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On Baire measures on D-topological spaces

by

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I. Introduction. Let X be a topological space and d a pseudometric on X. A subset $X_0 \subset X$ is d-discrete if there is some $c > 0$ such that $d(x, y) \geqslant c$ for any $x, y \in X_0$ with $x \neq y$. ([14], Ch. 15.)

A cardinal number $\mathfrak{m}$ is said to have *measure zero* if any nonnegative, countably additive, finite set function μ on the class of all subsets of a set M of cardinality $\mathfrak{m}$ is trivial, i.e. $\mu(M) = \mu(M_0)$ for some countable subset $M_0 \subset M$. $\aleph_0$ and $\aleph_1$ have measure zero (see G. Birkhoff [2], p. 187), while the cardinal of the continuum is not known to have measure zero. We say that a topological space X is a *D-space* if whenever d is a continuous pseudometric on X (see Gillman-Jerison [4], p. 234, 15 G(2)), all d-discrete subsets of X have cardinal of measure zero.

Any separable space X or any product $\prod_{\alpha \in I} X_\alpha$, with X_α separable and I not necessarily countable (which does not need to be separable if the cardinality of I exceeds that of the continuum; see Pondiczery [7] or Ross-Stone [9], p. 399) or any open subset of such a product or any Lindelöff regular space are a few examples of D-spaces. (In fact all these are spaces which satisfy the countable chain condition; see Sanin [10] or [9], p. 400.) If X is a D-space, then its closure in any space Y in which X can be embedded is also a D-space.

On the other hand even R_d (reals with discrete topology) is not known (assuming the usual Zermelo-Fraenkel axioms and the axiom of choice) to be a D-space.

The assumption that all topological spaces are D-spaces is equivalent to the assumption that all cardinals have measure zero (whether or not this is true is a well-known open problem of set theory) which is consistent with the usual axioms of set theory (see the remarks after the proof of theorem 1).

If X is a topological space, then $M_\sigma(X)$ (sometimes M_σ) will denote the linear space of all countably additive real valued (hence finite) set functions defined on the Baire sets of X (see below), $M_\sigma^+(X) = \{\mu \in M_\sigma(X);\ \mu(A) \geqslant 0$ for all Baire sets $A \subset X\}$ and $C(X)$ the space of bounded real continuous functions on X.

We should remark here that any $\mu \in M_\sigma^+(X)$ is regular with respect to zero sets in X (for definition see below), i.e. $\mu(A) = \sup\{\mu(Z);\, Z \subset A$ and Z a zero set$\}$ (see Varadarajan [13], theorem 18, p. 45) and is not necessarily regular with respect to compact Baire subsets of X even if X is locally compact non σ-compact (such X may even have cardinality $\aleph_1$).

If Y is completely regular (always Hausdorff), then $A \subset Y$ is conditionally compact iff $\bar{A}$, its closure in Y, is compact, i.e. iff any net in A contains a subnet which converges to some point of Y (see Kelley [19], Chapter 2, for nets).

$\sigma\big(C(X), M_\sigma(X)\big) = \sigma(C, M_\sigma)$ will denote the weakest topology on $C(X)$ which makes all linear functionals on $C(X)$ of type $\int f\,d\mu$ for μ in M_σ, continuous. A set $\mathcal{A} \subset C(X)$ is bounded iff $\sup\{|f(x)|;\, x \in X, f \in \mathcal{A}\} < \infty$.

It is the main purpose of this paper to prove the following two theorems about completely regular D-spaces which have in common the following: The assumption that they hold for all completely regular spaces X is *equivalent* to the assumption that all cardinals have measure zero.

THEOREM 1. *Let X be completely regular.*

(a) *If X is a D-space and $\mathcal{A} \subset C(X)$ is bounded and equicontinuous, then $\mathcal{A}$ is $\sigma(C, M_\sigma)$-conditionally compact.*

(b) *If X is not a D-space, then there is a bounded equicontinuous set $\mathcal{A} \subset C(X)$ which is not $\sigma(C, M_\sigma)$-conditionally compact.*

If X is σ-compact locally compact or complete separable metric, then more than part (a) of this theorem is known to be true. Theorem 1(a) asserts something new for those spaces X which admit Baire measures (i.e. elements of $M_\sigma^+(X)$) which are not supported by σ-compact sets. There are even locally compact spaces of cardinality $\aleph_1$ or subspaces of the real line which have this property (see the remarks after the proof of theorem 1).

THEOREM 2. *Let X be completely regular.*

(a) *If X is a D-space and μ_a a net in $M_\sigma^+(X)$ such that $\lim_a \int f\,d\mu_a = \int f\,d\mu$ for each f in $C(X)$ for some μ in $M_\sigma^+(X)$, then $\lim_a \sup_{f \in \mathcal{A}} \left|\int f\,d(\mu_a - \mu)\right| = 0$ for any bounded equicontinuous set $\mathcal{A} \subset C(X)$.*

(b) *If X is not a D-space, then there exists some μ in $M_\sigma^+(X)$, a net μ_a in $M_\sigma^+(X)$ and a bounded equicontinuous $\mathcal{A} \subset C(X)$ such that $\lim_a \int f\,d\mu_a = \int f\,d\mu$ for each f in $C(X)$ and this convergence is not uniform on $\mathcal{A}$.*

Theorem 2(a) has been proved by Ranga R. Rao [8] for the case where the net μ_a is a *sequence* μ_n, $n = 1, 2, \ldots$, and X is Lindelöff. It has been known by R. M. Dudley that theorem 2(a) is true for *sequences* μ_n and separable completely regular spaces X and that assuming it to be true for *sequences* $\mu_n \in M_\sigma^+(X)$ and all completely regular spaces X is "compatible with all axioms of set theory" (see [8], p. 664 footnotes 1, 2). The fact that theorem 2(a) is true for nets has a topological interpretation. We make use of theorem 2(a) as stated for *nets* in the proof of theorem 1(a) and in the applications in Section IV. The fact that theorem 2(a) is true for sequences is not enough in *each* of the applications given. It is also shown in the present work that theorem 2(a) is true for any completely regular X whenever the limit measure μ is supported by a subset $X_0 \subset X$ which is a D-space. This is the content of remark III.4. This remark is applied in Section IV.

One of the deepest results obtained by Varadarajan in [13] is that if (X, d) is metric and $M_\tau^+(X)$ is the set of $\mu \in M_\sigma^+(X)$ which "live" on closed separable subsets of X, then $M_\tau^+(X)$ equipped with $\sigma\big(M_\sigma(X), C(X)\big)$ is metrisable. We apply theorem 2 and give a much simpler and entirely different (see [13], pp. 62-65) proof to this theorem. In the spirit of theorems 1, 2 one gets from here the following, essentially known (in different terminology).

COROLLARY. *Let (X, d) be metric.*

(a) *If (X, d) is a D-space, then $M_\sigma^+(X)$ with $\sigma(M_\sigma, C)$ is metrisable.*

(b) *If (X, d) is not a D-space, then $M_\sigma^+(X)$ with $\sigma(M_\sigma, C)$ is not metrisable* (see Section IV.)

We also give in Section IV the following application to invariant means on topological groups G (not necessarily locally compact). Let $DP_\sigma(G) = \{\mu \in M_\sigma^+(G);\, \mu(G) = 1$ and μ "lives" on a D-subset of $G\}$ (see remark III.6 and Section IV). If $\mu \in M_\sigma(G)$ and $f \in C(G)$ let

$$(L_\mu f)(g) = \int f(hg)\,d\mu(h)\,, \qquad f_a(g) = f(ag)\,, \qquad f^a(g) = f(ga)\,,$$

for any $a, g \in G$. If p_a is the unit mass at $a \in G$, then $L_{p_a} f = f_a$. Let $\mathrm{LU}\,C(G) \subset C(G)$ be the left uniformly continuous functions on G. $\varphi \in \mathrm{LU}\,C(G)^*$ is left invariant if $\varphi(f_a) = \varphi(f)$ for any f in $\mathrm{LU}\,C(G)$ and any a in G. $\varphi \in \mathrm{LU}\,C(G)^*$ is D-topologically left invariant if $\varphi(L_\mu f) = \varphi(f)$ for any f in $\mathrm{LU}\,C(G)$ and any $\mu \in DP_\sigma(G)$. (Obviously, φ is then left invariant.)

This definition has been suggested to us by a definition of A. Hulanicki in [17] who uses in his, only μ's which are regular with respect to compact sets (and have hence σ-compact support) and locally compact groups G (1).

(1) Thanks are due to Professor Hulanicki for his kindness in sending us a preprint of his paper. It is this paper which convinced us to study the problems of the present work.

We prove, using theorem 2(a) and remark III.4 that if $\varphi \in LUC(G)^*$ is left invariant then φ is D-topologically left invariant (²) (and a fortiori topologically left invariant in the sense of Hulanicki). In particular if φ is a left invariant mean on $UC(R) = LUC(R)$ (R the real line) then $\varphi(f) = \varphi[\int f(x+t)\,d\mu(t)]$ for any Borel probability measure μ on R and any f in $UC(R)$.

It would be interesting to know whether $\varphi(f) = \varphi(L_\mu f)$ for any left invariant mean on $C(R_d)$ (R_d discrete additive reals) and any $\mu \in M_\sigma^+(R_d)$ with $\mu(R_d) = 1$. For what we know it still may be that the assumption that the continuum has cardinal of measure zero is not needed in order to show that invariant means on $C(R_d)$ satisfy $\varphi(f) = \varphi(L_\mu f)$ for any $\mu \in M_\sigma^+(R_d)$ with $\mu(R_d) = 1$.

Some more definitions and notations. All completely regular spaces will be assumed to be Hausdorff. If X is a topological space, then $A \subset X$ is a *zero set* iff $A = \{x; f(X) = 0\}$ for some f in $C(X)$. Complements of zero sets are *cozero* sets. If d is a continuous pseudometric on X, then A is *d-totally bounded* iff any $\varepsilon > 0$, A can be covered by finite union of zero-sets of d-diameter $\leqslant \varepsilon$. ([4], Chapters 1 and 15.) A *σ-field* (*field*) of subsets of X is a nonempty family of subsets of X closed under countable (finite) unions and complementation. The *Baire σ-field of X* is the σ-field generated by the zero sets of X. Its elements are called *Baire sets.* A *Baire measure* is an element of $M_\sigma^+(X)$. If $a \in X$ then $p_a \in M_\sigma^+(X)$ is the point measure at a, i.e. $\int f\,dp_a = f(a)$ for each f in $C(X)$. If $f \in C(X)$ then $\|f\| = \sup\{|f(x)|;\ x \in X\}$.

A Baire set $B \subset X$ is a *continuity set* for $\mu \in M_\sigma^+(X)$ if there is a cozero set U and a zero set Z such that $U \subset B \subset Z$ and $\mu(Z-U) = 0$. $\mathcal{A} \subset C(X)$ is equicontinuous if for any x in X and $\varepsilon > 0$ there is a neighborhood U of x such that $|f(y)-f(x)| < \varepsilon$ for each y in U and all f in $\mathcal{A}$. If d is a continuous pseudometric on X, then $\mathcal{A} \subset C(X)$ is *d-equiuniformly continuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ for each f in $\mathcal{A}$ if $x, y \in X$ and $d(x, y) < \delta$. If Y is a Banach space, then Y^* denotes its adjoint Banach space. We use the notations in [3] for such spaces. If X is a set $l_1(X)$ is the set of all real functions f on X which vanish outside a countable set and $\sum |f(x)| < \infty$.

If L is a set and L' a set of real functions on L, then $\sigma(L, L')$ denotes the weakest topology on L which makes all f in L' continuous. R (R_d) denotes the real line with the usual (discrete) topology.

(²) I. Namioka has kindly informed this author that he and independently Greenleaf, have meanwhile obtained this same result for the case where G is locally compact, using Hulanicki's definition of topological left invariance. Namioka's beautiful proof relies though heavily on the existence and especially uniqueness of the Haar measure and hence does not carry over topological groups which are not locally compact.

II. Proof of theorem 1.

THEOREM 1. (a) *Let X be a completely regular D-space. If $\mathcal{A} \subset C(X)$ is bounded and equicontinuous, then $\mathcal{A}$ is $\sigma(C, M_\sigma)$-conditionally compact.*

Proof. $M_\sigma(X)$ is a Banach subspace of $C(X)^*$ under the norm

$$\|\mu\| = \sup\left\{\left|\int f\,d\mu\right|:\ f \in C(X),\ \|f\| \leqslant 1\right\}$$

(see Varadarajan [13], p. 39 and theorem 18, p. 45). We can consider $\mathcal{A} \subset M_\sigma(X)^*$ (the adjoint Banach space of M_σ). Then for f in $C(X)$ we have

$$\|f\| = \sup\{|f(x)|;\ x \in X\} = \sup\left\{\left|\int f\,d\mu\right|;\ \mu \in M_\sigma,\ \|\mu\| \leqslant 1\right\}.$$

Hence $\mathcal{A} \subset M_\sigma^*$ is norm bounded. To finish the proof it is enough to show that if f_a is a net in $\mathcal{A}$ such that $\lim_a \int f_a\,d\mu = L(\mu)$ exists for each μ in M_α, then $L\mu = \int f\,d\mu$ for some fixed f in $C(X)$. This will show that the $\sigma(M_\sigma^*, M_\sigma)$-closure of $\mathcal{A}$ in M_σ^* (which is compact by Alaoglu's theorem; [3], p. 424) is included in $C(X)$, i.e. that the $\sigma(C, M_\sigma)$ closure of $\mathcal{A}$ is $\sigma(C, M_\sigma)$ compact.

Let hence $\lim_a \int f_a\,d\mu = L\mu$ for each μ in M_σ where f_a is a net in $\mathcal{A}$. We have

$$\lim_a f_a(x) = \lim_a \int f_a\,dp_x = L(p_x)$$

for x in X. Define $L(p_x) = f(x)$. There is some $K > 0$ such that $\|g\| \leqslant K$ for any g in $\mathcal{A}$ and hence $|f(x)| \leqslant K$ for any x in X. Furthermore for any $\varepsilon > 0$ and x in X there is a neighborhood U of x such that $|g(y)-g(x)| < \varepsilon$ for any y in U and g in $\mathcal{A}$. Hence $|f(y)-f(x)| = \lim_a |f_a(y)-f_a(x)| \leqslant \varepsilon$ for any y in U which shows that $f \in C(X)$. If we would know that Lebesgue's bounded convergence theorem would hold for nets for any μ in M_σ, the proof would be finished, since then

$$L\mu = \lim_a \int f_a\,d\mu = \int f\,d\mu$$

for each μ in M_σ. This theorem does not hold (²) even for locally compact X of power $\aleph_1$, and hence we cannot proceed this way. If we would know that each μ in M_σ has its support in a σ-compact set, then we could apply Arzela's theorem and finish the proof. This again is not always the case (³).

(³) Consider the following example (given by Varadarajan [13] to show examples of σ-smooth, non τ-smooth measures): Let X be the set of ordinals less than the first uncountable one Ω, with the order topology. X is locally compact pseudocompact and any $f \in C(X)$ is ultimately constant, i.e. there is a real constant c_f and some $x_f \in X$ such that $x \geqslant x_f$ implies $f(x) = c_f$. ([4], pp. 72-75.) Denote $c_f = \operatorname{Lim}_{x\to\Omega} f(x) = L(f)$. Then

We proceed as follows:

Let $\mu \in M_\sigma^+(X)$ be a fixed Baire measure on X such that $\mu(X) = 1$. There is a *net* $\{\mu_\beta\} \subset M_\sigma^+(X)$, each element of which is a convex combination of point measures, such that $\int h\,d\mu_\beta \to \int h\,d\mu$ for each $h \in C(X)$ ([13], theorem 10, p. 61). X is a D-space and hence by theorem 2(a) $\int g\,d\mu_\beta \to \int g\,d\mu$ uniformly in $g \in \mathcal{A}$. Furthermore $L\mu_\beta = \int f\,d\mu_\beta$ as readily checked (since μ_β are convex combinations of point measures).

Hence

$$\left|L(\mu) - \int f\,d\mu\right| \leqslant |L\mu - \mu_\beta)| + \left|\int f\,d(\mu_\beta - \mu)\right|$$
$$= \lim_\alpha \left|\int g_\alpha d(\mu - \mu_\beta)\right| + \left|\int f\,d(\mu_\beta - \mu)\right| \quad \text{for each } \beta.$$

There is now some β_0 such that $\left|\int g\,d(\mu_{\beta_0} - \mu)\right| < \varepsilon$ for each $g \in \mathcal{A}$ and $\left|\int f\,d(\mu_{\beta_0} - \mu)\right| < \varepsilon$. This implies that $L\mu = \int f\,d\mu$ and finishes the proof.

We see hence that this theorem is a consequence of theorem 2(a) as stated for *nets*. The fact that theorem 2(a) is true for sequences of Baire measures would not be enough for this application since one cannot find in general a *sequence* μ_n of convex combinations of point measures for which $\int h\,d\mu_n \to \int h\,d\mu$ for each $h \in C(X)$, is true.

Take for instance the example given in the footnote connected to this proof. Consider the measure μ of of this example. If μ_n is a *sequence* of convex combinations of point measures on X, such that $\int h\,d\mu_n \to \int h\,d\mu$ for each $h \in C(X)$ and if $\{X_n\}_1^\infty \subset X$ is the union of the supports of $\{\mu_n\}_1^\infty$, then there is some $x_0 \in X$ such that $x_0 \geqslant x_n$ for each n. The set $A = \{x \in X;\ x > x_0\} = \{x \in X;\ x \geqslant x_0 + 1\}$ is open and closed, hence the function f defined by $f(x) = 1$ iff $x \in A$ and 0 otherwise, is in $C(X)$. Now $0 = \int f\,d\mu_n \not\to \int f\,d\mu = 1$. Thus μ can be approximated in the $\sigma(M_\sigma, C)$ topology by *nets* of convex combinations of point measures but not by such sequences.

THEOREM 1(b). *If a completely regular non D-space X exists, then $C(X)$ contains a bounded equicontinuous subset $\mathcal{A}$ whose $\sigma(C, M_\sigma)$ closure (in $C(X)$) is not $\sigma(C, M_\sigma)$ compact.*

Proof. Assume at first that X is a discrete non D-space. We show then that $\mathcal{A} = \{f \in C(X);\ \|f\| \leqslant 1\}$ is not $\sigma(C, M_\sigma)$-compact. Let μ belong to $M_\sigma(X)$ and not to $l_1(X)$ and assume that $\mathcal{A}$ is $\sigma(C, M_\sigma)$-compact. Since $\sigma(C, l_1)$ is weaker than $\sigma(C, M_\sigma)$, we get that $\sigma(C, M_\sigma)$ and $\sigma(C, l_1)$

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by Glicksberg's theorem (see [13], p. 44) there is some μ in $M_\sigma^+(X)$ such that $L(f) = \int f\,d\mu$. For each a in X define f_a in $C(X)$ by $f_a(x) = 1$ if $x \geqslant a + 1$ and 0 otherwise. Then $\lim_a f_a(x) = 0$ for each x in X while $L(f_a) = \int f_a\,d\mu = 1$ for each a in X. In this case $\{f_a;\ a \in X\}$ is not an equicontinuous set and $\mu \in M_\sigma$ does not have σ-compact support. In fact $\mu(K) = 0$ for any compact Baire set $K \subset X$.

coincide on $\mathcal{A}$. Hence the linear functional $Af = \int f\,d\mu$ is $\sigma(C, l_1)$ continuous when restricted to the unit ball (and hence to any ball) of $C(X)$. Hence the intersection of $A^{-1}(0)$ with any norm closed ball of $C(X)$ is $\sigma(C, l_1)$-closed. But in our case, $C(X) = l_1(X)^*$ and hence by the theorem of Krein–Smulian ([3], p. 429) $A^{-1}(0)$ is $\sigma(C, l_1)$-closed. By [3], p. 422 Corollary 11, A is a $\sigma(C, l_1)$ continuous functional and by this same Corollary $A \in l_1(X)$, i.e. $\int f\,d\mu = \int f\,d\mu_1$ for some μ_1 in $l_1(X)$. If $X_1 = \{x;\ \mu_1\{x\} > 0\}$ and $X_2 = \{x;\ \mu_1\{x\} < 0\}$ and X_3 is any subset of $X - (X_1 \cup X_2)$, then $\mu(X_3) = \mu_1(X_3) = 0$ which shows that μ has countable support and hence $\mu = \mu_1 \in l_1(X)$. This contradiction shows that $\mathcal{A}$ is not $\sigma(C, M_\sigma)$-compact.

Assume now that X is a completely regular non D-space and let $X_0 \subset X$ be a d-discrete set, whose cardinal does not have measure zero, where d is a continuous pseudometric on X. Let X^* be the set of equivalence classes of X, for the equivalence relation $x \sim y$ iff $d(x, y) = 0$. If $x \in X$ then x^* denotes the equivalence class containing x. Let $d^*(x^*, y^*) = d(x, y)$. Then (X^*, d^*) is a metric space, the map $F\colon X \to X^*$ defined by $F(x) = x^*$ is continuous, and the set $X_0^* = F(X_0)$ is d^*-discrete. Let $\mathcal{A}_0^*$ be the set of all functions from X_0^* to the closed interval $[-1, 1.]$ Then $\mathcal{A}_0^*$ is uniformly bounded and d^*-equiuniformly continuous on X_0^* since X_0^* is d^*-discrete. We can hence apply theorem 1, p. 10 of M. Atsuji [1] and get that each $f_0^* \in \mathcal{A}_0^*$ has an extension $f^*\colon X^* \to [-1, 1]$ in such a way that the family $\mathcal{A}^* = \{f^*\colon f_0^* \in \mathcal{A}_0^*\}$ is d^*-equiuniformly continuous on X^*.

Let $\mathcal{A} = \{f(x) = f^*(F(x));\ f^* \in \mathcal{A}^*\}$. $\mathcal{A}$ is uniformly bounded and d-equiuniformly continuous, hence a fortiori equicontinuous on X. Furthermore, the family of restrictions to X_0 of functions in $\mathcal{A}$ coincides with the norm closed unit ball of $C(X_0)$.

Assume now that $\mathcal{A} \subset C(X)$ has $\sigma(C, M_\sigma)$ closure which is $\sigma(C, M_\sigma)$ compact and assume that f_α^0 is a net in $C(X_0)$ such that $\|f_\alpha^0\| \leqslant 1$ for each α. There is for each α some f_α in $\mathcal{A}$ which extends f_α^0 to all of X. We can find a subnet f_{α_β} such that $\lim_\beta \int f_{\alpha_\beta} d\mu = \int f\,d\mu$ for each $\mu \in M_\sigma(X)$, for some f in the $\sigma(C, M_\sigma)$ closure of $\mathcal{A}$. In particular, $f_{\alpha_\beta}(x) \to f(x)$ for x in X which shows that $\|f\| \leqslant 1$.

For $\mu_0 \in M_\sigma(X_0)$ let $\mu \in M_\sigma(X)$ be defined by $\mu(A) = \mu_0(A \cap X_0)$ for each Baire set $A \subset X$. Then $\int g\,d\mu = \int g^0 d\mu_0$ for any $g \in C(X)$ where g^0 is the restriction of g to X_0. Hence for any $\mu_0 \in M_\sigma(X_0)$

$$\int f^0 d\mu_0 = \int f\,d\mu = \lim_\beta \int f_{\alpha_\beta} d\mu = \lim_\beta \int f_{\alpha_\beta}^0 d\mu_0.$$

This shows that the set $\{h \in C(X_0);\ \|h\| \leqslant 1\}$ is $\sigma(C(X_0), M_\sigma(X_0))$ compact which contradicts the first part of the proof. We conclude hence that the $\sigma\big(C(X), M_\sigma(X)\big)$ closure of $\mathcal{A}$ in $C(X)$ is not $\sigma\big(C(M), M_\sigma(X)\big)$ compact, which finishes the proof.

Remarks. (1) It has been shown by S. Ulam [12], p. 149 and p. 141 (see also Keisler–Tarski [5], p. 267) that the assumption of nonexistence of inaccessible cardinals which are less than or equal to that of the continuum (which is implied by the continuum hypothesis [12], p. 141) implies that a discrete space is a D-space if and only if it has nonmeasurable cardinal (in terms of 0-1 valued measures as in [4], p. 161), i.e. if and only if it is real compact (Gillman–Jerison [4], p. 163). From the other side even without this assumption the space of ordinals less than the first uncountable one with the order topology is a locally compact pseudocompact D-space which is not real compact (see Gillman–Jerison [4], p. 237, 15Q and p. 229). If now $\mathfrak{m}$ is a cardinal of measure zero and $\mathfrak{n} \leqslant \mathfrak{m}$, then $\mathfrak{n}$ has measure zero. Hence if X is completely regular and has cardinal of measure zero, then X is a D-space. The above assumption (and in particular assuming the continuum hypotheses) together with the fact ([4], p. 164-165) that if $\mathfrak{m}$ is nonmeasurable so is $2^{\mathfrak{m}}$ and a sum of $\mathfrak{m}$ nonmeasurable cardinals is nonmeasurable, would imply that any completely regular X which arises from "concrete applications" is a D-space and our theorems hold for it. Without the above assumption we do not even know whether R_d is a D-space but, surprisingly enough, we know that $\prod_{t \in R_d} R_t$ (where R_t are copies of the reals with the usual topology) is a D-space.

It has been shown by D. Scott in [11] that the Zermelo–Frenkel axioms and Godel's constructibility axiom imply that all cardinals have measure zero and hence that, under these assumptions, all topological spaces are D-spaces (4).

(2) If X is discrete countable or of cardinality $\aleph_1$, then X is a D-space. In this case, $M_\sigma(X) = l_1(X)$ and $C(X) = l_1(X)^*$. Our theorem yields just Alaoglu's theorem, i.e. that the unit ball of $C(X)$ is $\sigma(C, l_1)$-compact.

(3) If X is compact then any bounded equicontinuous subset of $C(X)$ is even norm conditionally compact (see [3], p. 266) and hence our theorem is even weaker than a well-known result. The same is the case if X is locally compact and σ-compact. Since if τ_k is the topology of uniform convergence on compact subsets of X and $\mathcal{A} \subset C(X)$ is uniformly bounded and equicontinuous, then τ_k induced on $\mathcal{A}$ is stronger than $\sigma(C, M_\sigma)$ on $\mathcal{A}$ (since any $\mu \in M_\sigma^+(X)$ has σ-compact support). And as known $\mathcal{A}$ is in this case even τ_k conditionally compact.

(4) We acknowledge with thanks discussions we had with G. E. Sacks and A. B. Manaster.

(4) Let X be the locally compact space of ordinals given in footnote 3 and let μ and f_a be as constructed there. For this case τ_k is not stronger than $\sigma(C, M_\sigma)$ even on norm bounded subsets of $C(X)$. Since $f_a \to 0$ uniformly on compact subsets of X while $\int f_a \, d\mu = 1$. Theorem 1(a) asserts something new for this case.

(5) Let $X \subset [0, 1]$ be such that $\mu^*(X) = 1$ and $\mu_*(X) = 0$ where μ denotes the Lebesgue measure and μ^*, μ_* its respective outer and inner measure. Then as known ([15], p. 75) μ^* induces a probability measure on the Baire field of X.

Now $\mu^*(K) = 0$ for any compact $K \subset X$ (since K is Borel measurable and $\mu^*(K) = \mu_*\{(K) \leqslant \mu_*(X) = 0)$. Hence $Lf = \int f \, d\mu^*$, for $f \in C(X)$, is not continuous on the unit ball of $C(X)$ with τ_k. (In fact, if it would be so then by theorem 29 of Varadarajan [13], p. 53, $\mu_*(X-K) < \frac{1}{2}$ for some compact $K \subset X$, which cannot be.) Our theorem yields something new for this case.

(6) Let (Xd) be a nondiscrete metric space. Let $x_0 \in X$ be nonisolated and $x_n \in X$ be such that $d(x_0, x_n) = a_n > 0$ for each n and $a_n > a_{n+1} \to 0$. Let $f_n \in C(X)$ be defined by

$$f_n(x) = \begin{cases} 1 - \dfrac{1}{a_n} d(x_n x) & \text{if} \quad d(x_n x) \leqslant a_n, \\ 0 & \text{otherwise}. \end{cases}$$

One can check easily that $\|f_n\| = 1$ for each n, $f_n(x) \to 0$ for each x in X and $\{f_n(x)\} \subset C(X)$ is not equicontinuous. Now $\mathcal{A} = \{f_n\} \cup \{0\}$ is a $\sigma(C, M_\sigma)$ compact set by Lebesgue's convergence theorem. This example shows that for any metric nondiscrete D-space (Xd) the collection of bounded equicontinuous subsets of $C(X)$ is properly included in the collection of all $\sigma(C, M_\sigma)$ conditionally compact subsets of $C(X)$. If though a non D-metric space X exists, then using theorem 1(b) and the above defined $\mathcal{A}$ one gets that the above classes are not contained in each other. This fact has also the following interpretation (see [16], p. 62 for definitions): If (Xd) is a nondiscrete metric D-space then the Mackey topology τ_M on $M_\sigma(X)$ from the dual system (C, M_σ) is strictly stronger than the topology τ_{bc} of uniform convergence on bounded equicontinuous subsets of $C(X)$. (In fact for $\mathcal{A} = \{f_n\} \cup \{0\}$ given in this remark, $\int f dp_{x_n} \to \int f dp_{x_0}$ for each f in $C(X)$ but $\left|\int f_n d(p_{x_n} - p_{x_0})\right| = 1$ while from theorem 2(a) and the remark III-4 we have $\int f dp_{x_n} \to \int f dp_{x_0}$ uniformly on any bounded equicontinuous $\mathcal{A} \subset C(X)$.)

III. Proof of theorem 2. We shall need the following known fact:

Let B be a σ-field of subsets of $\Omega \neq \emptyset$ and μ a countably additive nonnegative (not necessarily finite) set function. Let $f_n \geqslant 0$, $f \geqslant 0$ be

B-measurable such that $\int f_n d\mu < \infty$, $\int f d\mu < \infty$, $f_n(x) \to f(x)$ μ-almost everywhere and $\int f_n d\mu \to \int f d\mu$. Then $\int |f_n - f| d\mu \to 0$. (If $a \vee b = \max(a, b)$ then $|f - f_n| = [2(f - f_n) \vee 0] - (f - f_n)$. Now $0 \leqslant (f - f_n) \vee 0 \leqslant f$, $(f - f_n) \vee 0 \to 0$ μ-almost everywhere and $\int (f - f_n) d\mu \to 0$ by assumption. Hence $\int |f - f_n| d\mu = 2 [\int [(f - f_n) \vee 0] d\mu - \int (f - f_n) d\mu] \to 0$ by Lebesgue's convergence theorem.)

LEMMA 1. *Let $f_a \geqslant 0$ be a net of functions and $f \geqslant 0$ a function, all defined on $\Omega = \{1, 2, 3, \ldots\}$, be such that*

$$\sum_j f_a(j) < \infty, \quad \sum_j f(j) < \infty, \quad f_a(j) \to f(j) \quad \textit{for each } j$$

and

$$\lim_a \sum_j f_a(j) = \sum_j f(j).$$

Then

$$\lim_a \sum_j |f_a(j) - f(j)| = 0.$$

Proof. Assume that this is false. Then $\sum_j |f_{a_\beta}(j) - f(j)| \geqslant d$ for a subnet f_{a_β} and some $d > 0$. There is a β_n' such that $\left|\sum_j (f_{a_\beta}(j) - f(j))\right| < 1/n$ if $\beta \geqslant \beta_n'$ and there is a $\beta_n \geqslant \beta_n'$ such that $|f_{a_{\beta_n}}(j) - f(j)| < 1/n$ for $j = 1, 2, \ldots, n$. Hence $\lim_n f_{a_{\beta_n}}(j) = f(j)$ for each j and $\lim_n \sum_j f_{a_{\beta_n}}(j) = \sum_j f(j)$ and $\sum_j |f_{a_{\beta_n}}(j) - f(j)| \geqslant d > 0$ which contradicts the above known fact.

COROLLARY 1. *Let B be a σ-field of subsets of $\Omega \neq \emptyset$, μ, μ_a be finite nonnegative countably additive on B and $A_j \in B$ be disjoint such that $\bigcup_{j=1}^{\infty} A_j = \Omega$. If $\lim_a \mu_a(A_j) = \mu(A_j)$, $j = 1, 2, \ldots$, and $\lim_a \mu_a(\Omega) = \mu(\Omega)$ then*

$$\lim_a \sum_j |\mu_a(A_j) - \mu(A_j)| = 0.$$

LEMMA 2. *Let (X, d) be a metric space. Let $\mu \in M_\sigma^+(X)$ satisfy: $\mu(S_0) = \mu(X)$ for some Baire set $S_0 \subset X$ for which (S_0, d) is a D-space. If μ_a is a net in $M_\sigma^+(X)$ such that $\int f d\mu_a \to \int f d\mu$ for each f in $C(X)$, then $\limsup_a \sup_{f \in \mathcal{A}} \left|\int f d(\mu_a - \mu)\right| = 0$ for any bounded equicontinuous $\mathcal{A} \subset C(X)$* [5].

Remarks (1). The metric space (X, d) is a D-space is equivalent to: *X contains a dense set whose cardinal has measure zero.* In fact for each n there is a subset $A_n \subset X$ maximal with respect to the property: if $x, y \in A_n$ and $x \neq y$, then $d(x, y) \geqslant 1/n$. If (X, d) is a D-space then A_n, and hence $\bigcup_{n=1}^{\infty} A_n$ (which is dense) has cardinal of measure zero. Conversely if X contains a dense set whose cardinal has measure zero and if d_1 is a continuous pseudometric on (X, d), then any d_1-discrete subset has cardinality which is less than or equal to that of the dense subset and hence has also cardinal of measure zero.

(2). Let Y be completely regular and $X \subset Y$. If X is a D-space, then so is $\overline{X}$ its closure in Y. In fact if e is a continuous pseudometric on $\overline{X}$ and $\{y_a\} \subset \overline{X}$ is such that $e(y_a, y_\beta) \geqslant c \geqslant 0$ if $a \neq \beta$ then there are x_a in X such that $e(y_a x_a) < c/3$. Hence $e(x_a x_\beta) \geqslant e(y_a y_\beta) - \frac{2}{3}c \geqslant c/3$ if $a \neq \beta$, which shows that $\overline{X}$ is a D-space. Hence we can and shall assume that S_0 is closed in this lemma.

Proof. By restricting μ to the Baire subsets of S_0 and applying Theorem III of Marczewski–Sikorski [6], p. 137, we get the existence of a separable closed $S \subset S_0$ for which $\mu(S) = \mu(S_0) = \mu(X)$ [6]. Let $\mathcal{A} \subset C(X)$ be bounded equicontinuous and $\varepsilon > 0$. Then as in Ranga R. Rao [8], p. 661, we can find for each s in S a neighborhood (*in* X) $N_s = \{y \in X;\ d(s, y) < \delta\}$ {δ may depend on s} such that $|f(y) - f(s)| < \varepsilon/2$ if $y \in N_s$ and $f \in \mathcal{A}$. N_s may and shall be chosen as a continuity set for μ. Since S is separable metric, we have $S \subset \bigcup_1^{\infty} N_{s_n}$ for some sequence $\{s_n\} \subset S$. The sets $A_1 = N_{s_1}$, $A_n = N_{s_n} - \bigcup_{j<n} N_{s_j}$, $n = 2, 3, \ldots$, have the properties: $\{A_i\}_1^{\infty}$ are disjoint continuity sets for μ (since the continuity sets for μ form a field), $S \subset \bigcup_1^{\infty} A_n$ and if $x, y \in A_n$ then $|f(x) - f(y)| < \varepsilon$ for each f in $\mathcal{A}$. We may and shall assume that $A_n \neq \emptyset$ for each n and pick a fixed

[5] This lemma has been proved for sequences μ_n and separable metric X by Ranga R. Rao in [8], p. 662. We follow his idea and adapt it to nets, using Corollary 1 and the theorem of Marczewski and Sikorski about the separable support of Baire measures on metric D-spaces.

[6] Let (Y, ϱ) be a metric D-space. Let $A_n \subset Y$ be set, maximal with respect to the property: if $x, y \in A_n$, $x \neq y$ then $\varrho(x, y) \geqslant 1/n$. ϱ is surely a continuous pseudometric and therefore A_n has cardinal of measure zero. Furthermore for any y in Y there is some x in A_n such tgat $\varrho(xy) < 1/n$. Therefore $A = \bigcup_1^{\infty} A_n$ is dense in Y and has cardinal of measure zero. By Marczewski-Sikorski [6], p. 136, Theorem II there is a basis for the topology of Y whose cardinal has measure zero and in particular the separability character of Y (i. e. the smallest cardinal of a basis for the topology of Y) has measure zero. At the referee's sugestion we bring here the statement of the theorem of Marczewski-Sikorski which we use in the proofs of the Theorems 1(a) and 2(a):

THEOREM III ([6] p. 137). *If the separability character of a metric space Y has measure zero and if ν is a finite Baire measure on Y (i.e. $\nu \in M_\sigma^+(Y)$) then:*

(a) *The union N of all open subsets of Y df ν-measure zero has also ν-measure zero.*

(b) *$\nu[U \cap (Y - N)] > 0$ for any non-void open $U \subset Y$. Therefore $Y - N$ is closed and separable and $\nu(Y) = \nu(Y - N)$.*

sequence $x_n \in A_n$ for each n. For any $\nu \in M_\sigma^+(X)$ denote by $\nu^* \in M_\sigma^+(X)$ the measure $\nu^* = \sum_1^\infty \nu(A_i) p_{x_i}$ where $\int f dp_x = f(x)$ for $f \in C(X)$ and $x \in X$. (We notice that S may have void interior and hence none of its subsets would be a continuity set for μ if $\mu \neq 0$.)

Denoting by $T = \bigcup_1^\infty A_i$ we have for any f in $\mathcal{A}$

$$\begin{aligned}\Big|\int f d(\mu_a - \mu_a^*)\Big| &\leqslant \Big|\int_T f d(\mu_a - \mu_a^*)\Big| + \Big|\int_{X-T} f d(\mu_a - \mu_a^*)\Big| \\ &\leqslant \sum_j \Big|\int_{A_j} f d(\mu_a - \mu_a^*)\Big| + K\mu_a(X-T) \\ &\leqslant \sum_j \int_{A_j} |f(x) - f(x_j)| d\mu_a + K\mu_a(X-T) \\ &< \varepsilon\mu_a(X) + K\mu_a(X-T)\,,\end{aligned}$$

where $K = \sup\{|f(x)|;\, x \in X, f \in \mathcal{A}\}$. We can now find some N such that $\mu(X - \bigcup_1^N A_i) < \varepsilon$. Since $X - \bigcup_1^N A_i$ is a continuity set for μ,

$$\limsup_a \mu_a(X-T) \leqslant \lim_a \mu_a(X - \bigcup_1^N A_i) = \mu(X - \bigcup_1^N A_i) < \varepsilon$$

(see Varadarajan [13], p. 56). This shows that there is some α_1 such that $\mu_a(X-T) < \varepsilon$ if $\alpha \geqslant \alpha_1$. Since $\mu_a(X) \to \mu(X)$, there is some $\alpha_0 \geqslant \alpha_1$ such that $\alpha \geqslant \alpha_0$ implies that $\mu_a(X) < \mu(X)+1$ and hence also that $\mu_a(X-T) < \varepsilon$.

Hence:

(*) $\quad \Big|\int f d(\mu_a - \mu_a^*)\Big| < \varepsilon\big(\mu(X)+1+K\big) \quad$ for $\quad \alpha \geqslant \alpha_0$ and any f in $\mathcal{A}$.

Now

(**) $\quad \Big|\int f d(\mu - \mu^*)\Big| = \Big|\int_T f d(\mu - \mu^*)\Big| \leqslant \sum_1 \int_{A_i} |f(x) - f(x_i)| d\mu < \varepsilon\mu(X)$

for any f in $\mathcal{A}$. Furthermore,

$$\begin{aligned}\Big|\int_X f d(\mu_a^* - \mu^*)\Big| &= \Big|\int_T f d(\mu_a^* - \mu^*)\Big| \leqslant \sum_i \Big|\int_{A_i} f d(\mu_a^* - \mu^*)\Big| \\ &= \sum_i \Big|f(x_i)\big(\mu_a(A_i) - \mu(A_i)\big)\Big| \leqslant K \sum_i |\mu_a(A_i) - \mu(A_i)|\,.\end{aligned}$$

Now A_i are continuity sets for μ and hence $\lim_a \mu_a(A_i) = \mu(A_i)$ for each i and $\lim_a \mu_a(X) = \mu(X)$ (see [13], p. 56). Hence using corollary 1 we can find some $\alpha_2 \geqslant \alpha_0$ such that

$\binom{*}{**}$ $\quad \Big|\int_X f d(\mu_a^* - \mu^*)\Big| < K\varepsilon \quad$ if $\quad \alpha \geqslant \alpha_2$ and $f \in \mathcal{A}$.

Combining now (*), (**) and $\binom{*}{**}$ we get that

$$\Big|\int f d(\mu_a - \mu)\Big| \leqslant \varepsilon\big(2\mu(X) + 2K + 1\big) \quad \text{for } \alpha \geqslant \alpha_2 \text{ and any } f \text{ in } \mathcal{A}\,.$$

This shows that $\limsup_{\alpha} \sup_{f \in \mathcal{A}} \Big|\int f d(\mu_a - \mu)\Big| = 0$ as required.

Remark (3). If instead of a net μ_a we would have had a *sequence* μ_n, then we could have found a separable closed subset $S \subset X$ such that *$\mu(X) = \mu(S)$ and $\mu_n(X) = \mu_n(S)$ for each n.* We could then have applied Ranga Rao's result to the restrictions of μ_n, μ to the set S. Since we deal with nets, we cannot find in general, a closed separable $S \subset X$ such that $\mu_a(S) = \mu_a(X)$ if $\alpha \geqslant \alpha_0$ for some α_0. In fact let $X = R_d$; then $p_0 + sp_s \to p_0$ if $s \to 0$ from the right, where $p_s(p_0)$ is the measure on R_d concentrated at s (0 respectively). If $s_0 > 0$ then the union of the supports of $p_0 + sp_s$ for $s \geqslant s_0$ is not contained in a countable set. We have hence to modify Ranga Rao's proof as above.

THEOREM 2. *Let X be completely regular, μ_a be a net in $M_\sigma^+(X)$ and $\mu \in M_\sigma^+(X)$ be such that $\lim_a \int f d\mu_a = \int f d\mu$ for each f in $C(X)$.*

(a) *If X is a D-space then for any bounded equicontinuous $\mathcal{A} \subset C(X)$*

$$\limsup_{\alpha} \sup_{f \in \mathcal{A}} \Big|\int f d(\mu_a - \mu)\Big| = 0\,.$$

(b) *If X is not a D-space then there is a bounded equicontinuous $\mathcal{A} \subset C(X)$ such that the convergence $\int f d\mu_a \to \int f d\mu$ is not uniform on $\mathcal{A}$.*

Proof. (a) Let $\mathcal{A} \subset C(X)$ be uniformly bounded and equicontinuous and define on X the pseudometric $d(xy) = \sup\{|f(x) - f(y)|;\, f \in \mathcal{A}\}$ (7). The equicontinuity of $\mathcal{A}$ implies that d is a continuous pseudometric. We define on X the equivalence relation $x \sim y$ iff $d(x, y) = 0$. If X^* is the class of all such equivalence classes, x^* being that which contains $x \in X$, then we define the metric d^* on X^* by $d^*(x^*, y^*) = d(x, y)$. (X^*, d^*) is a metric D-space and $F\colon X \to X^*$ defined by $F(x) = x^*$ is contiuuous onto as can easily be checked. Furthermore, since any $f \in \mathcal{A}$ is constant on the equivalence classes x^*, we can define f^* on X^* by $f^*(x^*) = f(x)$ for some x in x^* and f in $\mathcal{A}$. The class $\mathcal{A}^* = \{f^*;\, f \in \mathcal{A}\}$ is uniformly bounded and equicontinuous on (X^*, d^*) since $d^*(x^*, y^*) < \varepsilon$ implies $d(x, y) < \varepsilon$ and hence

$$|f^*(x^*) - f^*(y^*)| = |f(x) - f(y)| \leqslant d(x, y) < \varepsilon$$

(7) Thanks are due to R. M. Dudley for the idea to use this pseudometric and the theorem of Marczewski–Sikorski in this proof. The original proof of this theorem was more complicated and used theorem 15.17 in Gillman–Jerison [4], p. 227, instead.

for any f in $\mathcal{A}$. Also $f^*(F(x)) = f(x)$ for any x in X and f in $\mathcal{A}$. If $\nu \in M_\sigma(X)$ define $\nu^* \in M_\sigma(X^*)$ by $\nu^*(B^*) = \nu(F^{-1}(B^*))$ for any Baire set $B^* \subset X^*$. Then

$$\int g^*(x^*)\,d\nu^*(x^*) = \int g^*(F(x))\,d\nu(x)$$

for any g in $C(X)$ ([15], p. 163) and in particular

$$\int f^*(x^*)\,d\nu^*(x^*) = \int f^*(F(x))\,d\nu(x) = \int f(x)\,d\nu(x) \quad \text{for any } f \text{ in } \mathcal{A}\,.$$

Therefore

$$\int g^* d\mu^* = \int g^*(F(x))\,d\mu_a(x) \to \int g^*(F(x))\,d\mu(x) = \int g^* d\mu^*$$

for any $g^* \in C(X^*)$. Using now lemma 2 for (X^*, d^*) we get that

$$\lim_a \sup_{f \in \mathcal{A}} \left|\int f\,d(\mu_a - \mu)\right| = \lim_a \sup_{f^* \in \mathcal{A}^*} \left|\int f^* d(\mu_a^* - \mu^*)\right| = 0$$

which finishes part (a).

(b) Assume that $X_0 \subset X$ is a d-discrete set whose cardinal does not have measure zero, where d is some continuous pseudometric on X. Then $d(x, y) \geqslant \delta$ for x, y in X_0, $x \neq y$, for some $\delta > 0$. We can find now, as in the proof of theorem 1(b), a bounded equicontinuous class $\mathcal{A} \subset C(X)$ such that $\|f\| \leqslant 1$ for f in $\mathcal{A}$ and

$$\mathcal{A}_0 = \{f|_{X_0};\ f \in \mathcal{A}\} = \{g \in C(X_0);\ \|g\| \leqslant 1\}$$

where $f|_{X_0}$ is the restriction of $f \in C(X)$ to $X_0 \subset X$. There is by assumption a $\nu \in M_\sigma^+(X_0)$ such that $\nu \neq 0$ and $\nu\{x\} = 0$ for any x in X_0 ($(X_0 d)$ is discrete and hence all subsets of X_0 are Baire sets). There exists then, as known, a net ν_a of measures of type $\sum_1^k a_i p_{x_i}$ where $a_i \geqslant 0$ and $\int f dp_x = f(x)$ for $x \in X$ and $f \in C(X_0)$, such that $\int g\,d\nu_a \to g\,d\nu$ for any g in $C(X_0)$ ([13], p. 61, theorem 10). Define now μ_a, μ in $M_\sigma^+(X)$ by

$$\mu_a(B) = \nu_a(B \cap X_0) \quad \text{and} \quad \mu(B) = \nu(B \cap X_0)\,.$$

Then

$$\int f\,d\mu_a = \int f|_{X_0} d\nu_a \quad \text{and} \quad \int f\,d\mu = \int f|_{X_0} d\nu$$

for any f in $C(X)$. Therefore $\int f\,d\mu_a \to \int f\,d\mu$ for any f in $C(X)$. If

$$\lim_a \sup_{f \in \mathcal{A}} \left|\int f\,d(\mu_a - \mu)\right| = 0$$

then

$$\lim_a \sup_{g \in C(X_0), \|g\| \leqslant 1} \left|\int g\,d(\nu_a - \nu)\right| = 0$$

which implies that we can find a_n such that $\left|\int g\,d(\nu_{a_n} - \nu)\right| < 1/n$ for any g in $C(X_0)$ with $\|g\| \leqslant 1$. Hence $\lim_n \int g\,d\nu_{a_n} = \int g\,d\nu$ for any g in $C(X_0)$. If we denote by A the countable union of the (finite) supports of the measures ν_{a_n}, then A is countable and

$$\int l\,d\nu = \nu(X_0) = \lim_n \nu_{a_n}(X_0) = \lim_n \nu_{a_n}(A) = \lim_n \int l_A\,d\nu_{a_n} = \int l_A\,d\nu = \nu(A)$$

(where $l_A(y) = 1$ if $y \in A$ and 0 if $y \notin A$), since $l_A \in C(X_0)$. Since $\nu \neq 0$ and A is countable, $\nu\{x\} > 0$ for some x in A. This contradiction shows that

$$\lim_a \sup_{f \in \mathcal{A}} \left|\int f\,d(\mu_a - \mu)\right| = 0 \quad \text{is not true}\,,$$

which finishes the proof.

Remarks (4). Theorem 2(a) remains true if instead of assuming that X is a D-space we assume only that there is a set $S_\mu \subset X$ which is a D-space in the relative topology such that $\mu(S) = \mu(X)$ for any Baire set S containing S_μ (we have in mind for S_μ, σ-compact subsets of X, which as known need not be Baire sets). We show this as follows: In the notation of the proof of Theorem 2(a), $(F(S_\mu), d^*)$ is a metric D-space (since if e^* is a continuous pseudometric on $F(S_\mu)$ then $e^*(F(x), F(y))$ is a continuous pseudometric on S_μ). Hence $\overline{F(S_\mu)}$, the closure of $F(S_\mu)$ in X^* is a D-space. Moreover $\overline{F(S_\mu)}$ is a Baire set since (X^*, d^*) is metric. Using the fact that $\mu^*(\overline{F(S_\mu)}) = \mu^*(X^*)$ and lemma 2 we get that $\lim_a \sup_{f^* \in \mathcal{A}^*} \left|\int f^* d(\mu_a^* - \mu^*)\right| = 0$ which implies as in the proof of theorem 2(a) our assertion.

(5) Let X be discrete. Then using the previous remark one gets that the norm topology and the weak topology $\sigma(l_1(X), C(X))$ coincide on the positive cone of l_1 (any μ in l_1 has σ-compact support). This result is in fact proved much easier in corollary 1. Nevertheless the norm and weak topologies on $l_1(X)$ do not coincide unless X is finite ([13], p. 436 (9)).

(6) Let X be completely regular and

$$DM_\sigma^+(X) = \{\mu \in M_\sigma^+(X);\ \text{there is a } D\text{-subspace } S_\mu \subset X \text{ such that } \mu(S) = \mu(X) \text{ for any Baire set } S \text{ with } S_\mu \subset S\}\,.$$

$DM_\sigma^+(X) = M_\sigma^+(X)$ if X is a D-space. Let $DM_\sigma(X) = DM_\sigma^+(X) - DM_\sigma^+(X)$. (which is a linear subspace of $M_\sigma(X)$). Let τ_{be} be as defined in remark 6 of section II. Then τ_{be} coincides with $\sigma(M_\sigma, C)$ when both are restricted to $DM_\sigma^+(X)$. This is implied by remark III (4) (as pointed out in remark II(6), $\tau_{be} \neq \tau_M$ on $DM_\sigma^+(X)$ if X is a nondiscrete metric D-space).

Nevertheless τ_{be} and $\sigma(M_\sigma, C)$ do not coincide on the linear space $DM_\sigma(X)$ even for a countable (non-finite) discrete X. In this case we have $DM_\sigma(X) = l_1(X)$, τ_{be} is the norm topology and $\sigma(M_\sigma, C) = \sigma(l_1, l_1^*)$ is the weak topology on l_1.

The end of remark II(6) shows that we cannot replace in theorem 2(a), the condition "$\mathcal{A}$ is bounded equicontinuous" by $\mathcal{A}$ is $\sigma(C, M_\sigma)$ compact even if X is the closed unit interval $[0, 1]$.

(7) Theorem 2(a) does not remain true if $\mu_a, \mu \in M_\sigma^+(X)$ is replaced by $\mu_a, \mu \in M_\sigma(X)$. In fact let X be a countable infinite discrete space and consider the pair $(l_1(X), m(X))$. If Theorem 2(a) would be true for $\mu_a, \mu \in l_1(X)$ (even in the unit ball of $l_1(X)$) then $\mu_a \to \mu$, $(\sigma(l_1, m))$ if and only if $\|\mu_a - \mu\| \to 0$. This implies that the $\sigma(l_1, m)$ topology of the unit ball of l_1 is metric, which by [3], p. 426, Theorem 2 would imply that $m(X)$ is norm separable which is not true (unless X is a finite set).

IV. Applications.

(a) Proof of Varadarajan's theorem.

Lemma 1. *Let d be a pseudometric on X such that $d(x, y) \leqslant 1$ for any $x, y \in X$. Let A, B be subsets of X such that $d(A, B) > 0$ $(d(A, B) = \inf\{d(x, y);\ x \in A, y \in B\}$ and $d(x, A) = d(\{x\}, A))$. Let*

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Then

$$|f(x) - f(y)| \leqslant \frac{2}{[d(A, B)]^2} d(x, y).$$

Proof. If $a \in A$, $b \in B$, then $d(x, a) + d(x, b) \geqslant d(a, b) \geqslant d(A, B)$. Thus $d(x, A) + d(x, B) \geqslant d(A, B)$. Hence

$$\begin{aligned} &|f(x) - f(y)| \\ &\leqslant [d(A, B)]^{-2}\{d(x, A)[d(y, A) + d(y, B)] - d(y, A)[d(x, A) + d(x, B)]\} \\ &= [d(A, B)]^{-2}|d(x, A)d(y, B) - d(y, A)d(x, B)| \\ &= [d(A, B)]^{-2}|d(x, A)[d(y, B) - d(x, B)] - d(x, B)[d(x, A) - d(y, A)]| \\ &\leqslant [d(A, B)]^{-2}\cdot 2d(x, y) \end{aligned}$$

since

$$|d(y, A) - d(x, A)| \leqslant d(x, y).$$

Theorem 3. *Let (X, d) be a metric space. Then $DM_\sigma^+(X)$ equipped with $\sigma(M_\sigma, C)$ is metrisable.*

Proof. Let $e(x, y) = \min\{d(x, y), 1\}$. Then the metric e induces on X the original topology (see [19], p. 121). (Moreover, $f \in C(X)$ is uniformly continuous with respect to d if and only it is uniformly continuous with respect to e.) Let

$$\mathcal{A}_n = \{f \in C(X);\ |f(x) - f(y)| \leqslant 2n^2 e(x, y)\}$$

and for $\mu_1, \mu_2 \in M_\sigma^+(X)$ let

$$d_n(\mu_1, \mu_2) = \sup\left\{\left|\int f d(\mu_1 - \mu_2)\right|;\ f \in \mathcal{A}_n\right\}.$$

Obviously d_n are pseudometrics on $M_\sigma^+(X)$. Assume now that $\mu \in M_\sigma^+(X)$ and μ_a is a net in $M_\sigma^+(X)$ such that $\lim_a d_n(\mu_a, \mu) = 0$ for each n. Let $Z \subsetneqq X$ be closed (hence a Baire set) and $U_n = \{x \in X;\ d(x, Z) < 1/n\}$. Then $\mu(U_n) \to \mu(Z)$ and there is a n_0 such that $\mu(Z) > \mu(U_{n_0}) - \varepsilon$ and such that $U_{n_0} \neq X$. Let

$$f_n(x) = \frac{e(x, X - U_n)}{e(x, Z) + e(x, X - U_n)}.$$

By the previous lemma, $f_{n_0} \in \mathcal{A}_{n_0}$ and since the convergence $\int f d\mu_a \to \int f d\mu$ is even uniform on $\mathcal{A}_{n_0}$, we will have in particular that $\int f_{n_0} d\mu_a \to \int f_{n_0} d\mu$. Hence

$$\mu(Z) + \varepsilon > \mu(U_{n_0}) \geqslant \int f_{n_0} d\mu = \lim_a \int f_{n_0} d\mu_a \geqslant \limsup \mu_a(Z).$$

This shows that for any zero set $Z \subsetneqq X$

$$\limsup \mu_a(Z) \leqslant \mu(Z)$$

and since the constant one function on X belongs to $\mathcal{A}_n$, we get that $\mu_a(X) \to \mu(X)$. By Varadarajan [13], p. 56, we get that $\int f d\mu_a \to \int f d\mu$ for any f in $C(X)$.

Conversely if $\mu \in DM_\sigma^+(X)$ and μ_a is a net in $M_\sigma^+(X)$ and if $\int f d\mu_a \to \int f d\mu$ for each f in $C(X)$ then by remark III(4)

$$\limsup_a \sup_{f \in \mathcal{A}} \left|\int f d(\mu_a - \mu)\right| = 0$$

for any bounded equicontinuous $\mathcal{A}$ and in particular for the sequence of bounded equiuniformly continuous $\mathcal{A}_n \subset C(X)$. A metric $DM_\sigma^+(X)$ is for instance

$$d_0(\mu, \nu) = \sum_1^\infty \frac{1}{2^n}(d_n(\mu, \nu) \wedge 1).$$

Remarks (1). $\bigcup_{n=1}^\infty \mathcal{A}_n = \mathcal{A}$ contains only uniformly continuous functions and we have shown in fact more than stated, i.e.: if $\int f d\mu_a \to \int f d\mu$

for each $f \in \mathcal{A}$, then $\int f d\mu_a \to \int f d\mu$ for each $f \in C(X)$. In particular one gets Billingsley's result ([12], p. 252): If (X, d) is metric and $\int f d\mu_a \to \int f d\mu$ for each uniformly continuous bounded f then $\int f d\mu_a \to \int f d\mu$ for each $f \in C(X)$.

(2) If (X, d) is metric and not necessarily separable, then the above theorem implies that the set of $\mu \in M_\sigma^+(X)$ which "live" on closed separable subsets of X forms a metric space in the $\sigma(M_\sigma, C)$ topology.

(3) Let X be completely regular. Then $DM_\sigma^+(X) = M_\sigma^+(X)$ if and only if X is a D-space. In fact if X is not a D-space then by theorem 2(b) there is a $\mu \in M_\sigma^+(X)$ and a net $\mu_a \in M_\sigma^+(X)$ and a bounded equicontinuous $\mathcal{A} \subset C(X)$ such that $\int f d\mu_a \to \int f d\mu$ for $f \in C(X)$ and this convergence is not uniform on $\mathcal{A}$. Remark III(4) implies that $\mu \notin DM_\sigma^+(X)$, i.e. that $DM_\sigma^+(X) \neq M_\sigma^+(X)$.

COROLLARY 1. *Let (X, d) be metric.*

(a) *If (X, d) is a D-space then $M_\sigma^+(X)$ (with $\sigma(M_\sigma, C)$) is metric.*

(b) *If (X, d) is not a D-space then $M_\sigma^+(X)$ with $\sigma(M_\sigma, C)$ is not metric.*

Proof. We need to show only (b). Let $\mu \in M_\sigma^+(X)$. If $M_\sigma^+(X)$ would be metric (or only first countable), there would be a sequence μ_n of measures of type $\sum_1^n x_i p_{a_i}$, $x_i \geqslant 0$, $a_i \in X$ such that $\int f d\mu_n \to \int f d\mu$ for each f in $C(X)$. Let X_0 be the closure of the union of the supports of μ_n, $n = 1, 2, \ldots$ Then X_0 is separable closed and $\mu_n(X) = \mu_n(X_0)$ for all n. By [13], p. 56,

$$\mu(X) = \lim \mu_n(X) = \limsup \mu_n(X_0) \leqslant \mu(X_0) .$$

Hence

$$\mu(X_0) = \mu(X) \text{ and } \mu \in DM_\sigma^+(X), \quad \text{i.e.} \quad DM_\sigma^+(X) = M_\sigma^+(X)$$

which implies by the above remark that X is a D-space.

We consider now the relationship between the τ-smooth measures $M_\tau^+(X)$ of Varadarajan and $DM_\sigma^+(X)$.

DEFINITION (Varadarajan [13]). $\mu \in M_\sigma^+(X)$ is *τ-smooth* iff whenever f_a is a net in $C(X)$ such that $\|f_a\| \leqslant 1$ and $f_a(x) \downarrow 0$ for each x in X (i.e. if $a \geqslant \beta$, $f_a(x) \leqslant f_\beta(x)$, and $\lim_a f_a(x) = 0$ for each x in X), then $\lim_a \int f_a d\mu = 0$.

$$M_\tau^+(X) = \{\mu \in M_\sigma^+(X);\ \mu \text{ is } \tau\text{-smooth}\} .$$

It has been shown by Varadarajan ([13], p. 50 theorem 27 and the conclusion after it) that if (X, d) is metric then $\mu \in M_\tau^+(X)$ if and only if $\mu(X_0) = \mu(X)$ for some separable closed $X_0 \subset X$. Hence $M_\tau^+(X) \subset DM_\sigma^+(X)$ for metric X and hence

COROLLARY 2. *If X is metric so is $M_\tau^+(X)$ equipped with $\sigma(M_\sigma C)$ (since so is $DM_\sigma^+(X)$).*

This result is one of the deepest results obtained by Varadarajan in [13]. The above proof is much simpler (see [13], pp. 62-65).

The fact that we obtain that even $DM_\sigma^+(X)$ is metrisable should not mislead the reader to think that our result is stronger than that of Varadarajan. In fact:

LEMMA 2. *If (X, d) is metric then $DM_\sigma^+(X) = M_\tau^+(X)$ (i.e. if $\mu \in M_\sigma^+(X)$ has a support which is a D-subspace of X, then it already has a separable support).*

Proof. Let $\mu \in DM_\sigma^+(X)$. Let $X_1 \subset X$ be closed such that $\mu(X_1) = \mu(X)$ and (X_1, d) is a D-space. Let $Y \subset X_1$ be closed discrete and $e(x, y) = 1$ if $x \neq y$ and $e(x, x) = 0$ for x, y in Y. The function $e(x, y)$ is a continuous metric on the closed set Y (inducing on Y its original topology). A result of R. H. Bing (see [20]) implies that $e(x, y)$ can be extended to a metric on X_1, which induces on X_1 its original topology.

Since X_1 is a D-space and $Y \subset X_1$ is e-discrete, Y has cardinal of measure zero, i.e. if $\nu \in M_\sigma^+(Y)$ then $\nu(Y_0) = \nu(Y)$ for some countable $Y_0 \subset Y$. This shows that $M_\sigma(Y) = M_\tau(Y)$ for any closed discrete subset $Y \subset X_1$. Now μ can be considered as a Baire measure on the Baire sets of X_1 (since X_1 is closed, a Baire set of X_1 is a Baire set of X), i.e. $\mu \in M_\sigma(X_1)$. By Varadarajan [13], p. 51, theorem 28, $\mu \in M_\tau(X_1)$. Applying now [13], p. 50, theorem 27 and the conclusion after it we get that there is a separable closed $X_0 \subset X_1$ for which $\mu(X_0) = \mu(X_1) = \mu(X)$. Hence $\mu \in M_\tau^+(X)$ which finishes the proof.

Remark (4). The space of ordinals up to the first uncountable one provides an example of a locally compact X (of power $\aleph_1$) for which $DM_\sigma^+(X) \neq M_\tau^+(X)$. If X, f_a, μ are as in footnote 3, then $f_a(x) \downarrow 0$, $\int f_a d\mu = 1$ and $\mu \in M_\sigma^+(X)$. Thus $\mu \notin M_\tau^+(X)$ (this has been shown by Varadarajan [13]). Now X is a D-space and hence $M_\sigma^+(X) = DM_\sigma^+(X)$ and $\mu \in M_\sigma^+(X)$.

(b) Application to invariant means on topological groups. Let G be a topological group (always Hausdorff, but not necessarily locally compact) and $LUC(G) \subset C(G)$ be the space of left uniformly continuous functions, i.e. $f \in LUC(G)$ iff f is bounded and for any $\varepsilon > 0$ there is a neighborhood U of the identity $e \in G$ such that $|f(ug) - f(g)| < \varepsilon$ for any u in U and g in G. If $f \in C(G)$ and $a \in G$ define $f^a, f_a \in C(G)$ by

$$f^a(g) = f(ga) \quad \text{and} \quad f_a(g) = f(ag)$$

for any g in G.

If $\mu \in M_\sigma(G)$ and $f \in C(G)$ define

$$(L_\mu f)(g) = \int f^g d\mu = \int f(hg)\, d\mu(h) .$$

For instance, if $\mu = p_a$ for some a in G then

$$(L_{p_a} f) g = f(ag) , \quad \text{i.e.} \quad L_{p_a} f = f_a .$$

Denote by

$$P_\sigma(G) = \{\mu \in M^+_\sigma(G);\ \mu(G) = 1\} \quad \text{and} \quad DP_\sigma(G) = \{\mu \in DM^+_\sigma(G);\ \mu(G) = 1\}.$$

Hence $DP_\sigma(G) = P_\sigma(G)$ in case G is a D-space.

LEMMA 3. *Let G be a topological group. If $\mu \in DP_\sigma(G)$ then $L_\mu f \in \mathrm{LU}C(G)$ for any f in $\mathrm{LU}C(G)$.*

Proof. If $a \in G$ and $f \in \mathrm{LU}C(G)$, then $L_{p_a} f = f_a$ is in $\mathrm{LU}C(G)$ since if $\varepsilon > 0$ is given, there are neighborhoods of the identity U and V such that $|f(ug)-f(g)| < \varepsilon$ if $u \in U$ and $g \in G$ and $aV \subset Ua$. Hence if $v \in V$ then $|f_a(vg)-f_a(g)| < \varepsilon$. Consequently, if $\mu = \sum_{i=1}^{n} x_i p_{a_i}$ where $x_i > 0$, $\sum x_i = 1$ and $a_i \in G$ then $L_\mu f \in \mathrm{LU}C(G)$ for any $f \in \mathrm{LU}C(G)$.

If $\mu \in DP_\sigma(G)$ then there is a net $\mu_a \in P_\sigma(G)$ of measures of type $\sum x_i p_{a_i}$ such that $\int f d\mu \to \int f d\mu$ for any $f \in C(G)$ (see [13], p. 61). Hence $\int f d\mu_a \to \int f d\mu$ uniformly for f in $\mathcal{A}$ where $\mathcal{A} \subset C(G)$ is bounded and equicontinuous (remark III(4)). Let $f \in \mathrm{LU}C(G)$ and $\mathcal{A} = \{f^g;\ g \in G\}$. Thus $\mathcal{A}$ is clearly bounded and equicontinuous since for $\varepsilon > 0$ there is a neighborhood U of the identity such that

$$|f^g(uh) - f^g(h)| = |f(uhg) - f(hg)| < \varepsilon$$

for any u in U and h, g in G. Hence

$$L_{\mu_a} f(g) = \int f^g d\mu_a \to \int f^g d\mu = L_\mu f(g)$$

uniformly for g in G. Since $L_{\mu_a} f \in \mathrm{LU}C(G)$ and $\mathrm{LU}C(G)$ is norm closed, we get that $L_\mu f \in \mathrm{LU}C(G)$.

The following definition was suggested to us by a definition of A. Hulanicki [17].

DEFINITION. Let G be a topological group. Then

(1) $\varphi \in \mathrm{LU}C(G)^*$ is *left invariant* if $\varphi(f_a) = \varphi(f)$ for any $a \in G$ and $f \in \mathrm{LU}C(G)$.

(2) $\varphi \in \mathrm{LU}C(G)^*$ is *D-topologically left invariant* if $\varphi(L_\mu f) = \varphi(f)$ for any $\mu \in DP(G)$.

It is clear that (2) implies (1).

THEOREM 4. *Let G be a topological group. Then $\varphi \in \mathrm{LU}C(G)^*$ is left invariant if and only if φ is D-topologically left invariant.*

Proof. Let $\mu \in DP_\mu(X)$ and μ_a be a net of measures of type $\sum_{j=1}^{n} x_i p_{a_i}$ with $x_i \geq 0$, $\sum x_i = 1$ and $a_i \in G$ such that $\int f d\mu_a \to \int f d\mu$ for each f in $C(G)$.

Let $\varphi \in \mathrm{LU}C(G)^*$ be left invariant and $f \in \mathrm{LU}C(G)$. Then $L_{\mu_a} f$ is a finite linear combination of left translates of f and hence $\varphi(L_{\mu_a} f) = \varphi(f)$. Now

$$|\varphi(L_\mu f - f)| \leq |\varphi(L_\mu f - L_{\mu_a} f)| + |\varphi(L_{\mu_a} f - f)| = |\varphi(L_\mu f - L_{\mu_a} f)|$$
$$\leq \|\varphi\| \|L_\mu f - L_{\mu_a} f\| = \|\varphi\| \sup_s \left| \int f^s d(\mu_a - \mu) \right| < \varepsilon,$$

if a is big enough, since $\mathcal{A} = \{f^s;\ s \in G\}$ is bounded and equicontinuous (see remark III(4)). Thus $\varphi(L_\mu f) = \varphi(f)$ for any $\mu \in DP_\sigma(G)$ and $f \in \mathrm{LU}C(G)$.

Hulanicki considers in his definition of topological invariance only locally compact G's and measures μ which are regular with respect to compact subsets of G and have hence σ-compact support (such measures are always elements of $DP_\sigma(G)$). If $\varphi \in \mathrm{LU}C(G)^*$ is D-topologically left invariant in our sense, then φ is topologically left invariant in the sense of Hulanicki. Theorem 3 shows in particular that any left invariant $\varphi \in \mathrm{LU}C(G)^*$ is always topologically left invariant in the sense of Hulanicki. For example, if $G = R$ and $\varphi \in \mathrm{LU}C(R)^*$ satisfies $\varphi(f_a) = \varphi(f)$ for $f \in \mathrm{LU}C(R)$ and $a \in G$ then $\varphi\left[\int f(x+t) d\mu(t)\right] = \varphi f$ for any Borel probability measure μ on R (i.e. $\mu \in P_\sigma(R)$) and any bounded uniformly continuous f on R.

Question. Let $\varphi \in C(R)^*$ satisfy $\varphi(f) \geq 0$ if $f \geq 0$ and $\varphi(1) = 1$ and $\varphi(f_a) = \varphi(f)$ for $f \in C(R)$ and $a \in R$ (i.e. φ is an invariant mean on $C(R)$). Is it true that $\varphi\left[\int f(x+t) d\mu(t)\right] = \varphi(f)$ for any $f \in C(R)$? The answer seems to us to be negative. We do not know to prove it.

EXAMPLE. Let G_t, $t \in I$, be separable topological groups, and I any index set. Let $G = \prod G_t$ with the product topology. Then G is a D-space and any left invariant $\varphi \in \mathrm{LU}C(G)^*$ satisfies $\varphi\left[\int f(hg) d\mu(h)\right] = \varphi(f)$ for any *Baire probability measure* μ on G. It seems enlightening to take $G_t = Z_t$ as copies of the discrete additive integers and I a set of *any* cardinality.

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Sequents in many valued logic I

by

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The calculus of sequents for two-valued predicate logic is well known, either in the original formulation of Gentzen [1] or in any of the many variants in the literature. In this paper we show that there exists an analogous calculus for each finitely many valued predicate logic based on arbitrary connectives $F_1, \ldots, F_u$ and quantifiers $Q_1, \ldots, Q_w$.

1. Propositional calculus. Let $\boldsymbol{M} = \{0, 1, \ldots, M-1\}$ be the set of *truth-values* and for each k ($k = 1, \ldots, u$) let f_k be a *truth-function* of $r = r_k \geqslant 1$ arguments, i.e., a mapping of $\boldsymbol{M}\times\ldots\times\boldsymbol{M} = \boldsymbol{M}^r$ into $\boldsymbol{M}$. Let $\mathfrak{A}$ and $\{F_1, \ldots, F_u\}$ be disjoint sets of symbols, the elements of which are called *atomic statements* and *connectives* respectively. The set $\mathfrak{S}$ of *statements* is the smallest set of expressions which contains all atomic statements and which, for each connective F_k, contains $F_k\,\alpha_1\ldots\alpha_r$ whenever it contains $\alpha_1, \ldots, \alpha_r$. The *degree* of a statement is the number of occurrences of connectives in it. We will denote statements by the letters $\alpha, \beta, \gamma, \ldots$ and finite (possibly null) sequences of statements by $\Gamma, \Delta, \ldots$; in particular, the null sequence will be denoted by Λ.

A sequent is an expression of the form

$$\Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{M-2} \mid \Gamma_{M-1}\,. \tag{1}$$

We denote sequents by the letters $\Pi, \Sigma, \ldots$. If Π is the sequent (1) and Σ is the sequent

$$\Delta_0 \mid \Delta_1 \mid \ldots \mid \Delta_{M-2} \mid \Delta_{M-1}\,,$$

then $\Pi\Sigma$ will denote the sequent

$$\Gamma_0\Delta_0 \mid \Gamma_1\Delta_1 \mid \ldots \mid \Gamma_{M-2}\Delta_{M-2} \mid \Gamma_{M-1}\Delta_{M-1}\,.$$

If R is a subset of $\boldsymbol{M}$ then the sequent (1) may be written $|\Gamma|_R$ (or $|\Gamma|_m$ if $R = \{m\}$) provided

$$\Gamma_i = \begin{cases}\Gamma & \text{if} \quad i \in R\,,\\ \Lambda & \text{if} \quad i \notin R\,.\end{cases}$$