetrad l, n, L, N). Then the members of $\mathcal{L}$ with the effect $(l, n, L, N) \to (l', n', L', N')$ can be written as [4]

$$(4.1) \qquad \begin{aligned} l' &= al + bN, & N' &= cN + dl, \\ n' &= \lambda(cn - dL) - \rho(cN + dl), & L' &= \lambda(aL - bn) + \rho(bN + al), \end{aligned}$$

where $a, b, c, d, \rho, \lambda \in \mathbb{R}$ $(ac - bd \neq 0)$ and $\lambda = (ac - bd)^{-1}$. Such maps preserve the 2-space $l \wedge N$ and give rise to a 5-dimensional subgroup of $O(2, 2)$ (see, e.g., [24, 23, 6]). If one restricts further by imposing the condition that $ac - bd = 1$ $(\Leftrightarrow \lambda = 1)$ then these maps *fix* the bivector $l \wedge N$. Next the members of $\mathcal{L}$ which preserve a given null direction spanned by l (*null rotations about* l) are given by [4]

$$(4.2) \qquad \begin{aligned} &l' = Al, \quad N' = aN + bl, \quad L' = \frac{1}{a}L + cl, \\ &n' = \frac{1}{A}\left[n - bcl - acN - \frac{b}{a}L\right], \end{aligned}$$

where $A, a, b, c \in \mathbb{R}$ $(A \neq 0 \neq a)$. These transformations give rise to the group with Lie algebra $4(c)$ in [24].

LEMMA 4.4. *If* $f \in \mathcal{L}$, *then it has either exactly two or exactly four independent* (*real or complex*) *eigenvectors.*

Proof. Suppose f admits a unique, independent (necessarily real) eigenvector k, so that $f(k) = \gamma k$ with $\gamma \in \mathbb{R}$. Since f always admits an invariant 2-space V, Lemma 4.2 shows that V is either null or totally null and, in either case, k is null and from the first part of Lemma 4.1(b), $\gamma = \pm 1$. If V is null and, in the notation of Section 1 (with $k \equiv l$) represented by $l \wedge y$ and with (invariant) orthogonal complement $l \wedge s$, then one has $f(l) = \epsilon_1 l$, $f(y) = \epsilon_2 y + al$ and $f(s) = \epsilon_3 s + bl$ with $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$ and $a \neq 0 \neq b \in \mathbb{R}$.

There are two cases to consider: when $\epsilon_1 = \epsilon_2 = \epsilon_3$ and when ϵ_1 is different from at least one of ϵ_2 and ϵ_3. In the first case $by - as$ is an independent eigenvector of f, whilst in the second with, say, $\epsilon_1 = -\epsilon_2$, $\epsilon_1 y - (a/2)l$ is an independent eigenvector of f (and similarly if $\epsilon_1 = -\epsilon_3$ then $\epsilon_1 s - (b/2)l$ is an independent eigenvector of f). These contradictions show that this case is not possible.

Now suppose that V is totally null with, say, $V \equiv l \wedge N$ in the notation of Section 1. Then $k \in l \wedge N$ and so let $k \equiv l$ so that (4.2) applies to f with $A = \pm 1$ (from Lemma 4.1(b)). Then $l \wedge L$ is also invariant under f. To avoid the contradiction of having N and L eigenvectors, one first needs $b \neq 0 \neq c$ in (4.2) and then since f is non-singular, $a \neq 0$. Then to avoid further eigenvectors in $l \wedge N$ or $l \wedge L$, one needs $a = A = 1/a = \pm 1$. These lead to $f(y) = ay + \alpha l$ and $f(s) = as + \beta l$ (since $\sqrt{2}\,L = y + s$ and $\sqrt{2}\,N = y - s$ and with $\alpha, \beta \in \mathbb{R}$), and so $l \wedge y$ and $l \wedge s$ are invariant 2-spaces. If either α or β is zero then one has an extra independent eigenvector for f, and otherwise $\beta y - \alpha s$ is an independent eigenvector. This contradiction completes the proof of the first part of Lemma 4.4.

Now suppose that there are exactly three independent eigenvectors for f. Then either there is a complex conjugate pair of eigenvectors u and $\overline{u}$ plus an independent real one k, or there are exactly three independent real ones. In the former case, and to avoid four independent eigenvectors, the 2-space spanned by the real and imaginary parts of u must span a totally null 2-space, and then the last result in Lemma 4.2 yields a contradiction. So suppose that one may choose *real* vectors p, q and k as independent eigenvectors for f. If the (invariant) span of p, q and k is not null (that is, its orthogonal complement is not contained in it) then this orthogonal complement yields a fourth independent eigenvector and a contradiction follows. If it is null then its orthogonal complement, spanned say by l, is a null eigenvector for f, and one may choose, say, l, p and q as independent eigenvectors for f with $l.p = l.q = 0$. Then $p \wedge q$ is invariant and cannot be totally null (Lemma 2.2(iv)), and hence it is null. Let r be the unique null eigenvector in $p \wedge q$ so that l and r are independent and orthogonal (to avoid timelike invariant 2-spaces). Then $l \wedge r$ is totally null and $r.p = r.q = 0$. The contradiction that l, p, q are not independent follows. ∎

To achieve the full classification of the members of $\mathcal{L}$, note that if $l \wedge N$ is a totally null invariant 2-space for f in $\overset{+}{S}_m$, so that (4.1) holds, *and if another totally null invariant 2-space for f in $\overset{+}{S}_m$ exists*, it may be chosen as $n \wedge L$ and then $\rho = 0$ for f in (4.1). Then (Lemma 4.2) if $l \wedge N$ gives rise to a conjugate pair of complex eigenvectors (as may be arranged from (4.1)), so also does $n \wedge L$ and diagonalisability over $\mathbb{C}$ follows. Otherwise, if f admits real eigenvector(s) in $l \wedge N$, it can be shown that the number of independent such eigenvectors in $l \wedge N$ (one or two) equals that in $n \wedge L$, and if this number is two, diagonalisability over $\mathbb{R}$ follows.

The remainder of the argument concentrates on the situation when f is not diagonalisable (over $\mathbb{C}$ or $\mathbb{R}$), and we show that examples exist (with f admitting exactly two independent eigenvectors).

Suppose first that f admits a totally null invariant 2-space, say $F \equiv l \wedge N$ in $\overset{+}{S}_m$ so that (4.1) holds and which gives rise to a *conjugate complex* pair of eigenvectors. Then Lemma 4.2 shows that no real eigenvectors (and hence no invariant 2-spaces which are timelike, null or totally null *and* in $\overset{-}{S}_m$) exist.

Now let f be such that it *fixes* the bivector $F = l \wedge N$. Such maps belong to a 4-dimensional Lie subgroup of $\mathcal{L}$ whose corresponding Lie algebra, given in bivector form and in terms of some null basis l, n, L, N, is the span of the collection $(l \wedge N, l \wedge L, n \wedge N, l \wedge n + L \wedge N)$ and is represented by the real numbers $a, b.c, d$ and ρ in (4.1) with $ac - bd = 1$ [4].

Suppose further that f is the exponential of some member of this Lie algebra which has a non-zero component along $l \wedge N$ in the above basis. Then with F given as above and F' any other totally null member of $\overset{+}{S}_m$, one may choose a null basis l', n', L', N' so that $l' \wedge N' = l \wedge N$ and $F' = n' \wedge L'$ [23, 24]. In the resulting transformation (4.1), ρ is non-zero and the arbitrary F' is not invariant. Thus f admits no invariant totally null 2-spaces in $\overset{+}{S}_m$ other than F. (One could have chosen f just to fix the 2-space F and selected an appropriate member of the 5-dimensional group represented by (4.1) but it is perhaps simpler to restrict to the above subalgebra which is, conveniently, the product of the algebra spanned by F and that spanned by $\overset{-}{S}_m$.)

Finally, one shows that f cannot admit a spacelike invariant 2-space by assuming that it does admit one, say W, so that W and $W^{\perp}$ are spacelike and invariant. If so then W and $W^{\perp}$ must each give rise to a conjugate pair of complex eigenvectors (Lemma 4.2) with respective eigenvalue pairs $(z, \overline{z})$ and $(w, \overline{w})$ satisfying $|z| = |w| = 1$ and eigenvectors $r + is$ (for z) and $p + iq$ (for w) with $r.r = s.s = 1 = -p.p = -q.q$ and $r.s = p.q = 0$. If the eigenvalues arising from F are $(\chi, \overline{\chi})$, it follows from Lemma 4.1(b) that $|\chi| = 1$. Further, either χ differs from z *and* $\overline{z}$, or χ differs from w *and* $\overline{w}$, or $z = w$ (or $z = \overline{w}$). In the first of these cases one easily sees, from Lemma 4.1(c), that $l + iN$ is orthogonal to $r \pm is$ and hence $l.r = l.s = N.r = N.s = 0$, that is, $r, s \in l \wedge N$, which is a contradiction. The second case is similar. Thus the final case must hold and $r + is$ and $p + iq$ (or $p - iq$) span an eigenspace for f. By taking suitable combinations of these complex eigenvectors one may construct two independent, invariant, totally null 2-spaces for f in $\overset{+}{S}_m$ or in $\overset{-}{S}_m$, which is again a contradiction. Thus no spacelike invariant 2-spaces are admitted by f.

It follows that the original pair of complex eigenvectors for f are its only independent eigenvectors and the only possible Segre types for f are $\{22\}$ and $\{31\}$; it is easily checked that, in fact, f has Segre type $\{22\}$ with a (conjugate) pair of complex eigenvectors whose real and imaginary parts span $l \wedge N$.

Now suppose that f admits two independent real eigenvectors, say p and q. If these are the only two independent eigenvectors then the only types of invariant 2-spaces permitted are null or totally null, and hence one of these eigenvectors, say p, must be null and $p.q = 0$.

Suppose first that these conditions hold with p and q null and chosen, respectively, as l and N. Then (4.1) holds with $b = d = 0$ and (consistently with Lemma 4.1) one may choose $ac = 1$ ($\Rightarrow \lambda = 1$) and $\rho \neq 0$. Then it is easily checked directly that f admits no further independent eigenvectors and that its Segre type is $\{22\}$ with real eigenvalues which may be equal (and so $\{(22)\}$ is possible). Also from (4.1) or (4.2), $l \wedge L$ and $n \wedge N$ are invariant 2-spaces in $\overset{-}{S}_m$ but $l \wedge N$ is the only invariant such 2-space in $\overset{+}{S}_m$ (see the previous argument since $\rho \neq 0$ and $\lambda = 1$). Alternatively one may have two invariant totally null 2-spaces in $\overset{+}{S}_m$ with exactly one null eigenvector in each and with these eigenvectors spanning a totally null invariant 2-space in $\overset{-}{S}_m$. These two ways of looking at f merely reflect the symmetry between $\overset{+}{S}_m$ and $\overset{-}{S}_m$. [Note that, consistently with Lemma 4.1, one may also choose $b = d = 0$, $a = 1 = -c$, $\lambda = -1$ and $\rho \neq 0$ in (4.1) but this leads to two independent extra eigenvectors for f and Segre type $\{1111\}$ or some degeneracy of this type.]

Suppose now that p is null and q non-null. Since $p.q = 0$ still holds, choose $p \equiv l$ and $q \equiv y$ in the basis l, n, y, s so that $l \wedge y$ is a null invariant 2-space. Thus one has $f(l) = \mu l$ and $f(y) = \nu y$ with $\mu = \pm 1$, $\nu = \pm 1$ (Lemma 4.1). To show that l and y may be the only independent eigenvectors, consider the situation $f(l) = l$, $f(y) = y$, $f(s) = s + \gamma l$ and $f(n) = n + \gamma s + (\gamma^2/2)l$. It is now easily checked that l and y are the only independent eigenvectors and so the Segre type is either $\{(31)\}$ or $\{(22)\}$. However, the latter Segre type would lead to two invariant 2-spaces U and V for f satisfying $l \in U$, $y \in V$ and $U \cap V = \{0\}$, from which V must be null and hence admits another independent (null) eigenvector and a contradiction follows. Thus the Segre type is $\{(31)\}$. By retaining the above action of f on the basis with the single change given by $f(y) = -y$ one easily obtains the Segre type $\{31\}$. However, in this case, $\det A = -1$, whereas in the first (type $\{(31)\}$) case above, $\det A = 1$.

Note that if f satisfies $f(l) = l$, $f(y) = y$, $f(s) = -s+\gamma l$ then $s-(\gamma/2)l$ is an eigenvector with eigenvalue -1 and so f is diagonalisable over $\mathbb{R}$. Noting that the transformation $l \to \lambda l$, $n \to \lambda^{-1}n$, $L \to \mu L$, $N \to \mu^{-1}N$ (λ and μ non-zero, distinct members of $\mathbb{R}$ and $\mu\lambda \neq 1$) is in $\mathcal{L}$ and has type $\{1111\}$, one has the following result.

THEOREM 4.5. *A non-trivial member $f \in \mathcal{L}$ may be of Segre type* $\{1111\}$, $\{(11)11\}$, $\{(11)(11)\}$, $\{(111)1\}$, $\{11z\bar{z}\}$, $\{(11)z\bar{z}\}$, $\{z\bar{z}w\overline{w}\}$, $\{(zz)(\overline{zz})\}$, $\{22\}$ *with complex eigenvalues,* $\{22\}$ *or* $\{(22)\}$ *with real eigenvalues,* $\{31\}$ *or* $\{(31)\}$. *Types* $\{(111)1\}$, $\{11z\bar{z}\}$ *and* $\{31\}$ *can only occur if* $\det A = -1$.

This result differs from the corresponding result in the Lorentz case [5] in several ways which are easily seen by a simple comparison. In particular, and unlike the Lorentz case, the neutral signature case could have a non-trivial f with $\det A = 1$ admitting either exactly four independent real null eigenvectors, infinitely many distinct real null eigendirections, or no real null eigenvectors.

5. Sectional curvature. Let G_mM denote the 4-dimensional Grassmann manifold of all 2-dimensional subspaces of T_mM and let $\overline{G_mM}$ denote its open submanifold of all spacelike or timelike 2-spaces at m. Thus members of G_mM are in one-to-one correspondence with the blades of *simple* bivectors at m. Then the sectional curvature function σ_m at m is the real-valued function on $\overline{G_mM}$ given in terms of a representative simple bivector F for its argument by

$$\sigma_m(F) \equiv \frac{R_{abcd}F^{ab}F^{cd}}{2G_{abcd}F^{ab}F^{cd}} = \frac{R_{abcd}F^{ab}F^{cd}}{2F^{ab}F_{ab}} \tag{5.1}$$

where $G_{abcd} = \frac{1}{2}[g_{ac}g_{bd} - g_{ad}g_{bc}]$ and R_{abcd} are the components of the curvature tensor arising from g and its Levi-Civita connection. (Clearly the denominator of (5.1) vanishes if and only if F is null or totally null.) Of course, if g is positive definite then σ_m is defined on the whole of G_mM, whilst for Lorentz signature one must remove the null 2-spaces from G_mM first. It was showed in [13] that in the positive definite case and provided σ_m was at no point of M a constant function, the function σ_m uniquely determined g on M. Later it was shown independently in [7, 17] (see also [11, 5]) that, in the Lorentz case, and with a very special case excluded, a similar result was true. Here the situation for the neutral signature case will be briefly described (with further details being given elsewhere) and seen to be similar in many respects to the Lorentz case.

The general idea is to first consider the possible continuous extension of σ_m (with respect to the usual topologies) to some (any) member of $N_mM \equiv G_mM \setminus \overline{G_mM}$. This is certainly trivially possible in the event that (M, g)

satisfies the constant curvature condition at m. However, the existence of a continuous extension of σ_m to *any single member of* N_mM in this case turns out, by a proof similar to that given in [11], to be equivalent to the constant curvature condition at m. Briefly, this is a consequence of the fact that if F is a representative bivector for some member of N_mM satisfying $R_{abcd}F^{ab}F^{cd} \neq 0$ and if U is any open neighbourhood of F in G_mM then σ_m is unbounded on $U \cap \overline{G_mM}$, this latter set being non-empty since each member of N_mM is a limit point of $\overline{G_mM}$. It also relies on a classification of the Weyl conformal tensor for this signature and the properties of the principal null directions for this tensor [4]. It follows that if σ_m is at no point of M a constant function then it determines the subset N_mM at each $m \in M$, that is, the union of the sets of null and totally null 2-spaces at each m. Now if F and F' are representative members of points of N_mM, it is easily checked that the bivector $F + \lambda F'$ is, for some $0 \neq \lambda \in \mathbb{R}$, *simple and also represents a point of* N_mM if and only if the blades of F and F' intersect in a null direction at m (and then $F + \lambda F' \in N_mM$ for all $\lambda \in \mathbb{R}$). Thus σ_m, through N_mM, determines the null cone of g at m. It follows that if h is a smooth metric on M of arbitrary signature which has the same sectional curvature function as g on M (and which is at no point of M a constant function) then g and h are conformally related, so that $g = \phi h$ on M for a smooth function $\phi : M \to \mathbb{R}$ (and h has neutral signature). One can then substitute this result back into the statement that g and h have the same function σ_m, using (5.1), to get further information.

In fact, if it is assumed that the common (tensor type $(1,3)$) Weyl conformal tensor of g and h is nowhere zero on M then one can show that $\phi = 1$ and so $g = h$. This assumption is not needed but if it is dropped then one must carry out a topological decomposition of M, which leads to interesting special cases. In fact one can achieve a *disjoint* decomposition $M = W \cup V \cup K$ where W is an open subset of M on which $h = g$, K is a closed subset of M with empty interior, and V is an open subset of M which is conformally flat (for g and h) and on which g and h satisfy $g = \phi h$ with $\phi : V \to \mathbb{R}$ such that its gradient vanishes nowhere on V. The function ϕ is a function of a single variable and is forced to satisfy a certain second order differential equation on the subset V (and which can be solved conveniently [17]). On V the Ricci tensor, Ricc, with components $R_{ab} \equiv R^c{}_{acb}$, if not zero, is of Segre type $\{(211)\}$ (see (2.6)) and the curvature tensor is recurrent in the sense of Walker [22]. Then locally on V one may choose coordinates u, v, x, y so that this metric g is given by

$$ds^2 = H(u, x, y)du^2 + 2dudv + dx^2 - dy^2 \qquad (5.2)$$

for some function H of the specified variables. It is the analogue of the conformally flat plane waves [2, 21] used in general relativity theory in the

Lorentz case (and arises under similar circumstances for Lorentz signature [17, 5], differing from (5.2) only by changing the minus sign there to a plus sign). For neutral signature and also for the Lorentz case, if one imposes the Ricci flat condition (but insists that *Riem* is nowhere zero) on (M, g) then one achieves the neat result that the sectional curvature σ_m uniquely fixes g (and conversely) on M (cf. [5]).

6. Further brief general remarks. The geometry in and decomposition of T_mM described in Section 1 can be implemented to give a classification of the Weyl conformal tensor C. Such a classification has been given in full in [4] and independently (but only in partial form) in [14, 15]. Such a classification is algebraic and decomposes $C(m)$ uniquely into independent parts which may be said to "live in" $\overset{+}{S}_m$ and $\overset{-}{S}_m$. The classification then proceeds to study each of these parts separately by procedures not dissimilar from the techniques used in Sections 2–4. It further introduces the concept of a *principal totally null* 2-space and also of a *principal null direction* (the latter having proved useful [1] in the Petrov classification of the Weyl conformal tensor [16] in the Lorentz signature case used in general relativity theory).

The Weyl conformal theorem [25] states that the Weyl type $(1, 3)$ conformal tensors of two conformally related metrics are identical. A sensible converse of this theorem might be: Suppose a connected manifold M of dimension $n \geq 4$ admitting a metric g of arbitrary signature has a nowhere zero type $(1, 3)$ Weyl conformal tensor C on M. Then C uniquely determines the conformal class of the metric on M. This is, in fact, false unless $\dim M = 4$ and g is positive definite, and a complete description of the possible counter-examples can be written down at least in the 4-dimensional case [8, 9]. The counter-examples for the Lorentz and neutral signature cases are rather similar to each other and involve the metric (5.2). They also each involve the most specialised (non-zero) types in the classification of the tensor C of the previous paragraph and that of Petrov in the Lorentz case. The proof of these facts makes significant use of the techniques used in Section 1.

The holonomy structure of (M, g) with M connected, $\dim M = 4$ and g of neutral signature has been studied in [24, 23, 10]. This involved finding all the possible Lie subalgebras of the Lie algebra of the group $\mathcal{L}$ studied in Section 4 (and in these references emphasis was laid on those subalgebras arising as holonomy algebras for reasons given there) and considering their application to projective structure [24, 23] and recurrence theory [10]. A full list of Lie subalgebras for $\mathcal{L}$ has also been written down in [3] but the bivector language used in [24, 10] is more convenient for such purposes. Again the general techniques of Section 1 are crucial here.

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