Cones, products and fixed points

by

Ronald J. Knill* (New Orleans, Louisiana)

In 1930 Kuratowski asked [8] (1), *Si le continu X, ainsi que Y, possède la propriété que dans chaque transformation continu de ce continu en sous-ensemble il existe un point invariant, est-il vrai que $X \times Y$ possède la même propriété?* In setting the problem Kuratowski specified that X and Y be peanian continua. If one replaces the hypothesis peanian with one of contractibility, then the answer is *no*. If X is a contractible continuum the problem may be attacked through the topological cone $C(X)$, which is defined by identifying to a point the subset $X \times \{1\}$ of the product space of X with the closed unit interval I. By 3.1 the cone has the fixed point property if and only if the product $X \times I$ has the fixed point property. To show the negative answer to the modified Kuratowski problem, we consider in section 3 a contractible continuum B which has the fixed point property. However $C(B)$ has a retract, $C(B^0)$, which according to 2.6 does not have the fixed point property, hence neither does $C(B)$ nor $B \times I$. One may describe B^0 as the set of points of the closed unit disc in the complex plane, together with the points of the spiral $\{(1+2^{1-m})e^{\pi i m}\colon m \geqslant 0\}$. For a discussion of B^0 from another point of view, see [5].

We also direct ourselves to another question of fixed point theory for which B^0 serves as the basis of investigation. The question is of the relationship of higher order local connectedness and dimension to the fixed point property. Specifically one may ask:

(Q) *If X is a compact acyclic LC^{n-1} space of dimension $n+1$, where n is a natural number, does X in general have the fixed point property?*

If one changes the dimension condition to: dimension of X is less than $n+1$, the answer is *yes*, 2.7; and if dimension of X is greater than $n+1$, the answer for $n = 1$ is seen to be *no* by [2], and it is also *no* for $n > 1$, by statement (A), given below. If n were specified to be 0 in (Q) the question would become the well-known open question:

* Supported in part by National Science Foundation Grant No. GP3989.

(1) We are indebted to A. Lelek for this reference.

Do tree-like continua have the fixed point property?

Although we do not offer an answer to either (Q) or the latter question, we do consider a question analogous to (Q) for cones, showing, 2.6 and 2.7, that the following is true.

(A) *Suppose that n is a natural number and X is a compact acylic* LC^{n-1} *space. If the dimension of X is less than $n+1$, then the cone, $C(X)$, has the fixed point property; but if the dimension of X is not less than $n+1$, $C(X)$ need not have the fixed point property, even if X itself has the fixed point property.*

Thus even if the answer to (Q) is affirmative, little can be concluded about the fixed point properties of the cones over the spaces with which (Q) concerns itself. It still is an open question whether or not the cone over a tree-like continuum in general has the fixed point property. We conclude the paper with a discussion of a question of Bing's, section 4.

We wish to acknowledge the helpful comments of Mr. Jerome Dancis on a rough draft of this paper.

1. We will work entirely in the Hilbert space H of all square summable sequences $a=(a_0, a_1, a_2, \ldots)$. The norm is the usual one,

$$|a| = \Big(\sum_i (a_i)^2\Big)^{1/2},$$

and addition and scalar multiplication are coordinatewise. The *distance* between two points $a, b \in H$ is $|a-b|$. Let H_0 be the subspace of H of all sequences with null zeroth term, and let $v \in H \backslash H_0$ be the sequence

$$v = (1, 0, 0, \ldots).$$

For $p = 1, 2, \ldots$, let E^p consist of those points of H_0 with null nth terms, $n > p$. Henceforth a point a of E^p, $p \geqslant 2$, will be denoted by its cylindrical coordinates:

$$a = \langle 0, r_a, m_a, a_3, a_4, \ldots, a_p\rangle$$

where

$$r_a = (a_1^2 + a_2^2)^{1/2}$$

and

$$a_1 = r_a \cos m_a, \qquad a_2 = r_a \sin m_a.$$

If $0 \leqslant t \leqslant 1$ write $tv+(1-t)a$ as

$$tv+(1-t)a = \langle t; a\rangle,$$

or as

$$tv+(1-t)a = \langle t, r_a, m_a, a_3, a_4, \ldots\rangle,$$

and if X is a subset of H_0, let

$$C(v; X) = \{\langle t; x\rangle:\ x \in X,\ 0 \leqslant t \leqslant 1\}.$$

For compact X, $C(v; X)$ is homeomorphic with the cone $C(X)$, although for noncompact X this is not the case.

1.1. DEFINITION. The *canonical deformation*

$$d_s\colon H_0 \to H_0, \qquad -\infty < s < \infty$$

is defined by the rules

(i) *if* $r_a \leqslant 1$, *then*

$$d_s(a) = \langle 0, r_a, m_a+\pi s, 2^{-s}a_3, 2^{-s}a_4, \ldots\rangle;$$

(ii) *if* $1 \leqslant r_a$, *then*

$$d_s(a) = \langle 0, 1+(r_a-1)2^{-s}, m_a+\pi s, 2^{-s}a_3, 2^{-s}a_4, \ldots\rangle.$$

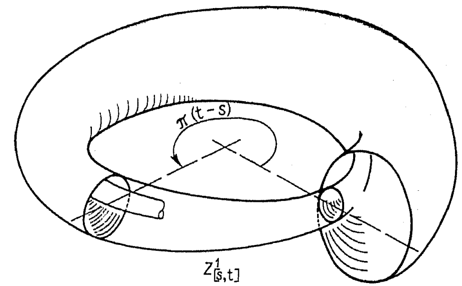

Fig. 1

We may denote $d_s(a)$ by $d(a, s)$.

1.2. Let $D = \{a \in E^2\colon r_a \leqslant 1\}$, $S^1 = \{a \in E^2\colon r_a = 1\}$. Let $c = \langle 0, 2, 0\rangle$ in E^2, let Z^n be the n-sphere:

$$Z^n = \{a \in E^{n+2}\colon m_a = 0,\ |a-c| = 1\},$$

and let Y^n be the solid sphere:

$$Y^n = \{a \in E^{n+2}\colon m_a = 0,\ |a-c| \leqslant 1\},$$

where $n = 0, 1, 2, \ldots$ Then for any real s, $d_s(Z^n)$ (resp., $d_s(Y^n)$) is a sphere (solid sphere) of radius 2^{-s} which meets D in the point $\langle 0, 1, \pi s\rangle \in S^1$. Note also that if $t \equiv s \pmod 2$ then $d_t(Z^n)$ is tangent to $d_s(Z^n)$ at $\langle 0, 1, \pi s\rangle$. For any real s, let $Z^n_s = d_s(Z^n)$, $Y^n_s = d_s(Y^n)$ and if J is an interval in the real line let

$$Y^n_J = \bigcup_{s\in J} Y^n_s, \qquad \text{and} \qquad Z^n_J = \bigcup_{s \in J} Z^n_s.$$

1.3. DEFINITION. For $n = 0, 1, \ldots$ let B^n be defined as

$$B^n = D \cup Z^n_{[0,\infty)}.$$

2. The properties of B^n.

2.1. Theorem. *Each B^n is compact and acyclic.*

Proof. By acyclic we mean that B^n has the Čech cohomology of a point. But

$$B^n = \bigcap_{s\geqslant 0} (D \cup Z^n_{[0,\infty]} \cup Y^n_{[s,\infty)})$$

and each of the spaces in the given family is evidently contractible, e.g., d_t, $0 \leqslant t \leqslant s$, deforms $D \cup Z^n_{[0,\infty)} \cup Y^n_{[s,\infty)}$ into $D \cup Y^n_{[s,\infty)}$ and the later is contractible. Then by the continuity axiom for Čech cohomology theory, B^n is acyclic.

2.2. Theorem. *For $n \geqslant 1$, B^n is LC^{n-1} at points of S^1, and locally contractible elsewhere.*

Proof. In fact, $D\backslash S^1$ is the open unit disk and $B^n\backslash D$ is readily seen to be homeomorphic with a closed half space in E^{n+1}, so B^n is locally contractible on $B^n\backslash S^1$. For a point x of S^1, the set

$$N^n_x = \bigcup \{Z^n_{m_x+2k}\colon\ k \geqslant 1,\ k \text{ an integer}\}$$

is a union of n-spheres tangent to x, and every sufficiently small neighborhood of x has the same homotopy type as N^n_x, so B^n is LC^{n-1} at x.

2.3. Theorem. *For $n = 0, 1, 2, \ldots$, B^n has the fixed point property.*

Proof. Suppose that $f\colon B^n \to B^n$, $n \geqslant 0$, is a continuous map. For $n = 0$, either f maps D into itself or into the other path component, $B^0\backslash D$, of B^0. In the former case f has a fixed point in D. In the latter, $B^0\backslash D$ is open so f maps all of B^0 into it, and so $f(B^0)$ is a closed connected subset J of $B^0\backslash D$. Such a subset of $B^0\backslash D$ is topologically an interval, so f has a fixed point in J.

For $n \geqslant 1$, suppose that f has no fixed points, and let p be the retraction of B^n onto D which, for $a \notin D$, is defined by $p(a) = \langle 0, 1, m_a\rangle$. Then $f(S^1)$ is not a subset of D, for otherwise pf would have a fixed point in D which would be a fixed point of f. So let $x_0 = \langle 0, 1, m_0\rangle$ be a point of S^1 such that $f(x_0)$ is not an element of D. Then choose a neighborhood N of $f(x_0)$ and positive numbers m, ε so that

(1) $m \equiv m_0 \pmod 2$ *and* $2^{1-m} < \inf_{x\in B^n} |f(x) - x|$;

(2) *N is an absolute retract, and*

$$f(Z^n_{[m,m+\varepsilon]}) \subset N, \quad \text{and} \quad Z^n_{[m,m+\varepsilon]} \cap N = \emptyset .$$

Let X be the contractible polyhedron defined by

$$X = D \cup Z^n_{[0,m]} \cup Y^n_m$$

and define maps

$$X \xrightarrow{f'} B^n \xrightarrow{q} X$$

as follows: For $a \in D \cup Z^n_{[0,m]}$, let $f'(a) = f(a)$ and let $q(a) = a$. For $a \in Z^n_{[m+\varepsilon,\infty)}$, let $q(a) = p(a)$. Since $f'(Z^n_m) \subset N$, and N is an AR, we may extend f' so that $f'(Y^n_m) \subset N$. Likewise we may extend q so that $q(Z^n_{[m,m+\varepsilon]})$ is a subset of $Y^n_m \cup (S^1 \cap Z^n_{[m,m+\varepsilon]})$. Now the composition

$$B^n \xrightarrow{f'q} B^n$$

has no fixed points, since:

(4) If $a \in B^n \cap X$, then $f'q(a) = f(a)$.

(5) If $a \in Z^n_{[m,m+\varepsilon]}$, then $f'q(a) \in N$, which is disjoint from $Z^n_{[m,m+\varepsilon]}$.

(6) If $a \in Z^n_{[m+\varepsilon,m]}$, then $f'q(a) = fp(a)$, and from (1),

$$|f(p(a)) - p(a)| > |p(a) - a|, \quad \text{so} \quad |f'q(a) - a| > 0 .$$

As a consequence, $gf'\colon X \to X$ has no fixed points either, contrary to the Lefschetz fixed point theorem. By contradiction, the theorem is proven.

2.4. Definition. For $\langle t; a\rangle \in C(v; H_0)$, let

$$a^- = \langle 0, r_a, m_a, -a_3, -a_4, -a_5, \ldots\rangle$$

$$\langle t; a\rangle^- = \langle t; a^-\rangle .$$

If X is a subset of E^n but *not of* E^{n-1}, $n \geqslant 3$, let

$$X^+ = \{a \in X\colon a_n \geqslant 0\}, \qquad X^- = \{a \in X\colon a_n \leqslant 0\} .$$

2.5. Theorem. *For $n \geqslant 1$, B^n is the union of two compact absolute retracts, $(B^n)^+$ and $(B^n)^-$.*

Proof. One need only to show that $(B^n)^+$ and $(B^n)^-$ are locally contractible [8] and contractible. We do this only for $(B^n)^+$ since the map, $a \to a^-$, takes $(B^n)^+$ homeomorphically onto $(B^n)^-$. Evidently $(B^n)^+$ is locally contractible at points of $(B^n)^+\backslash S^1$. To see that it is locally contractible at points of S^1 and contractible, it suffices to provide a strong deformation retraction

$$k_t\colon (B^n)^+\backslash\{0\} \to (B^n)^+ \qquad 0 \leqslant t \leqslant 1$$

of $(B^n)^+\backslash\{0\}$ onto S^1, and to note that S^1 is locally contractible. Since $(Z^n)^+$ is an n-cell, there is a contraction

$$h_t\colon (Z^n)^+ \to (Z^n)^+, \qquad 0 \leqslant t \leqslant 1$$

such that $h_t\langle 0, 1, 0\rangle = \langle 0, 1, 0\rangle$ for $0 \leqslant t \leqslant 1$. Then define

$$k_t\colon (B^n)^+\backslash\{0\} \to (B^n)^+, \qquad 0 \leqslant t \leqslant 1$$

by letting

$$k_t(a) = \langle 0, t + (1-t)r_a, m_a\rangle$$

if $a \in D\backslash\{0\}$, and letting

$$k_t(a) = d_s h_t d_{-s}(a)$$

if $a \in (Z^n_s)^+$, $0 \leqslant s$. Then for $a \in (B^n)^+\backslash\{0\}$, $k_0(a) = a$, $k_1(a) \in S^1$, and for $a \in S^1$, $k_t(a) = a$ for $0 \leqslant t \leqslant 1$, so the theorem is proven.

2.6. THEOREM. *For each $n \geqslant 0$ there is a map of $C(v; B^n)$ into itself which has no fixed points.*

Proof. First note that since $C(v; S^1)$ is a retract of $C(v; D)$, then also $C(v; Z^n_{[0,\infty)})$ is a retract of $C(v; B^n)$ for $n = 0, 1, 2, \ldots$; so it suffices to define a fixed point free map g^n of $C(v; Z^n_{[0,\infty)})$ into itself for each n. This will be done inductively so that g^n agrees with g^{n-1} on $C(v; Z^{n-1}_{[0,\infty)})$ for $n > 0$.

First define a map

$$g\colon \bigcup_{n=0}^{\infty} C(v; Z^n_{[1,\infty)}) \to \bigcup_{n=0}^{\infty} C(v; Z^n_{[0,\infty)}),$$

so that g maps $C\big(v; (Z^n_{[1,\infty)})^+\big)$ into $C\big(v; (Z^n_{[0,\infty)})^-\big)$ for each $n \geqslant 1$, so that $g(x)^- = g(x^-)$ for each x, and g has no fixed points, as follows. For $a \in Z^n_{[1,\infty)}$, and any $n \geqslant 0$, let

1. $g(\langle t; a\rangle) = \langle 0; d(a^-, 1-8t)\rangle, \quad 0 \leqslant t \leqslant \frac{1}{4},$
2. $g(\langle t; a\rangle) = \langle 4t-1; d(a^-, -1)\rangle, \quad \frac{1}{4} \leqslant t \leqslant \frac{1}{2},$
3. $g(\langle t; a\rangle) = \langle 3-4t; e\rangle, \quad \frac{1}{2} \leqslant t \leqslant \frac{3}{4},$
4. $g(\langle t; a\rangle) = e, \quad \frac{3}{4} \leqslant t \leqslant 1,$

where e is $\langle 0, 3, 0\rangle$, the endpoint of the spiral $Z^0_{[0,\infty)}\backslash S^1$. Then g has no fixed points since g either changes the modulus m_x of x or its distance from v.

Define g^n by induction on $n = 0, 1, 2, \ldots$ so that it satisfies

(i) g^n agrees with g on $C(v; Z^n_{[1,\infty)})$, and if $n \geqslant 1$, g^n agrees with g^{n-1} on $C(v; Z^{n-1}_{[0,\infty)})$.

(ii) $g^n\Big(C\big(v; (Z^n_{[0,\infty)})^+\big)\Big) \subset \Big(C\big(v; (Z^n_{[0,\infty)})^-\big)\Big)$, for $n \geqslant 1$.

(iii) $g^n(x^-) = \big(g^n(x)\big)^-$ for $x \in C(v; B^n)$, $n \geqslant 1$.

(iv) g^n has no fixed points.

For $n = 0$, let g^0 agree with g on $C(v; Z^0_{[1,\infty)})$. To complete the definition of g^0, note that it suffices to define it on

$$K = \bigcup_{0 \leqslant t \leqslant 1} C\big(v; d_t(e)\big).$$

Regard K as a simplicial complex with vertices $u_0 = d_1(e)$, $u_1 = \langle 1/4, d_1(e)\rangle$, $u_2 = \langle 1/2, d_1(e)\rangle$, $u_3 = \langle 3/4, d_1(e)\rangle$, $u_4 = v$, $u_5 = \langle 3/4, e\rangle$, and $u_6 = e$. The two dimensional simplexes of K are $(u_0\, u_5\, u_6)$ together with $(u_i\, u_{i+1}\, u_5)$,

$i = 0, 1, 2, 3$. Here, "$(a\, b \ldots)$" means "the closed simplex spanned by $a, b, \ldots$". If we let $L = C\big(v; d_1(e)\big)$, then L is a subcomplex of K determined by the simplexes $(u_i\, u_{i+1})$, where $i = 0, 1, 2, 3$, and $g|L$ is defined. It may be regarded as a simplicial map into a complex, M, which is determined by simplexes $(u_4\, u_6)$ and $(u_6\, u_7)$, where $u_7 = d_2(e)$. So extend g^0 to all of $C(v; Z^0_{[0,\infty)})$ by letting it agree on K with the simplicial map into M which takes u_5 into u_6, and u_6 into u_7, and is as previously specified on u_0, u_1, u_2, u_3, and u_4. By definition g^0 satisfies (i) and it satisfies (ii) and (iii) vacuously. As for (iv), g^0 has no fixed points on $C(v; Z^0_{[1,\infty)})$ since g does not, and g^0 has none elsewhere, i.e. in K, for $g(K) \subset M$, and on $M \cap K$, $g(u_0\, u_6) = u_7$ which is not an element of $(u_0\, u_6)$, $g(u_5\, u_6) = (u_6\, u_4)$ which meets $(u_5\, u_6)$ only in u_6, and $g(u_4\, u_5) = u_6$.

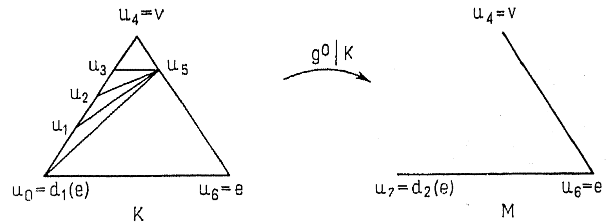

Fig. 2

Assuming g^{n-1} has been defined for some $n \geqslant 1$ so as to satisfy (i)-(iv), define g^n so that it agrees with g^{n-1} and g where these are defined. Then g^n maps

$$C(v; Z^{n-1}_{[0,1]}) \cup C\big(v; (Z^n_{[1,\infty)})^+\big) \text{ into } C\big(v; (Z^n_{[0,\infty)})^-\big).$$

By 2.5, $C\big(v; (B^n)^-\big)$ is an absolute retract and since $C\big(v; (Z^n_{[0,\infty)})^-\big)$ is a retract of $C\big(v; (B^n)^-\big)$, it, too, is an absolute retract. Thus g^n may be extended to $C\big(v; (Z^n_{[0,\infty)})^+\big)$ so as to satisfy (ii). The definition of g^n is then completed by requiring it to satisfy (iii). Then g^n satisfies (iv) for it maps $C\big(v; (Z^n_{[0,\infty)})^+\big)$ into $C\big(v; (Z^n_{[0,\infty)})^-\big)$, and *vice versa*, and cannot have any fixed points outside of the intersection of these two sets, $C(v; Z^{n-1}_{[0,\infty)})$. Since by (i), g^n has no fixed points in $C(v; Z^{n-1}_{[0,\infty)})$, it follows that g^n has no fixed points so (iv) holds for g^n.

2.7. THEOREM. *If an $X \subset H_0$ is either zero dimensional or is compact acyclic, of covering dimension n, $n \geqslant 1$, and* LC^{n-1}, *then the cone with base X has the fixed point property.*

Proof. For X zero dimensional this is easy to see. Suppose that f is a continuous map of $C(v; X)$ into itself. Then $f(v) = \langle t; x_0\rangle$ for some point x_0 of X. Let t_0 be the minimum of the values of t for which

$$f(\langle t; x_0\rangle) = \langle t'; x_0\rangle$$

for some $t' \leqslant t$. If $t'_0 = 1$ then $f(v) = v$. If $t'_0 < 1$ then since X is totally disconnected, there is an $\varepsilon > 0$ such that for

$$t_0 - \varepsilon \leqslant t \leqslant t_0,$$

we have

$$f(t; x_0) = \langle t'; x_0\rangle,$$

for some t'. By the definition of t_0, $t' \geqslant t$ if $t_0 - \varepsilon \leqslant t \leqslant t_0$. By the continuity of f, then, $t'_0 = t_0$, so $\langle t_0; x_0\rangle$ is a fixed point of f.

For X compact acyclic, of dimension $n \geqslant 1$, and LC^{n-1}, X is an HLC^* space ([9], [10]): it is HLC^{n-1} as an LC^{n-1} space, and since it is acyclic and has covering dimension n, the n dimensional Čech homology groups of all compact subsets vanish, so X is HLC^n. The dimension condition then gives that X is HLC^*, hence so also is $C(v; X)$. By the Lefschetz-Begle theorem ([1], [9]), $C(v; X)$ has the fixed point property.

3. *A compact contractible space B such that B, but not $B \times I$, has the fixed point property.*

In outline one shows, 3.1, that for compact contractible space X, $X \times I$ has the fixed point property iff $C(X)$ does. Then one defined B so that $C(v; B^0)$ is a retract of $C(v; B)$; then $C(v; B)$ and $B \times I$ do not have the fixed point property. We conclude with the proof that B has the fixed point property.

3.1. THEOREM. *If X is a compact contractible space then the topological cone $C(X)$, with base X, has the fixed point property if and only if $X \times I$ has the fixed point property.*

Proof. $C(X)$ is the quotient space of $X \times I$ formed by identifying all of $X \times \{1\}$ to a point, p. For $0 \leqslant t < 1$, it is convenient to regard $X \times [0, t]$ as a subspace of $C(X)$. Since X is contractible, there is a map r: $C(X) \to X$ such that $r(x, 0) = x$ for each x in X. Define a retraction r' of $C(X)$ onto $X \times [0, \frac{1}{2}]$ by letting $r'(p) = (r(p), \frac{1}{2})$, and for $(x, t) \in X \times [0, 1)$, letting

$$r'(x, t) = \begin{cases} (x, t) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ (r(x, 2t-1), \frac{1}{2}) & \text{if} \quad \frac{1}{2} \leqslant t < 1. \end{cases}$$

Since $X \times I$ is homeomorphic with a retract of $C(X)$, then if $C(X)$ has the fixed point property, so does $X \times I$. Conversely if $C(X)$ does not have the fixed point property and f: $C(X) \to C(X)$ has no fixed points then there is a $t_1 < 1$ such that $f(p) = (x_1, t_1)$. Since X is compact, there is a $t_0, t_1 \leqslant t_0 < 1$, such that $f(x, t) \in X \times [0, t_0]$ for $(x, t) \in X \times [t_0, 1)$. It is no restriction to assume that $t_0 = \frac{1}{2}$. Then define a map f' on $X \times I$ by letting $f'(x, 1) = f(p)$, $f'(x, t) = f(x, t)$ for $\frac{1}{2} \leqslant t < 1$, and $f'(x, t) = r'f(x, t)$ for $0 \leqslant t \leqslant \frac{1}{2}$, where $x \in X$. Then f' has no fixed points, for if $0 \leqslant t < \frac{1}{2}$, then $f'(x, t)$ is either $f(x, t)$ or is in $X \times \{\frac{1}{2}\}$, and if $\frac{1}{2} \leqslant t < 1$, $f'(x, t) = f(x, t)$, and if $t = 1$, $f'(x, 1) = (x_1, t_1)$.

3.2. DEFINITION. Let R be the set of points in E^3 of the form

$$a = \langle 0, r, M, (2-r)2^{-M}\rangle$$

where $1 \leqslant r \leqslant 2$, and $1 \leqslant M$. This value of M will be denoted by M_a. Let $2D$ (respectively, $2S^1$) be the set of points a in E^2 with $r_a \leqslant 2$ (respectively, $r_a = 2$). Define the set B in E^3 by letting

$$B = R \cup 2D.$$

Evidently the closure of R is in B so B is compact.

PROPOSITION. *The set B is contractible in itself.*

Proof. Since $2D$ is contractible, it suffices to define a homotopy

$$h_t\colon B \to B, \quad 0 \leqslant t \leqslant 1,$$

such that h_0 is the identity of B and $h_1(B) \subset 2D$. For a point $\langle 0, r, m, w\rangle$ of B, $0 \leqslant r \leqslant 1$, and for $0 \leqslant t \leqslant 1$ let

$$h_t(\langle 0, r, m, w\rangle) = \langle 0, (t+1)r, m, w\rangle,$$

and let

$$h_t(\langle 0, r, m, w\rangle) = \langle 0, 2t+(1-t)r, m, (1-t)w\rangle,$$

if $1 \leqslant r \leqslant 2$. One readily checks that h_t is appropriately defined to complete the proof.

3.4. THEOREM. *Neither the cone $C(v; B)$ nor the product space $B \times [0, 1]$ have the fixed point property.*

Proof. By 3.1 and 3.3 it suffices to show that $C(v; B)$ does not have the fixed point property.

Note that the set

$$B^{0\prime} = \{a \in B\colon r_a \leqslant 1\}$$

is homeomorphic to B^0. By 2.6, $C(v; B^0)$ does not have the fixed point property, so it suffices to show that $C(v; B^{0\prime})$ is a retract of $C(v; B)$. Such a retraction is the map which is the identity on $C(v; B^{0\prime})$, which takes $C(v; 2S^1)$ into the vertex v, and is extended linearly, that is, it takes points of $C(v; B)$ of the form $\langle t, r, m, w\rangle$, $1 < r < 2$, onto points of the form: $\langle (2-r)t+r-1, 1, m, w/(2-r)\rangle$.

3.5. THEOREM. *Every continuous function f of B into itself has a fixed point.*

Proof. Suppose that f has no fixed points. Recall the homotopy h_t: $B \to B$, $0 \leqslant t \leqslant 1$, defined in the proof of 3.3. We claim that there is an arc J (= a set homeomorphic to I) in $2D$ from a point a^1 of S^1 to a point a^2 of $2S^1$ such that

(i) $J \cap D = \{a^1\}$ and $J \cap 2S^1 = \{a^2\}$, and

(ii) for every $a \in J$ and $0 \leqslant t \leqslant 1$, $a \neq h_t f(a)$.

Indeed let F be the closed set

$$F = \{a \in B\colon a = h_t f(a) \text{ for some } t,\ 0 \leqslant t \leqslant 1\}.$$

Note $h_t(0) = 0$ for all t, $0 \leqslant t \leqslant 1$, so $0 \notin F$. The existence of J is proven if we can show that the origin, 0 is in the unbounded component of the complement of F in E^2, or equivalently, $F \cap E^2$ is contractible to a point in $E^2 \setminus \{0\}$. To define a contraction

$$k_t\colon F \cap E^2 \to E^2 \setminus \{0\}, \qquad 0 \leqslant t \leqslant 1,$$

define first a fixed point free map

$$g\colon 2D \to E^2$$

by letting $g(a) = f(a)$ if $a \in 2D \cap f^{-1}(2D)$, and letting

$$g(a) = \langle 0, f(a)_3 + 2, M_{f(a)} \rangle$$

if $a \in 2D \cap f^{-1}(R)$. Then g is readily seen to be well defined, and g is continuous for $2D \cap f^{-1}(2D)$ and $2D \cap f^{-1}(R)$ are closed sets (the latter being closed since $f(2D)$ is locally connected, so meets R in a closed set). Now define k_t by letting

$$k_t(a) = \left(1 - t + \frac{t r_a}{r_a - r_{f(a)}}\right)\big(ta - g(ta)\big)$$

for $a \in F \cap E^2$ and $0 \leqslant t \leqslant 1$. By the definition of F, f maps each point a of $F \cap E^2$ radially inward so $r_a - r_{f(a)} > 0$ and $k_t(a)$ is well defined. Moreover,

$$k_1(a) = \left(\frac{r_a}{r_a - r_{f(a)}}\right)\big(a - f(a)\big) = a.$$

Since f has no fixed points neither does g, so $k_t(a) \neq 0$ for all t, a. Thus k is a contraction in $E^2 \setminus \{0\}$ of $F \cap E^2$ to a point (the point is $-g(0)$), and the existence of J is proven. Let A be the annulus of all points $a \in E^2$ with $1 \leqslant r_a \leqslant 2$.

Then evidently there is a retraction

$$p\colon A \to J \cup 2S^1.$$

Define a retraction

$$p'\colon A \cup R \to J \cup 2S^1$$

by letting

$$p'(a) = p(\langle 0, r_a, M_a, 0\rangle) \quad \text{for} \quad a \in A \cup R.$$

We claim that there is a number $M > 1$ that satisfies the two conditions:

(iii) For $0 \leqslant t \leqslant 1$ and every $a \in 2S^1$ such that $f(a) \in R$ and $M_{f(a)} > M$, we have

$$p' h_t f(a) \neq a.$$

(iv) For every $a \in R$ such that $\langle 0, r_a, m_a, 0\rangle \in J$ and $M_a > M$, a is not in F.

To see (iii), suppose that

$$\varepsilon = \min\{|f(a) - a|\colon a \in B\}.$$

Since B is compact, ε is positive. Since p' is the identity on $2S^1$, we may choose a number δ with $0 < \delta < \varepsilon/2$ such that if the distance from a point a of $A \cup R$ to $2S^1$ is less than δ, then $|p'(a) - a| < \varepsilon/2$. Since $f(2S^1)$ is locally connected, there is an $M_1 > 1$ so large that if $a \in 2S^1$ and $f(a) \in R$ with $M_{f(a)} > M_1$, then the distance from $f(a)$ to $2S^1$ is less than δ. Then for $0 \leqslant t \leqslant 1$, the distance from $h_t f(a)$ to either $f(a)$ or $2S^1$ is less than δ, so

$$|p' h_t f(a) - f(a)| \leqslant |p' h_t f(a) - h_t f(a)| + |h_t f(a) - f(a)| < \varepsilon/2 + \delta < \varepsilon.$$

Since the distance from a to $f(a)$ is at least ε, then $p' h_t f(a) \neq a$, so any $M \geqslant M_1$ would satisfy (iii). To choose M so that it also satisfies (iv), one need only to require that 2^{-M} is less than the distance from J to F.

Since J is an arc, there is a continuous real valued function u on J such that $u(a) \equiv m_a \pmod 2$ and $u(a) > M$ for $a \in J$. Define a map

$$q\colon J \cup 2S^1 \to R$$

by letting q be the identity on $2S^1$, and letting

$$q(a) = \langle 0, r_a, m_a, (2 - r_a)2^{-u(a)}\rangle,$$

for $a \in J$. Then q is a homeomorphism of $J \cup 2S^1$ onto a subset L of R and $R \setminus L$ has two components. Let R' be the component of $R \setminus L$ which contains $\langle 0, 1, 1, 1/2\rangle$, and let $K = R' \cup J$. Then the interior of K is a topological open 2-cell and the set

$$X = K \cup 2D$$

is a contractible polyhedron. Note from (iv) that $q(J)$ is disjoint from F, so for $x \in X$ we may define a number $d(x)$ as the least of the number 1 and of the quotient of the distance from x to $q(J)$ by the distance from $q(J)$ to F. Now define a continuous transformation f' of X into itself by letting

1. $f'(x) = h_{1-d(x)} f(x)$, if $h_{1-d(x)} f(x)$ is contained in X,
2. $f'(x) = q p' h_{1-d(x)} f(x)$, otherwise.

Then f' has no fixed points: In case 1, if $d(x) = 1$ then $f'(x) = f(x) \neq x$, and if $0 \leqslant d(x) < 1$ then $x \notin F$ so $f'(x) \neq x$. In case 2 we have $f'(x) \in L = q(J) \cup 2S^1$ and

$$M_{f(x)} = M_{h_t f(x)} > M,$$

since $h_{1-d(x)}f(x) \notin X$. If $x \in q(J)$, then $d(x) = 0$ so $f'(x) = h_1 f(x) \neq x$. If $x \in 2S^1$ then (iii) and the fact that q is the identity on $2S^1$ imply that $f'(x) \neq x$.

This is contrary to Lefschetz's theorem to the effect that every continuous transformation of a contractible polyhedron into itself has a fixed point. The assumption that f had no fixed points was false and B has the fixed point property.

4. This paragraph is to provide an answer to a question of Bing's posed privately. When apprised of B and its pathologies, he asked:

May one attach a three cell A to B so that $B \cup A$ does not have the fixed point property?

The answer is *yes*: Let

$$A = \{a \in E^3:\ 0 \leqslant r_a \leqslant 2,\ -1 \leqslant a_3 \leqslant 0\}.$$

Then $B \cup A$ does not have the fixed point property. We only sketch the argument. Consider the equivalence relation $\sim$ generated on $C(v; B^0)$ by the identification of $\langle t; d_s(e)\rangle$ with $\langle t, 1, \pi s\rangle$, for $\frac{3}{4} \leqslant t \leqslant 1$ and $0 \leqslant s$. Then the quotient space $C(v; B^0)/\sim$ is homeomorphic with $B \cup A$, with a homeomorphism that maps $C(v; D)$ onto A, and the remaining points into B. Note that if $a \in Z^0_{[0,\infty)}$ and $\frac{3}{4} \leqslant t \leqslant 1$, then $g^0(\langle t; a\rangle) = e$ so g^0 induces a fixed point free map of $C(v; Z^0_{[0,\infty)})/\sim$ into itself. The later is a retract of $C(v; B^0)/\sim$, so we are finished.

References

[1] E. G. Begle, *A fixed point theorem*, Annals of Math. 2 (51) (1950), pp. 544-550.

[2] K. Borsuk, *Sur un continu acyclique qui se laisse transformer topologiquement et lui même sans points invariants*, Fund. Math. 24 (1935), pp. 51-58.

[3] — *Problem 54*, Colloq. Math. 1 (1948), p. 332.

[4] E. H. Connell, *Properties of fixed point spaces*, Proc. Amer. Math. Soc. 10 (1959), pp. 974-979.

[5] J. E. Keisler, *Dissertation*, p. 50, University of Michigan 1959.

[6] S. Kinoshita, *On some contractible continua without the fixed point property*, Fund. Math. 40 (1953), pp. 96-98.

[7] V. Klee, *An example related to the fixed point property*, Nieuw. Arch. Wisk. (3) VIII (1960), pp. 81-82.

[8] K. Kuratowski, *Problem 49*, Fund. Math. 15 (1930), p. 356.

[9] S. Lefschetz, *Topics in topology*, Ann. of Math. Studies No. 10, Princeton 1942.

[10] — *Algebraic topology*, Amer. Math. Soc. Colloq. Publ. Vol. XXVII, New York, 1942.

[11] R. L. Wilder, *Topology of manifolds (Chapter VI)*, Amer. Math. Soc. Colloq. Publ. Vol. XXXII, Providence 1949.

Reçu par la Rédaction le 4. 11. 1965

On subgroups of the homeomorphism group of the Cantor set*,**

by

Arnold R. Vobach (Athens, Georgia)

1. Introduction. Two recent strong results in the attempt to link algebraic and topological structures are due to M. T. Wechsler and J. V. Whittaker. In the first of these, [4], isomorphism of homeomorphism groups is shown to be equivalent to homeomorphism of spaces—but only for spaces satisfying a strong homogeneity condition. In the second, [5], the same equivalence is obtained for compact manifolds. In view of the machinery required for these results, it might be wondered if there are not groups associated with compact metric spaces which, by sharing a common milieu, more easily guarantee space homeomorphisms. This is the case, as indicated in the main theorem below. The common setting is the Cantor set, and the groups in question are subgroups, associated with maps of the Cantor set onto the spaces, of the homeomorphism group of the Cantor set.

2. Preliminary results. The following definition and lemmas, due to M. K. Fort, Jr., [2], will be helpful in later constructions:

DEFINITION. Let X be a separable metric space. Let F be a function on X such that each $F(x)$ is a non-empty compact subset of a metric space Y. Then, F is *upper semi-continuous* at p if, corresponding to each open set U of Y for which $F(p) \subset U$, there is a neighbourhood V of p such that if $x \in V$ then $F(x) \subset U$. The function F is upper semi-continuous on X if F is upper semi-continuous at each point of X.

LEMMA. *If $F_1, F_2, F_3, \ldots$ is a sequence of upper semi-continuous set-valued functions, $F_1(x) \supset F_2(x) \supset F_3(x) \supset \ldots$ and $F(x) = \bigcap_{n=1}^{\infty} F_n(x)$ for each $x \in X$, then F is also upper semi-continuous.*

LEMMA. *If f is a function on X into Y and for each $x \in X$, $F(x)$ is the set whose only member is $f(x)$, then f is continuous if and only if F is upper semi-continuous.*

* This research was partly supported by NSF Grant No. GP3915.

** Presented to the Society September 3, 1965.