On Wallman compactifications

by

R. M. Brooks* (Minneapolis, Minnesota)

1. Introduction. In 1938 Wallman [9] showed that if L is a distributive lattice with a zero and a unit, then L generates a compact T_1-space wL (the space of ultrafilters of L). He applied this to the special case where L was the lattice of all closed subsets of a T_1-space. In 1948 Samuel [8] obtained similar results for g.l.b.-semi-lattices (he called them "directed sets"), and used his techniques to study uniform spaces. Several other authors (e.g., Ky Fan and Gottesman [3], Banaschewski [1], and Frink [4]) have considered the problem of obtaining compactifications of topological spaces as spaces of ultrafilters or maximal ideals (defined in a special way in [1]) in lattices. They have considered lattices with special properties, often for the purpose of obtaining compactifications having special topological properties.

This paper is directed toward the study of ultrafilter spaces of lattices of subsets of a set (often a topological space, in which case the lattice is related to the topology). The main reason for our general treatment is that there is a natural application to the study of topological algebras and the lattices one obtains there do not in general satisfy the strong restrictions of [1], [3], and [4]. Also there are applications to measure theory (cf. [6]), and in such applications the topology of the underlying set (if there is any) is often unrelated to the lattice.

The main problem we consider is the following. If X is a topological space and (T, σ) is a pair, where T is a compact Hausdorff space and σ is a continuous map of X onto a dense subspace of T, does there exist a lattice $\mathfrak{L}$ of subsets of X such that $T = w\mathfrak{L}$ (the space of ultrafilters of $\mathfrak{L}$)? This problem is shown to be equivalent to the special case when X is a completely regular Hausdorff space and σ is a homeomorphism. Our results consist of some sufficient conditions on X and T in order that such a lattice exists, and a method of realizing T (in the general case) as the space of all filters in a certain lattice which are maximal with respect to some natural auxiliary property.

* This work was supported in part by NSF Grant G 24295.

The first part of the paper is devoted to a study of the basic properties of lattice compactifications of a set X, especially the relationship between the separation properties of a given lattice $\mathfrak{L}$ and the resulting properties of $w\mathfrak{L}$ and the embedding map of X into $w\mathfrak{L}$. We also consider here a certain algebra of functions on X defined by $\mathfrak{L}$ and its relationship to $w\mathfrak{L}$.

Next we consider the problem of uniqueness of a lattice compactification or, equivalently, the non-uniqueness of lattices yielding the same compactification, and give conditions under which two lattices one contained in the other have the same ultrafilter space. This section has a direct application to the study of certain commutative topological algebras.

We then turn to the main problem, first reducing to the special case, then exhibiting a criterion to use in deciding whether a given lattice $\mathfrak{L}$ on X yields a given compactification T of X as its ultrafilter space. This criterion is applied to obtain sufficient conditions (on X and T) for obtaining T as an ultrafilter space.

In the last section we show that a Hausdorff compactification T of X is always a quotient space of a Wallman compactification $w\mathfrak{L}$ of X, where $\mathfrak{L}$ is the lattice of zero-sets of the algebra $\mathfrak{A}$ of continuous real- (or complex-) valued functions on X which have continuous extensions to T. (T is always a quotient space of βX, but this space need have little relation to $w\mathfrak{L}$.) We then define the concept of a $*$-filter in $\mathfrak{L}$ and show that T "is" the space X^* of all maximal $*$-filters in $\mathfrak{L}$, where the space X^* is equipped with a topology defined exactly like that of an ultrafilter space.

2. Basic properties. We consider a set X and a lattice $\mathfrak{L}$ of subsets of X such that X and the null set $\emptyset$ are members of $\mathfrak{L}$. We shall use capital Latin letters to denote subsets of X and script letters to denote families of subsets. We shall refer to $\mathfrak{L}$ as "a lattice on X".

DEFINITION 2.1. A *filter in* $\mathfrak{L}$ is a subset $\mathcal{F}$ of $\mathfrak{L}$ satisfying the conditions:

(i) *$\mathcal{F}$ is closed under finite intersections.*

(ii) $\emptyset \notin \mathcal{F}$.

(iii) *If $A \in \mathcal{F}$ and $B \in \mathfrak{L}$, then $A \cup B \in \mathcal{F}$,*

or, equivalently,

(iii′) *If $A \in \mathcal{F}$, $B \in \mathfrak{L}$, and $A \subset B$, then $B \in \mathcal{F}$.*

DEFINITION 2.2. An *ultrafilter in* $\mathfrak{L}$ is a maximal (relative to the partial order on the collection of filters in $\mathfrak{L}$ given by inclusion) filter in $\mathfrak{L}$.

We note the following characterization of ultrafilters given by Samuel ([8], p. 105) for a g.l.b.-semi-lattice, which holds for lattices.

THEOREM 2.1. *A necessary and sufficient condition for a filter $\mathcal{F}$ in $\mathfrak{L}$ to be an ultrafilter is that for each $A \in \mathfrak{L}-\mathcal{F}$ there exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.*

A second easily-proved theorem we also state without proof.

THEOREM 2.2. *An ultrafilter $\mathcal{U}$ in $\mathfrak{L}$ is $\cup$-prime (i.e., if A, $B \in \mathfrak{L}$ and $A \cup B \in \mathcal{U}$, then one of A and B is a member of $\mathcal{U}$).*

We denote by $w\mathfrak{L}$ the set of all ultrafilters in $\mathfrak{L}$ and for each $A \in \mathfrak{L}$ we define $C(A) = \{\mathcal{U} \in w\mathfrak{L}\colon A \in \mathcal{U}\}$. For reference we state without proof some of the elementary properties of the mapping $A \to C(A)$ (cf. [9]).

THEOREM 2.3. *The mapping $A \to C(A)$ is a lattice homomorphism of $\mathfrak{L}$ into the power set of $w\mathfrak{L}$. Specifically, $C(X) = w\mathfrak{L}$, $C(\emptyset) = \emptyset$, $C(A \cup B) = C(A) \cup C(B)$, $C(A \cap B) = C(A) \cap C(B)$, for each pair A, B in $\mathfrak{L}$.*

We define the topology $\mathcal{T}(\mathfrak{L})$ in $w\mathfrak{L}$ by taking the collection $\{C(A)\colon A \in \mathfrak{L}\}$ as a base for the closed sets. This is the topology $\mathcal{T}_F$ of Samuel's paper [8], p. 117. Thus, for each $A \in \mathfrak{L}$, $C(A)$ is closed in $w\mathfrak{L}$, and every closed subset F of $w\mathfrak{L}$ is the intersection of all the sets $C(A)$ which contain F. An equivalent way of defining the topology $\mathcal{T}(\mathfrak{L})$ is as follows:

THEOREM 2.4. *For each filter $\mathcal{F}$ in $\mathfrak{L}$, the set $C(\mathcal{F}) = \{\mathcal{U} \in \mathfrak{L}\colon \mathcal{F} \subseteq \mathcal{U}\}$ is closed, and every closed set in $w\mathfrak{L}$ is of this form.*

Proof. The fact that each set $C(\mathcal{F})$ is closed is proved in [8], p. 117. If F is any closed set in $w\mathfrak{L}$, then the family $\mathcal{F} = \{A \in \mathfrak{L}\colon F \subseteq C(A)\}$ is a filter in $\mathfrak{L}$ and $F = C(\mathcal{F})$.

The space $w\mathfrak{L}$ equipped with the topology $\mathcal{T}(\mathfrak{L})$ is a compact T_1 space ([9], p. 116). We note here that the family of sets $C(A)$, $A \in \mathfrak{L}$, forms a base for the $\mathcal{T}(\mathfrak{L})$-closed sets of $w\mathfrak{L}$, whereas in Samuel's case ($\mathfrak{L}$ is a g.l.b.-semi-lattice) the family forms only a subbase. Some further observations on the relation between Samuel's compactifications and those obtained from lattices: (i) If $\mathcal{S}$ is a g.l.b.-semi-lattice on X, then the family $\mathfrak{L}$ of all finite unions of members of $\mathcal{S}$ is a lattice on X, and (ii) the space $w\mathfrak{L}$ is homeomorphic to the space of ultrafilters of $\mathcal{S}$ with the topology $\mathcal{T}_F$ ([8], p. 117).

DEFINITION 2.3. $\mathfrak{L}$ is a lattice on X. We say that

(i) $\mathfrak{L}$ satisfies *property* (α) ($\mathfrak{L}$ is an α-lattice) provided that for each $A \in \mathfrak{L}$ and $x \in X-A$ there exists $B \in \mathfrak{L}$ such that $x \in B$ and $A \cap B = \emptyset$.

(ii) $\mathfrak{L}$ satisfies *property* (β) ($\mathfrak{L}$ is a *β-lattice*) provided that for each pair x, y of distinct points in X there exists $A \in \mathfrak{L}$ such that $x \in A$ and $y \in X-A$.

(iii) $\mathfrak{L}$ is a *normal lattice* provided that for each pair A, B of disjoint elements of $\mathfrak{L}$ there exists a pair A_1, B_1 of members of $\mathfrak{L}$ such that $A \subset A_1$, $B \subset B_1$, $A \cap B_1 = \emptyset$, $B \cap A_1 = \emptyset$, and $A_1 \cup B_1$ belongs to every ultrafilter in $\mathfrak{L}$.

(iv) $\mathfrak{L}$ is a *γ-lattice* provided $\mathfrak{L}$ satisfies (α), (β), and is normal.

THEOREM 2.5. *Let $\mathfrak{L}$ be a lattice on X and for each $x \in X$ define $\mathfrak{U}_x = \{A \in \mathfrak{L}: x \in A\}$. Then*

(i) *A necessary and sufficient condition that $\mathfrak{U}_x$ be an ultrafilter for each $x \in X$ is that $\mathfrak{L}$ be an α-lattice.*

(ii) *If $\mathfrak{L}$ is an α-lattice, then the mapping $x \to \mathfrak{U}_x$ of X into $w\mathfrak{L}$ is one-to-one if and only if $\mathfrak{L}$ is a β-lattice.*

(iii) *$w\mathfrak{L}$ is Hausdorff if and only if $\mathfrak{L}$ is normal.*

(iv) *If $\mathfrak{L}$ is an α-lattice, then in Definition 2.3 (iii) the statement "$A_1 \cup B_1$ belongs to every ultrafilter in $\mathfrak{L}$" is equivalent to "$A_1 \cup B_1 = X$".*

(v) *If $\mathfrak{L}$ is an α-lattice, then the image of X under the mapping $x \to \mathfrak{U}_x$ is dense in $w\mathfrak{L}$.*

Proof. (i) follows immediately from the characterization of ultrafilters given in Theorem 2.1. (ii) is obvious. (iii) is proved in [9] (cf. footnote 12 [9], p. 119), (iv) is clear in view of the fact that if $\mathfrak{L}$ is an α-lattice then the only member of $\mathfrak{L}$ which belongs to every ultrafilter in $\mathfrak{L}$ is X. (v): If $\mathfrak{U} \in w\mathfrak{L}$ and N is an open set in $w\mathfrak{L}$ containing $\mathfrak{U}$, then there exists $A \in \mathfrak{L}$ such that $\mathfrak{U} \in w\mathfrak{L} - C(A) \subset N$. If $x \in X - A$, then $\mathfrak{U}_x \in w\mathfrak{L} - C(A)$.

We shall consider from this point on only α-lattices in order that the ultrafilter space have some relation to the set X. We shall denote by φ the mapping $x \to \mathfrak{U}_x$ of X into the ultrafilter space.

THEOREM 2.6. *The mapping φ: $X \to w\mathfrak{L}$ satisfies:*

(i) *For each $B \in \mathfrak{L}$, $\varphi(B) = C(B) \cap \varphi(X)$.*

(ii) *For each $B \in \mathfrak{L}$, $\varphi(X - B) = \varphi(X) - \varphi(B) = \varphi(X) \cap [w\mathfrak{L} - C(B)]$.*

(iii) $\mathrm{Cl}[\varphi(B)] = C(B)$ *for each $B \in \mathfrak{L}$, where* Cl *denotes the closure operator in $w\mathfrak{L}$.*

(iv) *For each $F \subseteq X$, $\mathrm{Cl}[\varphi(F)] = \bigcap \{C(A): F \subseteq A\}$.*

(v) *If $F \subseteq X$, $A \in \mathfrak{L}$, then $\varphi(A \cap F) = \varphi(A) \cap \varphi(F)$.*

Proof. (i): The containment $\varphi(B) \subset C(B) \cap \varphi(X)$ is clear. If $\mathfrak{U} \in C(B) \cap \varphi(X)$, then $B \in \mathfrak{U}$. If $\mathfrak{U} = \mathfrak{U}_x$ implies $x \in X - B$, then fixing one such x we obtain (by (α)) an element A of $\mathfrak{L}$ which contains x and fails to meet B. Then $A \in \mathfrak{U} = \mathfrak{U}_x$ and $B \notin \mathfrak{U}$, a contradiction. Thus, if $\mathfrak{U} \in C(B) \cap \varphi(X)$, there exists $x \in B$ such that $\mathfrak{U} = \mathfrak{U}_x$, and $\mathfrak{U} \in \varphi(B)$.

(ii): By (i), $\varphi(X) - \varphi(B) = \varphi(X) - [C(B) \cap \varphi(X)]$. If $\mathfrak{U}$ is in the right-hand set, then $\mathfrak{U} \in \varphi(X)$ and $\mathfrak{U} \notin C(B)$. Therefore, $\mathfrak{U} \in \varphi(X) \cap [w\mathfrak{L} - C(B)]$. The other containment is clear and we have $\varphi(X) - \varphi(B) = \varphi(X) \cap [w\mathfrak{L} - C(B)]$. If $\mathfrak{U} \in \varphi(X) - \varphi(B)$, then $\mathfrak{U} = \mathfrak{U}_x$ for some $x \in X$ and $\mathfrak{U} \neq \mathfrak{U}_y$ for all $y \in B$. Hence, $x \in X - B$ and we have $\varphi(X) - \varphi(B) \subset \varphi(X - B)$. If this containment were proper, there would exist $\mathfrak{U} \in \varphi(X - B) \cap \varphi(B)$. From this one obtains $x \in X - B$, $y \in B$ such that $\mathfrak{U} = \mathfrak{U}_x = \mathfrak{U}_y$. Since $B \in \mathfrak{L}$ and $x \in X - B$, there exists $A \in \mathfrak{L}$ such that $x \in A$ and $A \cap B = \emptyset$. Hence $A \in \mathfrak{U}_x = \mathfrak{U}_y$, and $B \in \mathfrak{U}_y$; a contradiction. We conclude that $\varphi(X) - \varphi(B) = \varphi(X - B)$.

(iii): It is clear that $\mathrm{Cl}[\varphi(B)] \subset C(B)$. If $\mathfrak{U} \in w\mathfrak{L} - \mathrm{Cl}[\varphi(B)]$, then since $\{\mathfrak{U}\} = \bigcap \{C(A): A \in \mathfrak{U}\}$ and the family $\{C(A): A \in \mathfrak{U}\}$ is a descending family of compact subsets of $w\mathfrak{L}$ whose intersection is contained in the open set $w\mathfrak{L} - \mathrm{Cl}[\varphi(B)]$, there exists $A \in \mathfrak{U}$ such that $C(A) \subset w\mathfrak{L} - \mathrm{Cl}[\varphi(B)]$. This implies that $A \cap B = \emptyset$ and $B \notin \mathfrak{U}$. Hence $\mathfrak{U} \notin C(B)$.

(iv): For each $F \subseteq X$, $\mathrm{Cl}[\varphi(F)] = \bigcap \{C(A): \varphi(F) \subset C(A)\}$. We show that $\varphi(F) \subset C(A)$ if and only if $F \subset A$. One implication is obvious. Conversely, if $F \not\subset A$, then there exists $x \in F - A$, and $x \in X - A$ implies $\varphi(x) \in \varphi(X) - \varphi(A) = \varphi(X) - C(A)$. Thus $\varphi(F) \not\subseteq C(A)$.

(v): It is clear that $\varphi(A \cap F) \subseteq \varphi(A) \cap \varphi(F)$. If $\mathfrak{U} \in [\varphi(A) \cap \varphi(F)] - \varphi(A \cap F)$, then $\mathfrak{U} \in \varphi(F)$ and if $\mathfrak{U} = \mathfrak{U}_x$, then $x \notin A \cap F$. We fix $x \in F$ such that $\mathfrak{U} = \mathfrak{U}_x$. By ($\alpha$) we have the existence of $B \in \mathfrak{L}$ such that $x \in B$ and $A \cap B = \emptyset$. But then $B \in \mathfrak{U}_x = \mathfrak{U}$ and $\mathfrak{U} \in \varphi(A) \subseteq C(A)$. This implies that both A and B are members of $\mathfrak{U}$, a contradiction.

We now assume that X is a topological space and $\mathfrak{L}$ is an α-lattice on X. We give conditions on $\mathfrak{L}$ which yield connections between the topology of X and that of $w\mathfrak{L}$.

THEOREM 2.7. *If $\mathfrak{L}$ is an α-lattice on X, a topological space, then*

(i) *φ is continuous if and only if each element of $\mathfrak{L}$ is closed in X.*

(ii) *φ is a homeomorphism if and only if each element of $\mathfrak{L}$ is closed in X, $\mathfrak{L}$ is a β-lattice, and $\mathfrak{L}$ forms a base for the closed sets of X.*

Proof. (i) We assume each A in $\mathfrak{L}$ is closed in X, fix $x \in X$ and an open set N in $w\mathfrak{L}$ containing $\varphi(x)$. There exists $A \in \mathfrak{L}$ such that $\varphi(x) \in \varphi(X) \cap [w\mathfrak{L} - C(A)] \subseteq N \cap \varphi(X)$. By (ii) of Theorem 2.6, we have $\varphi(x) \in \varphi(X - A) \subseteq N \cap \varphi(X)$, and $X - A$ is open in X. Conversely, if φ is continuous, then for each $A \in \mathfrak{L}$, $\varphi^{-1}[C(A) \cap \varphi(X)]$ is closed in X. But this set is A.

(ii) If $\mathfrak{L}$ satisfies the conditions, then φ is continuous and one-to-one. We fix F, a closed subset of X. Then $F = \bigcap \{A \in \mathfrak{L}: F \subseteq A\}$, and $\varphi(F) = \bigcap \{\varphi(A): F \subseteq A\} = \bigcap \{C(A) \cap \varphi(X): F \subseteq A\}$. This last set is closed in $\varphi(X)$. The proof of the converse is essentially the same.

We note that a γ-lattice of closed subsets of X which forms a base for the closed sets of X is a *normal base*, as defined by Frink [4], in case X is T_1.

DEFINITION 2.4. Let X be a set and $\mathfrak{L}$ an α-lattice on X. A scalar-valued function (real-valued of complex-valued) f on X is said to be *$(X, \mathfrak{L})$-continuous* provided that for each $\varepsilon > 0$ there exists a finite family $\{A_1, A_2, \ldots, A_n\} \subseteq \mathfrak{L}$ such that $\bigcap \{A_i: i = 1, 2, \ldots, n\} = \emptyset$ and $\theta(f, X - A_i) < \varepsilon$ for $i = 1, 2, \ldots, n$, where for $E \subseteq X$, $\theta(f, E) = \sup\{|f(x) - f(y)|: x, y \in E\}$.

We denote by $C(X, \mathfrak{L})$ the family of all real-valued $(X, \mathfrak{L})$-continuous functions on X, and by $C_C(X, \mathfrak{L})$ the family of all complex-valued $(X, \mathfrak{L})$-continuous functions on X. This type of continuity is that which Frink discussed and called "*Z-uniform continuity*" in [4] where Z is his symbol for a lattice. We note the following regarding this concept. If $\mathfrak{L}$ is an α-, β-lattice and we define for each finite family $\{A_1, \ldots, A_n\} \subseteq \mathfrak{L}$ satisfying $\bigcap A_i = \emptyset$, $V(A_1, \ldots, A_n) = \bigcup\{(X-A_i)\times(X-A_i)\colon i = 1, \ldots, n\}$, then the collection of all such sets in X forms the base for a separated uniform structure on X if and only if $\mathfrak{L}$ is normal. The proof of this fact is identical to that given by Samuel in the case where $\mathfrak{L}$ is the lattice of all closed subsets of X ([8], p. 126). In the case where $\mathfrak{L}$ is normal, the family of all real-valued functions on X uniformly continuous relative to this structure is exactly $C(X, \mathfrak{L})$.

THEOREM 2.8. (i) *$C(X, \mathfrak{L})$ is a uniformly closed real subalgebra of $B(X)$ (the algebra of all bounded real-valued functions on X). Moreover, if one defines for each $f \in C(X, \mathfrak{L})$ and $x \in X$, $f^\wedge(\varphi(X)) = f(x)$, then $f^\wedge$ is a well-defined continuous function on $\varphi(X)$ and extendible to a continuous function on $w\mathfrak{L}$. The mapping $f \to f^\wedge$ so defined is an isomorphism and isometry of $C(X, \mathfrak{L})$ onto $C(w\mathfrak{L})$ (the algebra of all real-valued continuous functions on $w\mathfrak{L}$).*

(ii) *The same statements as in* (i) *for $C_C(X, \mathfrak{L})$, where in addition we have the fact that $C_C(X, \mathfrak{L})$ is closed under conjugation.*

(iii) *If $\mathcal{U} \in w\mathfrak{L}$ and $f \in C(X, \mathfrak{L})$, then $\bigcap\{f(A)^-\colon A \in \mathcal{U}\}$ contains a single point λ of $\boldsymbol{R}$ and $f^\wedge(\mathcal{U}) = \lambda$.*

Proof. (i): For the proof of (i) we shall show that $f^\wedge$ is well-defined and "uniformly continuous" on $\varphi(X)$ and refer the reader to [4], p. 605, for the proof of extendibility, since once $f^\wedge$ is defined on $\varphi(X)$ the details of the proof in the general case are essentially the same as that given by Frink when $\mathfrak{L}$ is a special lattice. To show that $f^\wedge$ is well-defined we show that if x and y are elements of X such that $f(x) \neq f(y)$, then $\mathcal{U}_x \neq \mathcal{U}_y$. If $f(x) \neq f(y)$, we fix $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2}|f(x)-f(y)|$. There exists a family $\{A_1, \ldots, A_n\} \subseteq \mathfrak{L}$ such that $\bigcap A_i = \emptyset$ and $\theta(f, X-A_i) < \varepsilon$ for each $i \leqslant n$. There exists $j \leqslant n$ such that $x \in X-A_j$. Since $|f(x)-f(y)| > \theta(f, X-A_j)$, $y \in A_j$, and $\mathcal{U}_x \neq \mathcal{U}_y$. Thus $f^\wedge$ is well-defined on $\varphi(X)$, and since for each $A \in \mathfrak{L}$, $\varphi(X-A) = \varphi(X)-\varphi(A)$, it is clear that for each $\varepsilon > 0$ there is a family $\{A_1, \ldots, A_n\}$ in $\mathfrak{L}$ such that $\bigcup\{\varphi(X-A_i)\} = \bigcup\{\varphi(X)-\varphi(A_i)\} = \varphi(X)$ and $\theta\big(f^\wedge, \varphi(X)-\varphi(A_i)\big) < \varepsilon$ for each $i \leqslant n$. This is essentially the starting point of Frink's argument. The remaining claims of parts (i) and (ii) follow immediately, once the extendibility is proved.

(iii): Since each f in $C(X, \mathfrak{L})$ is bounded, the family $\{f(A)^-\colon A \in \mathcal{U}\}$ is a descending family of compact subset of $\boldsymbol{R}$ (the real line), and has a non-empty intersection. We fix λ in this set. For each $A \in \mathcal{U}$, we have $f(A)^- \subseteq f^\wedge[C(A)]$. We choose $\mathcal{U}_A \in C(A)$ such that $f^\wedge(\mathcal{U}_A) = \lambda$. The set $\{\mathcal{U}_A\colon A \in \mathcal{U}\}$ converges to $\mathcal{U}$ and $f^\wedge$ is continuous. Hence, $f^\wedge(\mathcal{U}) = \lambda$.

3. Equivalent compactifications. We shall consider in this section the question of uniqueness of a lattice compactification and dually the non-uniqueness of the lattice yielding a given compactification. We consider a fixed set X and α-lattices $\mathfrak{L}$ and $\mathcal{M}$ on X. The ultrafilters in $\mathfrak{L}$ will be denoted by $\mathcal{U}$, those in $\mathcal{M}$ by $\mathcal{V}$. For $A \in \mathfrak{L}$, $C(A) = \{\mathcal{U} \in w\mathfrak{L}\colon A \in \mathcal{U}\}$ and for $F \in \mathcal{M}$, $D(F) = \{\mathcal{V} \in w\mathcal{M}\colon F \in \mathcal{V}\}$. The natural maps of X into $w\mathfrak{L}$ and $w\mathcal{M}$ will be denoted by φ and ψ, respectively. In lieu of closure operators with $\mathfrak{L}$ and $\mathcal{M}$ subscripts we shall use $^-$ to denote closure in $w\mathfrak{L}$ and $*$ to denote closure in $w\mathcal{M}$.

DEFINITION 3.1. We say that $\mathfrak{L}$ and $\mathcal{M}$ are *X-equivalent* provided there exists a homeomorphism σ of $w\mathfrak{L}$ onto $w\mathcal{M}$ such that $\sigma\varphi = \psi$ on X.

THEOREM 3.1. *With $X, \mathfrak{L}, \mathcal{M}, \varphi$, and ψ as defined above and $\mathfrak{L} \subseteq \mathcal{M}$, the following statements are equivalent.*

(i) *If $F, H \in \mathcal{M}$, then $F \cap H = \emptyset$ if and only if $(\varphi F)^- \cap (\varphi H)^- = \emptyset$.*

(ii) *If $F, H \in \mathcal{M}$, then $\varphi(F \cap H)^- = (\varphi F)^- \cap (\varphi H)^-$.*

(iii) *$\mathfrak{L}$ and $\mathcal{M}$ are X-equivalent.*

Proof. (i) *implies* (ii): We first verify (ii) for $A \in \mathfrak{L}$, $F \in \mathcal{M}$. One containment is clear. We fix $\mathcal{U} \in w\mathfrak{L}-\varphi(A \cap F)^-$ and show $\mathcal{U} \notin (\varphi A)^- \cap (\varphi F)^-$. $\{\mathcal{U}\} = \bigcap\{C(B)\colon B \in \mathcal{U}\}$, $w\mathfrak{L}-\varphi(A \cap F)^-$ is open, and $\{C(B)\colon B \in \mathcal{U}\}$ is a descending family of compact sets, so there exists $B \in \mathcal{U}$ such that $C(B) \cap \varphi(A \cap F)^- = \emptyset$, and it follows that $B \cap (A \cap F) = \emptyset$ or $(B \cap A) \cap F = \emptyset$. Thus, by (i), we have $\varphi(A \cap B)^- \cap (\varphi F)^- = \emptyset$. However, $\varphi(A \cap B)^- = C(A) \cap C(B)$. Hence, $C(B) \cap C(A) \cap (\varphi F)^- = \emptyset$. Since $\mathcal{U} \in C(B)$, we must have either $\mathcal{U} \notin C(A)$ or $\mathcal{U} \notin (\varphi F)^-$. In either case $\mathcal{U} \notin (\varphi A)^- \cap (\varphi F)^-$.

We now assume that F and H are members of $\mathcal{M}$ and fix $\mathcal{U} \in w\mathfrak{L}-\varphi(F \cap H)^-$. There exists $A \in \mathcal{U}$ such that $(CA) \cap \varphi(F \cap H)^- = \emptyset$. Hence, $A \cap (F \cap H) = \emptyset$, or $(A \cap F) \cap (A \cap H) = \emptyset$. By (i), $\varphi(A \cap F)^- \cap \varphi(A \cap H)^- = \emptyset$. From the first part of this theorem, we obtain $C(A) \cap (\varphi F)^- \cap (\varphi H)^- = \emptyset$. But $\mathcal{U} \in C(A)$, so $\mathcal{U} \notin (\varphi F)^- \cap (\varphi H)^-$.

(ii) *implies* (iii): For $\mathcal{U} \in w\mathfrak{L}$ we consider the subset $\sigma\mathcal{U}$ of $\mathcal{M}$ defined by $\sigma\mathcal{U} = \{F \in \mathcal{M}\colon \mathcal{U} \in (\varphi F)^-\}$. It is clear that $\sigma\mathcal{U}$ is a filter in $\mathcal{M}$, and if $F \in \mathcal{M}-\sigma\mathcal{U}$, then $\mathcal{U} \notin (\varphi F)^-$ and there exists $A \in \mathcal{U}$ such that $C(A) \cap (\varphi F)^- = \emptyset$. But $A \in \mathcal{U}$ and $\mathfrak{L} \subseteq \mathcal{M}$ imply that $A \in \sigma\mathcal{U}$. Hence, $\sigma\mathcal{U}$ is an ultrafilter in $\mathcal{M}$ (Theorem 2.1). Thus $\mathcal{U} \to \sigma\mathcal{U}$ defines a function from $w\mathfrak{L}$ to $w\mathcal{M}$, which we denote by σ. From the containment $\mathfrak{L} \subseteq \mathcal{M}$ it follows that σ is one-to-one.

We show first that $\sigma\varphi = \psi$. If $x \in X$, then $\sigma\varphi(x) = \sigma\mathcal{U}_x = \{F \in \mathcal{M}\colon \mathcal{U}_x \in (\varphi F)^-\}$ is an ultrafilter in $\mathcal{M}$ and it is easily verified (using (iv) of Theorem 2.6) that $\mathcal{V}_x \subseteq \sigma\mathcal{U}_x$. We must therefore have $\mathcal{V}_x = \sigma\mathcal{U}_x$ or $\sigma\varphi(x) = \varphi(x)$.

If $\mathcal{V}$ is an ultrafilter in $\mathcal{M}$, then $\{(\varphi F)^-\colon F \in \mathcal{V}\}$ is a descending family in $w\mathcal{L}$ and there exists $\mathcal{U} \in w\mathcal{L}$ which belongs to the intersection of this family. If $\mathcal{U}_1 \neq \mathcal{U}$, there exist A, B in $\mathcal{U}_1$, $\mathcal{U}$, respectively, such that $A \cap B = \emptyset$. Since $B \in \mathcal{U}$, we have $\mathcal{U} \in (\varphi B)^-$ and $\mathcal{L} \subseteq \mathcal{M}$ implies $B \in \mathcal{V}$, and $\mathcal{U}_1 \notin (\varphi B)^-$. Thus, $\mathcal{U}$ is the unique point in the intersection. The fact that $\sigma\mathcal{U} = \mathcal{V}$ follows from (a) $\sigma\mathcal{U}$ is a filter in $\mathcal{M}$, and (b) $\mathcal{V} \subseteq \sigma\mathcal{U}$.

We now have a mapping σ of $w\mathcal{L}$ onto $w\mathcal{M}$ which is one-to-one and satisfies $\sigma\varphi = \psi$. We verify the topological properties. If $F \in \mathcal{M}$, then $(\varphi F)^- = \sigma^{-1}[D(F)]$, and if $A \in \mathcal{L}$, then $D(A) = \sigma[C(A)]$. From these facts and the definition of the topologies in $w\mathcal{L}$ and $w\mathcal{M}$ it follows that σ is continuous and closed.

(iii) *implies* (i): If σ is a homeomorphism of $w\mathcal{L}$ onto $w\mathcal{M}$ which satisfies $\sigma\varphi = \psi$, then for each $F \in \mathcal{M}$ $(\varphi F)^- \subseteq \sigma^{-1}[D(F)]$. If F and H are disjoint members of $\mathcal{M}$, then $D(F) \cap D(H) = \emptyset$ and $\sigma^{-1}[D(F)] \cap \sigma^{-1}[D(H)] = \emptyset$. Thus, $(\varphi F)^- \cap (\varphi H)^- = \emptyset$.

We note that these conditions for the X-equivalence of two lattices are similar to some of the uniqueness conditions for βX, when X is a completely regular Hausdorff space, given on p. 86 of [4]. The following theorem and example show the property of extendibility is not in general equivalent to the conditions of Theorem 3.1.

THEOREM 3.2. *If X, $\mathcal{L}$, $\mathcal{M}$, φ, and ψ are as in Theorem* 3.1, *then the following statements hold.*

(i) *If $\mathcal{L}$ and $\mathcal{M}$ are X-equivalent, then $C(X, \mathcal{L}) = C(X, \mathcal{M})$.*

(ii) *If $\mathcal{L}$ and $\mathcal{M}$ are both normal and if $C(X, \mathcal{L}) = C(X, \mathcal{M})$, then $\mathcal{L}$ and $\mathcal{M}$ are X-equivalent.*

Proof. (i): We denote by σ the homeomorphism of $w\mathcal{L}$ onto $w\mathcal{M}$ satisfying $\sigma\varphi = \psi$. The mappings φ and ψ induce isomorphisms φ^* and ψ^* of $C(w\mathcal{L})$ onto $C(X, \mathcal{L})$ and $C(w\mathcal{M})$ onto $C(X, \mathcal{M})$, respectively. Also, σ induces an isomorphism σ^* of $C(w\mathcal{M})$ onto $C(w\mathcal{L})$. It is clear that $C(X, \mathcal{L}) \subseteq C(X, \mathcal{M})$. If $f \in C(X, \mathcal{M})$, then $f = \psi^* f^\wedge$ for a unique $f^\wedge \in C(w\mathcal{M})$. We shall show that $f = \varphi^*\sigma^* f^\wedge$, the latter being a member of $C(X, \mathcal{L})$. For each $x \in X$ we have $\varphi^*\sigma^* f^\wedge(x) = \sigma^* f^\wedge(\varphi x) = f^\wedge(\sigma\varphi x) = f^\wedge(\psi x) = \psi^* f^\wedge(x) = f(x)$.

(ii): If both $\mathcal{L}$ and $\mathcal{M}$ are normal, then $w\mathcal{L}$ and $w\mathcal{M}$ are compact Hausdorff spaces. We have isomorphisms φ^* of $C(w\mathcal{L})$ onto $C(X, \mathcal{L})$ and φ^* of $C(w\mathcal{M})$ onto $C(X, \mathcal{M})$, defined by the equations $\varphi^* f^\wedge(x) = f^\wedge(\varphi x)$, $\psi^* g^\wedge(x) = g^\wedge(\psi x)$ for $f^\wedge \in C(w\mathcal{L})$, $g^\wedge \in C(w\mathcal{M})$, and $x \in X$. We define τ from $C(w\mathcal{M})$ into $C(w\mathcal{L})$ by $\tau = \varphi^{*-1}\psi^*$. Since $C(X, \mathcal{L}) = C(X, \mathcal{M})$, the definition is valid and τ is an isomorphism of $C(w\mathcal{M})$ onto $C(w\mathcal{L})$. Theorem IV. 6.26 of [2] implies the existence of a homeomorphism σ of $w\mathcal{L}$ onto $w\mathcal{M}$, and from an examination of the proof of that theorem one sees that σ can be chosen to satisfy $\tau g^\wedge(\mathcal{U}) = g^\wedge(\sigma\mathcal{U})$ for each $g^\wedge \in C(w\mathcal{M})$ and $\mathcal{U} \in w\mathcal{L}$. If $g^\wedge \in C(w\mathcal{M})$, then $\psi^* g^\wedge = \varphi^*\varphi^{*-1}\psi^* g^\wedge = \varphi^*(\tau g^\wedge)$. We fix $x \in X$. Then for each $g^\wedge \in C(w\mathfrak{M})$ we have $g^\wedge(\psi x) = \psi^* g^\wedge(x) = \varphi^*(\tau g^\wedge)(x) = \tau g^\wedge(\varphi x) = g^\wedge(\sigma\varphi x)$. Since $C(w\mathcal{M})$ separates the points of $w\mathcal{M}$, we must have $\sigma\varphi = \psi$. Hence, $\mathcal{L}$ and $\mathfrak{M}$ are X-equivalent.

EXAMPLE 3.1. Let X be a completely regular Hausdorff, non-normal space. Let $\mathcal{L}$ be the lattice of zero-sets of all continuous real-valued functions on X and $\mathcal{M}$ the lattice of all closed subsets of X. Then $\mathcal{L} \subseteq \mathcal{M}$, $\mathcal{L}$ is a γ-lattice on X and $\mathcal{M}$ is an α-, β-lattice. $\mathcal{M}$ is not a normal lattice, since X is not a normal space. It is well known that $w\mathcal{L}$ is βX, the Stone-Čech compactification of X and $w\mathcal{M}$ is the "classical" Wallman compactification given in [9]. Since $C(X, \mathcal{L})$ is the algebra of all functions on X extendible to $w\mathcal{L}$ $(= \beta X)$, we have $BC(X) \subseteq C(X, \mathcal{L}) \subseteq C(X, \mathcal{M})$, where $BC(X)$ is the algebra of all bounded continuous real-valued functions on X. Since $\mathcal{M}$ is the lattice of all closed subsets of X, it is easily verified that $C(X, \mathcal{M}) \subseteq BC(X)$. Thus, $C(X, \mathcal{L}) = C(X, \mathcal{M})$, but $\mathcal{L}$ and $\mathcal{M}$ are not X-equivalent.

We note that Frink ([4], p. 606) has given conditions for the extendibility to $w\mathcal{L}$ of a function on X to a compact Hausdorff space when $\mathcal{L}$ is a normal base (his terminology, see remarks after Theorem 2.7 above). These conditions are essentially the same for α-lattices and the consideration of such functions leads to a theorem analogous to Theorem 3.2 and an example similar to Example 3.1.

4. Sufficient conditions.

In this section we give a partial solution to the problem of determining whether every compactification of a topological space is a space of ultrafilters. Unfortunately, the conditions we have obtained thus far are fairly restrictive, and for this reason the results of Section 5 are much more satisfactory, though the characterization given there is not exactly that which is desired.

There are several applications of lattice compactifications to topological algebras in which one does not have a homeomorphic embedding of X in the compact space T. We therefore give first a reduction theorem.

DEFINITION 4.1. If X is a topological space, a *Hausdorf compactification* of X (resp., a *generalized Hausdorff compactification* of X) is a pair (T, σ), where T is a compact Hausdorff space and σ is a homeomorphic (resp., continuous) mapping of X onto a dense subspace of T.

DEFINITION 4.2. *If X is a topological space and* (T, σ) is a Hausdorff compactification of X), we shall say that (T, σ) is a *Wallman compactification of X* provided there exists an α-lattice $\mathcal{L}$ on X and a homeomorphism τ of $w\mathcal{L}$ onto T such that $\tau\varphi = \sigma$, where φ is the natural embedding of X into $w\mathcal{L}$.

QUESTION 4.1. *Is it true that if X is a completely regular Hausdorff space, then every Hausdorff compactification of X is a Wallman compactification of X?*

QUESTION 4.2. *Is it true that if X a topological space, then every generalized Hausdorff compactification of X is a Wallman compactification of X?*

THEOREM 4.1. *An affirmative answer to Question 4.1 implies an affirmative answer to Question 4.2, and conversely.*

Proof. It is clear that an affirmative answer to Question 4.2 implies an affirmative answer to Question 4.1. To prove the converse we fix a generalized Hausdorff compactification (T, σ) of X. Then $\sigma(X)$ is a dense subspace of T and (T, i) is a Hausdorff compactification of the completely regular Hausdorff space $\sigma(X)$, where i is the identity map on $\sigma(X)$. Assuming an affirmative answer to Question 4.1, we have the existence of an α-lattice $\mathcal{L}$ on $\sigma(X)$ and a homeomorphism τ of $w\mathcal{L}$ onto T such that $\tau\varphi = i$. We let $\mathcal{M} = \{B \subseteq X\colon$ there exists $A \in \mathcal{L}$ such that $B = \sigma^{-1}(A)\}$. It is easily verified that (a) $\mathcal{M}$ is an α-lattice on X, (b) the family $\varrho(\mathcal{U}) = \{A \in \mathcal{L}\colon \sigma^{-1}(A) \in \mathcal{U}\}$ is an ultrafilter in $\mathcal{L}$ for each ultrafilter $\mathcal{U}$ in $\mathcal{M}$, (c) the function $\varrho\colon w\mathcal{M} \to w\mathcal{L}$ defined in (b) is one-to-one, onto, and satisfies $\varrho\psi = \varphi\sigma$, where ψ is the natural mapping of X into $w\mathcal{M}$. Moreover, ϱ is continuous, hence a homeomorphism. This follows from an examination of the basic neighborhoods in $w\mathcal{L}$ and $w\mathcal{M}$ and the definition of ϱ. We therefore have the following situation:

(4.1)
$$\begin{tikzcd}
T & & w\mathcal{L} \arrow[ll, "\tau"'] & & w\mathcal{M} \arrow[ll, "\varrho"'] \\
& \sigma(X) \arrow[ul, "i"] \arrow[ur, "\varphi"'] & & & \\
& & X \arrow[ul, "\sigma"] \arrow[uurr, "\psi"'] & &
\end{tikzcd}$$

The functions τ, φ, i and ϱ are homeomorphisms; σ and ψ are continuous; and $i = \tau\varphi$, $\varphi\sigma = \varrho\psi$. We let $\eta = \tau\varrho$. Then η is a homeomorphism of $w\mathcal{M}$ onto T and $\eta\psi = \sigma$.

For the remainder of the paper we shall consider Question 4.1 and shall, therefore, assume that the space X is completely regular and if (T, σ) is a Hausdorff compactification of X we shall for notational convenience identify X and $\sigma(X)$ and consider X to be a dense subspace of T. We shall use "*" to denote closure in T. Finally, if $\mathcal{L}$ is an α-lattice on X we shall write "$T = w\mathcal{L}$" in lieu of the more cumbersome statement "there

exists a homeomorphism τ of $w\mathcal{L}$ onto T such that $\tau\varphi(x) = x$ for each $x \in X$, where φ is the natural mapping of X into $w\mathcal{L}$".

THEOREM 4.2 (1). *Suppose that X is a completely regular Hausdorff space, T is a Hausdorff compactification of X, and $\mathcal{L}$ is a lattice of closed subsets of X. A necessary and sufficient condition in order that $T = w\mathcal{L}$ is that $\mathcal{L}$ satisfy:*

(i) (Star) *If $A, B \in \mathcal{L}$, then $(A \cap B)^* = A^* \cap B^*$.*

(ii) (α^*) *If $A \in \mathcal{L}$ and $t \in T - A^*$, then there exists $B \in \mathcal{L}$ such that $t \in B^*$ and $A \cap B = \emptyset$.*

(iii) (β^*) *If t_1 and t_2 are distinct points of T, then there exists $A \in \mathcal{L}$ such that $t_1 \in A^*$ and $t_2 \in T - A^*$.*

(iv) (H) *If t_1 and t_2 are distinct points of T, then there exist $A, B \in \mathcal{L}$ such that $t_1 \in T - A^*$, $t_2 \in T - B^*$, and $A \cup B = X$ (or, equivalently, $(T - A^*) \cap (T - B^*) = \emptyset$).*

Proof. We note first that (β^*) is implied by (H), but leave it separate, since the full strength of (H) is not needed until the last statement of the proof of the sufficiency. We prove the sufficiency of conditions (i)-(iv). We note that (α^*) and (β^*) imply (α) and (β), respectively, so $(w\mathcal{L}, \varphi)$ is a (generalized) compactification of X and φ is one-to-one. If $\mathcal{U}$ is an ultrafilter in $\mathcal{L}$, then $\{A^*\colon A \in \mathcal{U}\}$ is a descending family of closed subsets of T and has non-empty intersection. It is readily verified using (β^*) and (Star) that this intersection consists of a single point. We define $\sigma(\mathcal{U})$ to be that point. If $t \in T$, then the family $\mathcal{U} = \{A \in \mathcal{L}\colon t \in A^*\}$ is an ultrafilter in $\mathcal{L}$, and $\sigma(\mathcal{U}) = t$. Thus, σ is one-to-one, onto, and $\sigma\varphi(x) = x$ for each $x \in X$. In order to show that σ is homeomorphism we note that $\sigma[C(A)] = A^*$. Thus, σ induces on T a second topology $\mathcal{S}$ with base for the $\mathcal{S}$-closed sets $\{A^*\colon A \in \mathcal{L}\}$. It is clear that $\mathcal{S}$ is coarser than the given topology $\mathcal{T}$ on T, and that σ is a homeomorphism with respect to the $\mathcal{S}$-topology on T. Property (H) is just the statement that the $\mathcal{S}$-topology is Hausdorff. Thus, $\mathcal{S} = \mathcal{T}$ and the sufficiency of (i)-(iv) is proved.

To prove the necessity we note that if σ is the homeomorphism of $w\mathcal{L}$ onto T such that $\sigma\varphi(x) = x$ for all $x \in X$, then $\sigma[C(A)] = A^*$ for each $A \in \mathcal{L}$, and $\mathcal{L}$ is a γ-lattice of closed subsets of X which forms a base for the closed sets of X. It is then merely a routine matter to verify conditions (i)-(iv).

In the following $\mathfrak{A}$ will denote the algebra of all real-valued continuous functions on X which have continuous extensions to T. If $f \in \mathfrak{A}$, we shall denote its (unique) extension to T by f' and shall in general use a prime to denote elements of $C(T)$ or of $C(S)$ for any space S homeo-

(1) The author is indebted to G. Keller of the University of Minnesota for suggesting the possibility and usefulness of such a criterion.

morphic to T. We note that $\mathfrak{A}$ is a uniformly closed subalgebra of $BC(X)$, contains the constant functions, and separates points from closed sets of X (i.e., if A is a closed subset of X and if $x \in X-A$, then there exists $f \in \mathfrak{A}$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in A$), and is isomorphic (and isometric) to $C(T)$. $\mathfrak{A}_0$ will be the family of all functions f in $\mathfrak{A}$ which satisfy the condition: for each $t \in T-X$ there exists a neighborhood U of t such that f' is constant on U. $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$ and each f in $\mathfrak{A}_0$ has the property that if $t \in T-X$, then f' is constant on the component of t in $T-X$. This follows from the following elementary characterization of connectivity. A space S is connected if and only if for each pair s, t of points of S and each open cover $\mathcal{G}$ of S there exists a finite subfamily $\{G_1, \ldots, G_n\}$ of $\mathcal{G}$ such that $s \in G_1$, $t \in G_n$ and $G_i \cap G_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. Finally, $\mathfrak{L}_0$ is the collection of all zero-sets of elements of $\mathfrak{A}_0$.

THEOREM 4.3. *If X is a locally compact Hausdorff space, T a Hausdorff compactification of X, and $\mathfrak{A}$, $\mathfrak{A}_0$, and $\mathfrak{L}_0$ as defined above; then $T = w\mathfrak{L}_0$ if and only if $T-X$ is totally disconnected.*

Proof. We prove first the sufficiency of the condition on $T-X$, and will verify (i)-(iv) of Theorem 4.2.

(i): If $A, B \in \mathfrak{L}_0$ then $A = Z(f)$ and $B = Z(g)$ for some pair f, g of elements of $\mathfrak{A}_0$. Let $t \in A^* \cap B^*$. If $t \in X$, then $t \in A \cap B \subseteq (A \cap B)^*$. If $t \in T-X$, then $f'(t) = g'(t) = 0$ and there exists a neighborhood U of t on which f' and g' vanish. If V is any open set containing t, then $U \cap V$ is also and there exists $x \in X \cap U \cap V$. But then $f(x) = g(x) = 0$ and $x \in V \cap Z(f) \cap Z(g)$. Hence, $t \in (A \cap B)^*$.

(ii) If $A \in \mathfrak{L}_0$ and $t \in T-A^*$, then for some $f \in \mathfrak{A}_0$, $A = Z(f)$ and $f'(t) \neq 0$. If we let $g = f - f'(t)$, then $g \in \mathfrak{A}_0$, $t \in Z(g)^*$, and $Z(g) \cap Z(f) = \emptyset$. To finish the proof it suffices to verify (iv). There are three cases to consider: (a) t_1 and t_2 are in X, (b) $t_1 \in X$, $t_2 \in T-X$, (c) t_1 and t_2 are in $T-X$. If (a) both t_1 and t_2 are elements of X, we choose open (in T) sets U_1 and U_2 such that $t_1 \in U_1$, $\{t_2\} \cup (T-X) \subseteq U_2$, and $U_1^* \cap U_2^* = \emptyset$. (Since X is locally compact, $T-X$ is closed in T.) There exist $f_1, f_2 \in \mathfrak{A}$ such that $f_i(t_i) = 1$ and f_i is identically zero on $T-U_i$, $i = 1, 2$. Then f_1 and f_2 are in $\mathfrak{A}_0$, $t_i \in T-Z(f_i)^*$, $i = 1, 2$, and $Z(f_1) \cup Z(f_2) = X$. If (b) $t_1 \in X$ and $t_2 \in T-X$, we choose U_1 and U_2 open (in T) sets such that $t_1 \in U_1$, $T-X \subseteq U_2$, and $U_1^* \cap U_2^* = \emptyset$, then proceed as in (a). If (c) both t_1 and t_2 are in $T-X$, then there exist subsets T_1 and T_2 of $T-X$ which are closed in $T-X$ and such that $t_i \in T_i$, $i = 1, 2$, $T_1 \cap T_2 = \emptyset$, and $T_1 \cup T_2 = T-X$ ($T-X$ is compact and totally disconnected). Since $T-X$ is closed in T, T_1 and T_2 are disjoint closed subsets of T and there exist open sets U_1 and U_2 such that $T_i \subseteq U_i$, $i = 1, 2$, and $U_1^* \cap U_2^* = \emptyset$. The remainder is as in (a).

The necessity follows from the observation made preceding the statement of this theorem. If $T-X$ is not totally disconnected, there exist non-trivial connected sets in $T-X$ and each f', $f \in \mathfrak{A}_0$, is constant on such sets, so $\mathfrak{L}_0$ fails to satisfy (β^*).

COROLLARY 4.3. *If X is a locally compact (non-compact) Hausdorff space, then the one-point compactification of X is a Wallman compactification of X.*

We note that the lattice $\mathfrak{L}_0$ in this case is the family of all zero sets of continuous functions on X which are constant on the complement of some compact subset of X (constant on a neighhborhood of the "point at infinity"), the result attributed to the author (without proof) by Frink ([4], p. 606). We note also that this result was obtained earlier by Ky Fan and Gottesman by means of their normal bases for open sets.

5. A representation theorem for Hausdorff compactifications. We consider in this section a fixed completely regular Hausdorff space X and a fixed Hausdorff compactification T of X. We shall use the same notation and terminology given after Theorem 4.1. It is well known that T is a quotient space of βX. However, βX need have little other relation to T. We first show that T is a quotient space of $w\mathfrak{L}$, where $\mathfrak{L}$ is the lattice of zero-sets of elements of $\mathfrak{A}$, the algebra of all continuous real-valued functions on X which are extendible to T.

It is readily verified that $\mathfrak{L}$ is an α-, β-lattice of closed subsets of X and that $\mathfrak{L}$ forms a base for the closed sets of X. Therefore, $\varphi(x \to \mathcal{U}_x)$ is a homeomorphism of X into $w\mathfrak{L}$. We have not been able to determine whether $w\mathfrak{L}$ is always Hausdorff, although all of our examples have this additional property.

THEOREM 5.1. $\mathfrak{A} \subseteq C(X, \mathfrak{L})$.

Proof. We use here the fact that $\mathfrak{A}$ is a lattice with respect to the operations $\max(f, g)$ and $\min(f, g)$. We denote by λ the function on X which maps X into the single point λ of $\mathbf{R}$. We fix $f \in \mathfrak{A}$, $\varepsilon > 0$ and let a and b be real numbers such that $a < \inf\{f(x)\colon x \in X\} \leqslant \sup\{f(x)\colon x \in X\} < b$, and choose a partition $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b$ of $[a, b]$ such that $\lambda_i - \lambda_{i-1} < \varepsilon/4$ for each i: $1 \leqslant i \leqslant n$. For each $i \leqslant n-2$ we define

$$g_i = [\lambda_{i+2} - \min(\lambda_{i+2}, f)][\max(\lambda_i, f) - \lambda_i]\,.$$

The functions g_i, $i = 0, 1, \ldots, n-2$, are in $\mathfrak{A}$ and

$$Z(g_i) = f^{-1}(-\infty, \lambda_i] \cup f^{-1}[\lambda_{i+2}, \infty) = X - f^{-1}(\lambda_i, \lambda_{i+2})\,.$$

It follows that

$$\bigcap \{Z(g_i)\colon i = 0, 1, \ldots, n-2\} = \emptyset \text{ and } \theta\big(f, X - Z(g_i)\big) < \varepsilon \text{ for each } i\,.$$

From Theorems 5.1 and 2.8 we have that each $f \in \mathfrak{A}$ is extendible to a continuous function $f\hat{}$ on $w\mathfrak{L}$ and $f\hat{}(\varphi x) = f(x)$ for each $x \in X$. We define a relation $\mathcal{R}$ on $w\mathfrak{L}$ by

$$(5.1)\qquad \mathcal{R} = \{(\mathcal{U}, \mathcal{V}) \in w\mathfrak{L} \times w\mathfrak{L}\colon f\hat{}(\mathcal{U}) = f\hat{}(\mathcal{V}) \text{ for each } f \in \mathfrak{A}\}.$$

It is clear that $\mathcal{R}$ is a closed relation on $w\mathfrak{L}$ and for each $x \in X$ the class $\mathcal{R}[\mathcal{U}_x]$ containing $\mathcal{U}_x$ contains only the single point $\mathcal{U}_x$. We denote by σ the projection map of $w\mathfrak{L}$ onto $w\mathfrak{L}/\mathcal{R}$ and endow $w\mathfrak{L}/\mathcal{R}$ with the quotient topology.

THEOREM 5.2. (i) *$w\mathfrak{L}/\mathcal{R}$ is a compact Hausdorff space*, (ii) *$C(w\mathfrak{L}/\mathcal{R})$ and $\mathfrak{A}\hat{}$ (the family of all extensions of functions in $\mathfrak{A}$ to $w\mathfrak{L}$) are isomorphic*, (iii) *T and $w\mathfrak{L}/\mathcal{R}$ are X-homeomorphic (there exists a homeomorphism τ of T onto $w\mathfrak{L}/\mathcal{R}$ such that $\tau(x) = \sigma\varphi(x)$ for each $x \in X$).*

Proof. (i) Since σ is continuous, $w\mathfrak{L}/\mathcal{R}$ is compact, and for each $f \in \mathfrak{A}$ the function f' defined on $w\mathfrak{L}/\mathcal{R}$ by $f'(\sigma\mathcal{U}) = f\hat{}(\mathcal{U})$ is continuous (cf. [7], p. 95). From the definition of $\mathcal{R}$ it is clear that the family of all such functions separates the points of $w\mathfrak{L}/\mathcal{R}$. Thus, this space is Hausdorff. Also, the family $\mathfrak{A}'$ of functions f', $f \in \mathfrak{A}$, is a uniformly closed subalgebra of $C(w\mathfrak{L}/\mathcal{R})$ which contains the constant functions. Thus, by the Stone-Weierstrass theorem, $\mathfrak{A}' = C(w\mathfrak{L}/\mathcal{R})$. The mapping $f\hat{} \to f'$ of $\mathfrak{A}\hat{}$ onto $\mathfrak{A}'$ is clearly an isomorphism, and the composition $f(\to f\hat{}) \to f'$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}' = C(w\mathfrak{L}/\mathcal{R})$. But $\mathfrak{A}$ is isomorphic to $C(T)$, and by Theorem IV, 6.26 of [2] T and $w\mathfrak{L}/\mathcal{R}$ are homeomorphic. Since $\mathcal{R}[\mathcal{U}_x] = \{\mathcal{U}_x\}$, it follows that there is an X-homeomorphism of T onto $w\mathfrak{L}/\mathcal{R}$.

We now define a mapping of the family of filters in $\mathfrak{L}$ into itself. In the special case $\mathfrak{A} = BC(X)$ this mapping was considered in [5], Exercise 2L. Most of the properties of this mapping which we exhibit were given by Samuel ([8], p. 121) for the case $\mathfrak{L} =$ the power set of X.

DEFINITION 5.1. For $f \in \mathfrak{A}$, $\varepsilon > 0$, $E_\varepsilon(f) = \{x\colon |f(x)| \leqslant \varepsilon\}$. $E_\varepsilon(f) = Z[\max(|f|, \varepsilon) - \varepsilon] \in \mathfrak{L}$.

DEFINITION 5.2. If $\mathcal{F} \subseteq \mathfrak{L}$, $\mathcal{F}^*$ is the family $\{A \in \mathfrak{L}\colon$ there exists $f \in \mathfrak{A}$, $\varepsilon > 0$ such that $A = E_\varepsilon(f)$ and $E_\delta(f) \in \mathcal{F}$ for each $\delta > 0\}$.

THEOREM 5.3. *For each filter $\mathcal{F}$ in $\mathfrak{L}$, the family $\mathcal{F}^*$ is a filter in $\mathfrak{L}$ satisfying $\mathcal{F}^* \subseteq \mathcal{F}$.*

Proof. We first observe that we can (and shall) always choose non-negative functions to represent sets $A \in \mathfrak{L}$ in the form $E_\varepsilon(f)$, since $E_\varepsilon(f) = E_\varepsilon(|f|)$. We shall verify (i) $\emptyset \notin \mathcal{F}^*$, (ii) if $A, B \in \mathcal{F}^*$, then there exists $C \in \mathcal{F}^*$ such that $C \subseteq A \cap B$, (iii) if $A \in \mathcal{F}^*$, $B \in \mathfrak{L}$, and $A \subseteq B$, then $B \in \mathcal{F}^*$. It is clear that $\mathcal{F}^* \subseteq \mathcal{F}$ and (i) is immediate.

(ii): If $A, B \in \mathcal{F}^*$, then there exist $f, g \in \mathfrak{A}$, $\varepsilon, \delta > 0$ such that $A = E_\varepsilon(f)$, $B = E_\delta(g)$, and $E_\eta(f)$, $E_\eta(g) \in \mathcal{F}$ for each $\eta > 0$. $A \cap B \supseteq E_{\min(\varepsilon,\delta)}(f) \cap E_{\min(\varepsilon,\delta)}(g) \supseteq E_{\min(\varepsilon,\delta)}(f+g)$. We let $C = E_{\min(\varepsilon,\delta)}(f+g)$, and show $C \in \mathcal{F}^*$.

For $\eta > 0$ $E_\eta(f+g) \supseteq E_\gamma(f) \cap E_\gamma(g)$, where $\gamma = \eta/2$, and each of these sets is an element of $\mathcal{F}$. It follows that $C \in \mathcal{F}^*$.

(iii) We first note that if $f, h \in \mathfrak{A}$ and if f has the property that $E_\delta(f) \in \mathcal{F}$ for each $\delta > 0$, then so does fh. This follows from the containment $E_\eta(f) \subseteq E_\delta(fh)$, where $\eta = \delta/\sup\{|h(x)|\colon x \in X\}$. We fix $A \in \mathcal{F}^*$, $B \in \mathfrak{L}$ such that $A \subseteq B$. There exist $f, g \in \mathfrak{A}$, $\varepsilon > 0$ such that $A = E_\varepsilon(f)$, $B = Z(g)$, and $E_\delta(f) \in \mathcal{F}$ for all $\delta > 0$. We shall construct $h \in \mathfrak{A}$ such that $Z(g) = E_\varepsilon(fh)$. Since $E_\varepsilon(f) \subseteq Z(g)$, we have also $E_\varepsilon(f)^* \subseteq Z(g)^*$. We let $T_1 = E_\varepsilon(f)^*$, $T_2 = [T - E_\varepsilon(f)^*]^*$. We define h_1' on T_1 by $h_1'(t) = 1$ for all $t \in T_1$, and h_2' on T_2 by $h_2'(t) = g'(t) + \varepsilon f'(t)^{-1}$ for each $t \in T_2$. The first function is clearly continuous, and the continuity of the second function will follow once we establish that f' cannot vanish on T_2. In fact, if $f'(t) < \varepsilon$, then $t \in \operatorname{int}(E_\varepsilon(f)^*)$. It is clear that T_2 does not meet the interior of $E_\varepsilon(f)^*$, since $T_1 \cap T_2$ is precisely the boundary of T_1. We define $h'(t)$ to be $h_1'(t)$ if $t \in T_1$ and to be $h_2'(t)$ if $t \in T_2$. It suffices to show that h_1' and h_2' agree on $T_1 \cap T_2$. If $t \in T_1 \cap T_2$, $t \in \operatorname{Bdy}(E_\varepsilon(f)^*)$ and $f'(t) = \varepsilon$, $g'(t) = 0$. It follows that $h_1'(t) = h_2'(t)$ and h' is continuous on T. We let h be the restriction of h' to X. Then $h \in \mathfrak{A}$ and $Z(g) = E_\varepsilon(fh)$.

THEOREM 5.4. *The mapping $\mathcal{F} \to \mathcal{F}^*$ satisfies* (i) $\mathcal{F}^* \subseteq \mathcal{F}$, (ii) $\mathcal{F}^{**} = \mathcal{F}^*$, (iii) *if $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F}^* \subseteq \mathcal{G}^*$*, (iv) *if $\mathcal{U}$ is an ultrafilter in $\mathfrak{L}$ and if $\mathcal{U}^* \subseteq \mathcal{F}$, then $\mathcal{F}^* \subseteq \mathcal{U}$*, (v) *if $x \in X$, then $\mathcal{U}_x^*$ is contained in a unique ultrafilter $(\mathcal{U}_x)$ of $\mathfrak{L}$.*

Proof. (i), (ii), and (iii) are clear. (iv). If $\mathcal{U}$ is an ultrafilter in $\mathfrak{L}$ and $\mathcal{F}$ a filter in $\mathfrak{L}$ such that $\mathcal{F}^* \not\subseteq \mathcal{U}$, then $\mathcal{U}^* \not\subseteq \mathcal{F}$. If $\mathcal{F}^* \not\subseteq \mathcal{U}$, then there exist $f, g \in \mathfrak{A}$, $\varepsilon > 0$ such that $E_\delta(f) \in \mathcal{F}$ for each $\delta > 0$ and $E_\varepsilon(f) \cap Z(g) = \emptyset$. We fix $\delta\colon 0 < \delta < \varepsilon$. Then $E_\delta(f') \cap Z(g)^* = \emptyset$. If not, then there exists $t \in Z(g)^*$ such that $f'(t) \leqslant \delta$. We choose $\eta > 0$ such that $\delta + \eta < \varepsilon$. There exists a neighborhood U of t such that $f'(s) < \delta + \eta$ for each $s \in U$. Since $t \in Z(g)^*$, there exists $x \in Z(g) \cap U$, and $f(x) < \varepsilon$, $g(x) = 0$. This contradicts the fact that f, g, and ε where chosen so that $E_\varepsilon(f) \cap Z(g) = \emptyset$. Since $E_\delta(f') \cap Z(g)^* = \emptyset$ there exists $h \in \mathfrak{A}$ such that h' is zero on $Z(g)^*$, h' is one on $E_\delta(f')$, and $0 \leqslant h' \leqslant 1$. Then $Z(g) \subseteq Z(h)$ and $E_\eta(h) \in \mathcal{U}^*$ for each $\eta > 0$. We fix $\eta < 1$. Then $E_\eta(h) \cap E_\delta(f) = \emptyset$ and $\mathcal{U}^* \not\subseteq \mathcal{F}$.

(v). We fix $x \in X$ and $\mathcal{U} \in w\mathfrak{L}$ such that $\mathcal{U} \neq \mathcal{U}_x$. Then there exists $A \in \mathcal{U}$ such that $x \in X - A$. There exists $f \in \mathfrak{A}$ such that $A = Z(f)$, $f \geqslant 0$, and $f(x) = 1$. The set $B = E_{1/2}(f)$ is in $\mathcal{U}^*$ and $x \notin B$. Thus, $\mathcal{U}^* \not\subseteq \mathcal{U}_x$, and $\mathcal{U}_x^* \not\subseteq \mathcal{U}$ by (iv).

DEFINITION 5.3. A filter $\mathcal{F}$ in $\mathfrak{L}$ is called a *$*$-filter* provided $\mathcal{F} = \mathcal{F}^*$.

THEOREM 5.5. *If $\mathcal{U}$ is an ultrafilter in $\mathfrak{L}$, then $\mathcal{U}^*$ is a maximal $*$-filter. Conversely, if $\mathcal{F}$ is a maximal $*$-filter, then $\mathcal{F} = \mathcal{U}^*$ for some ultrafilter $\mathcal{U}$ in $\mathfrak{L}$.*

Proof. If $\mathcal{U}$ is an ultrafilter in $\mathfrak{L}$ and $\mathcal{U}^* \subseteq \mathcal{F} = \mathcal{F}^*$, then $\mathcal{F}^* \subseteq \mathcal{U}$ (Theorem 5.4 (iv)), and $\mathcal{F}^* = \mathcal{F} \subseteq \mathcal{U}^*$ (Theorem 5.4 (iii)). Conversely, if $\mathcal{F}$ is a maximal *-filter, then since $\mathcal{F}$ is a filter in $\mathfrak{L}$, there exists an ultrafilter $\mathcal{U}$ in $\mathfrak{L}$ such that $\mathcal{F} \subseteq \mathcal{U}$. By (iii) of Theorem 5.4, $\mathcal{F}^* \subseteq \mathcal{U}^*$.

We let X^* $(=(X, \mathfrak{L})^*$ if reference to $\mathfrak{L}$ is necessary) denote the set of all maximal *-filters in $\mathfrak{L}$, and define $\pi\colon w\mathfrak{L} \to X^*$ by $\pi(\mathcal{U}) = \mathcal{U}^*$. The function π maps $w\mathfrak{L}$ onto X^*. We endow X^* with the quotient topology induced by π.

Theorem 5.6. *If X^* is endowed with the quotient topology induced by π, then the mapping $\psi = \pi\varphi$ of X into X^* is a homeomorphism of X onto a dense subspace of X^*. Moreover, X^* and T are X-homeomorphic.*

Proof. Since φ is one-to-one on X and π is one-to-one on $\varphi(X)$ (Theorem 5.4 (v)), it follows that ψ is one-to-one. The continuity of ψ and denseness of $\psi(X)$ are clear. We defer the closedness of this map. We have $\sigma\colon w\mathfrak{L} \to T$ a continuous closed map and $\pi\colon w\mathfrak{L} \to X^*$ continuous. Both T and X^* are endowed with the respective quotient topologies, so it suffices to show that the decompositions of $w\mathfrak{L}$ induced by σ and π are the same (i.e., $\sigma(\mathcal{U}) = \sigma(\mathcal{V})$ if and only if $\pi(\mathcal{U}) = \pi(\mathcal{V})$). If $\sigma(\mathcal{U}) \neq \sigma(\mathcal{V})$, then there exists $f \in \mathfrak{A}$ such that $f^\wedge(\mathcal{U}) \neq f^\wedge(\mathcal{V})$ and we may assume that $f^\wedge(\mathcal{U}) = 0$, $f\ (\mathcal{V}) = 1$, and $0 \leqslant f^\wedge \leqslant 1$. Then $\mathcal{U} \in Z(f^\wedge) = \bigcap \{E_\varepsilon(f)^-\colon \varepsilon > 0\}$, where "$^-$" denotes closure in $w\mathfrak{L}$. For each $\varepsilon > 0$, $E_\varepsilon(f) \in \mathcal{U}^*$ and for $\varepsilon < 1$, $\mathcal{V} \notin E_\varepsilon(f)^- = C[E_\varepsilon(f)]$. Hence, $E_\varepsilon(f) \notin \mathcal{V}$ and $\mathcal{U}^* \not\subseteq \mathcal{V}$. This implies $\mathcal{U}^* \neq \mathcal{V}^*$ $(\pi(\mathcal{U}) \neq \pi(\mathcal{V}))$. Conversely, if $\mathcal{U}^* \neq \mathcal{V}^*$, there exist $A \in \mathcal{U}^*$, $B \in \mathcal{V}$ such that $A \cap B = \emptyset$, and there exist $f, g \in \mathfrak{A}$, $\varepsilon > 0$ such that $A = E_\varepsilon(f)$, $B = Z(g)$ and $f^\wedge(\mathcal{U}) = 0$. If $f^\wedge(\mathcal{V}) = 0$, then $\mathcal{V} \in E_\delta(f)^-$ for each $\delta > 0$. But $\mathcal{V} \in Z(g)^-$ and this set is disjoint from $E_\varepsilon(f)^-$. Thus, $f^\wedge(\mathcal{U}) \neq f^\wedge(\mathcal{V})$ and $\sigma(\mathcal{U}) \neq \sigma(\mathcal{V})$. A homeomorphism τ is induced from X^* to T by $\tau(\pi\mathcal{U}) = \sigma\mathcal{U}$. For $x \in X$, $\tau\pi\mathcal{U}_x = \sigma\mathcal{U}_x = x$ and τ is, therefore, an X-homeomorphism. It now follows that ψ is a homeomorphism of X into X^*.

The preceding would be a more satisfying result if the topology of the space X^* were defined as in an ultrafilter space. We recall (Theorem 2.4) that the topology in an ultrafilter space $w\mathfrak{L}$ is given by the following: For each filter $\mathcal{F}$ in $\mathfrak{L}$ the set $C(\mathcal{F}) = \{\mathcal{U} \in w\mathfrak{L}\colon \mathcal{F} \subseteq \mathcal{U}\}$ is closed, and every closed set is of this form. We show below that the topology of X^* can be described in the fashion.

Lemma 5.7.1. *If F is a closed set in $w\mathfrak{L}$ and if $F = \{\mathcal{U} \in w\mathfrak{L}\colon \mathcal{F} \subseteq \mathcal{U}\}$ for some filter $\mathcal{F}$ in $\mathfrak{L}$, then $\pi^{-1}\pi F = \{\mathcal{U} \in w\mathfrak{L}\colon \mathcal{F}^* \subseteq \mathcal{U}\}$.*

Proof. It is clear that $\pi^{-1}\pi F \subseteq \{\mathcal{U} \in w\mathfrak{L}\colon \mathcal{F}^* \subseteq \mathcal{U}\}$. If $\mathcal{U} \in w\mathfrak{L} - \pi^{-1}\pi F$, then $\mathcal{U}^* \in X^* - \pi F$ and πF is closed in X^*. There exists $f \in \mathfrak{A}$, $0 \leqslant f \leqslant 1$, such that f' is identically zero on πF and $f'(\mathcal{U}^*) = 1$. Then $f^\wedge$ is zero on $\pi^{-1}\pi F$ and $f^\wedge(\mathcal{U}) = 1$. Thus, $F \subseteq Z(f^\wedge) \subseteq E_\varepsilon(f)^-$ for each $\varepsilon > 0$, and $E_\varepsilon(f) \in \mathcal{F}$ for all such ε. This implies that each $E_\varepsilon(f) \in \mathcal{F}^*$ and for $\varepsilon < 1$ we have $E_\varepsilon(f) \notin \mathcal{U}$. Hence, $\mathcal{F}^* \subseteq \mathcal{U}$ and $\mathcal{U} \in w\mathfrak{L} - \{\mathcal{V} \in w\mathfrak{L}\colon \mathcal{F}^* \subseteq \mathcal{V}\}$.

Lemma 5.7.2. *The quotient toplogy in X^* can be described as follows: For each *-filter $\mathcal{F}$ in $\mathfrak{L}$ the set $C^*(\mathcal{F}) = \{\mathcal{U}^* \in X^*\colon \mathcal{F} \subseteq \mathcal{U}^*\}$ is closed, and every closed subset of X^* is of this form.*

Proof. If $\mathcal{F}$ is a *-filter in $\mathfrak{L}$, then $C(\mathcal{F})$ is closed in $w\mathfrak{L}$. It is easily verified that $C(\mathcal{F}) = \pi^{-1}C^*(\mathcal{F})$. Hence, $C^*(\mathcal{F})$ is closed in X^*. If F is closed in X^*, then $\pi^{-1}(F)$ is closed in $w\mathfrak{L}$ and there exists a filter $\mathcal{F}$ in $\mathfrak{L}$ such that $\pi^{-1}(F) = C(\mathcal{F})$. By Lemma 5.7.1, $\pi^{-1}\pi(\pi^{-1}F) = C(\mathcal{F}^*)$. But $\pi^{-1}F = \pi^{-1}\pi(\pi^{-1}F)$, so $F = \pi\pi^{-1}F = C^*(\mathcal{F}^*)$.

We now collect these facts, proved above, as

Theorem 5.7. *If X is a completely regular Hausdorff space and T is a Hausdorff compactification of X, then* (i) *T is a quotient space of $w\mathfrak{L}$, where $\mathfrak{L}$ is the lattice of zero sets of the algebra of all real-valued continuous functions on X extendible to T,* (ii) *T is X-homeomorphic to X^*, the space of all maximal *-filters in $\mathfrak{L}$, endowed with the topology $\mathcal{T}^*$ defined by taking as closed sets all sets on the form $\{\mathcal{U}^* \in X^*\colon \mathcal{F} \subseteq \mathcal{U}^*\}$, where $\mathcal{F}$ is a *-filter in $\mathfrak{L}$,* (iii) *The topology $\mathcal{T}^*$ is the quotient topology induced on X^* by the map from $w\mathfrak{L}$ which identifies ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U}^* = \mathcal{V}^*$.*

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UNIVERSITY OF MINNESOTA
Minneapolis, Minnesota

Reçu par la Rédaction le 8. 3. 1966