since $h_{1-d(x)}f(x) \notin X$. If $x \in q(J)$, then $d(x) = 0$ so $f'(x) = h_1 f(x) \neq x$. If $x \in 2S^1$ then (iii) and the fact that q is the identity on $2S^1$ imply that $f'(x) \neq x$.

This is contrary to Lefschetz's theorem to the effect that every continuous transformation of a contractible polyhedron into itself has a fixed point. The assumption that f had no fixed points was false and B has the fixed point property.

4. This paragraph is to provide an answer to a question of Bing's posed privately. When apprised of B and its pathologies, he asked:

May one attach a three cell A to B so that $B \cup A$ does not have the fixed point property?

The answer is *yes*: Let

$$A = \{a \in E^3\colon\ 0 \leqslant r_a \leqslant 2,\ -1 \leqslant a_3 \leqslant 0\}.$$

Then $B \cup A$ does not have the fixed point property. We only sketch the argument. Consider the equivalence relation $\sim$ generated on $C(v; B^0)$ by the identification of $\langle t; d_s(e)\rangle$ with $\langle t, 1, \pi s\rangle$, for $\frac{3}{4} \leqslant t \leqslant 1$ and $0 \leqslant s$. Then the quotient space $C(v; B^0)/\sim$ is homeomorphic with $B \cup A$, with a homeomorphism that maps $C(v; D)$ onto A, and the remaining points into B. Note that if $a \in Z^0_{[0,\infty)}$ and $\frac{3}{4} \leqslant t \leqslant 1$, then $g^0(\langle t; a\rangle) = e$ so g^0 induces a fixed point free map of $C(v; Z^0_{[0,\infty)})/\sim$ into itself. The later is a retract of $C(v; B^0)/\sim$, so we are finished.

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On subgroups of the homeomorphism group of the Cantor set*,**

by

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1. Introduction. Two recent strong results in the attempt to link algebraic and topological structures are due to M. T. Wechsler and J. V. Whittaker. In the first of these, [4], isomorphism of homeomorphism groups is shown to be equivalent to homeomorphism of spaces—but only for spaces satisfying a strong homogeneity condition. In the second, [5], the same equivalence is obtained for compact manifolds. In view of the machinery required for these results, it might be wondered if there are not groups associated with compact metric spaces which, by sharing a common milieu, more easily guarantee space homeomorphisms. This is the case, as indicated in the main theorem below. The common setting is the Cantor set, and the groups in question are subgroups, associated with maps of the Cantor set onto the spaces, of the homeomorphism group of the Cantor set.

2. Preliminary results. The following definition and lemmas, due to M. K. Fort, Jr., [2], will be helpful in later constructions:

Definition. Let X be a separable metric space. Let F be a function on X such that each $F(x)$ is a non-empty compact subset of a metric space Y. Then, F is *upper semi-continuous* at p if, corresponding to each open set U of Y for which $F(p) \subset U$, there is a neighbourhood V of p such that if $x \in V$ then $F(x) \subset U$. The function F is upper semi-continuous on X if F is upper semi-continuous at each point of X.

Lemma. *If $F_1, F_2, F_3, \ldots$ is a sequence of upper semi-continuous set-valued functions, $F_1(x) \supset F_2(x) \supset F_3(x) \supset \ldots$ and $F(x) = \bigcap_{n=1}^{\infty} F_n(x)$ for each $x \in X$, then F is also upper semi-continuous.*

Lemma. *If f is a function on X into Y and for each $x \in X$, $F(x)$ is the set whose only member is $f(x)$, then f is continuous if and only if F is upper semi-continuous.*

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Let $H(C)$ be the group of homeomorphisms of the Cantor set C onto itself. Let M be a compact metric space and p a map of C onto M.

DEFINITION. $G(p, M) = \{h \in H(C) \mid \forall x \in C, p(x) = ph(x)\}$.

$G(p, M)$ is a subgroup of $H(C)$.

From among the multitude of maps of C onto M, it is desirable to study those with the following characteristic:

DEFINITION. A map p of C onto a compact metric space M is a *standard map* if, for each pair of points x and y such that $p(x) = p(y)$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset G(p, M)$ such that $x_n \to x$ and $h_n(x_n) \to y$.

LEMMA. *Given a compact metric M, there is a standard map of C onto M.*

Proof. First we shall construct a map $p\colon C \to M$ and then show that it has the desired property: Let us consider a sequence of finite closed covers of M, $\{T_i^1\}_{i=1}^{n(1)}$, $\{T_i^2\}_{i=1}^{n(2)}$, ..., with the properties:

(1) Mesh $\{T_i^k\}_{i=1}^{n(k)} < 1/k$, and

(2) $T_i^k \cap T_j^k \neq \emptyset$ is the union of two or more of the elements of $\{T_i^{k+1}\}_{i=1}^{n(k+1)}$.

What remains in the construction of p is the usual routine procedure, [3]. It is interesting to note that, with what follows, the second condition on the covers, above, is sufficient to make p a standard map. In this sense, standard maps are very natural.

We divide the interval $[0, 1]$ into $2n(1)$ equal subintervals. Label every second one of these, end points included, as $E_1^1, E_2^1, \ldots, E_{n(1)}^1$. Suppose that we have the interval $E^k_{i(1),i(2),\ldots,i(k)}$, where the sequence $i(1), \ldots, i(k)$ is such that $T^1_{i(1)} \supset T^2_{i(2)} \supset \ldots \supset T^k_{i(k)}$. We divide it into $2m(i(k))$ equal subintervals, where $m(i(k))$ is the number of elements of $\{T_i^{k+1}\}_{i=1}^{n(k+1)}$ which are contained in $T^k_{i(k)}$. Denote every second one of these intervals by

$$E^{k+1}_{i(1),i(2),\ldots,i(k),j(1)},\ E^{k+1}_{i(1),\ldots i(k),j(2)},\ \ldots,\ E_{i(1),\ldots,i(k),j(m(i(k)))}$$

where the $j(r)$'s, $r = 1, \ldots, m(i(k))$ are the subscripts, from $\{T_i^{k+1}\}$, of the elements of this collection which are contained in $T^k_{i(k)}$.

Set $S_k = \bigcup E^k_{i(1),\ldots i(k)}$ for all sequences $i(1), \ldots, i(k)$ for which $T^1_{i(1)} \supset T^2_{i(2)} \supset \ldots \supset T^k_{i(k)}$. Set $C = \bigcap_{k=1}^{\infty} S_k$. C is a Cantor set.

Next, define the upper semi-continuous set-valued function $F_1\colon C \to M$ by $F_1(x) = T^1_{i(1)}$ for $x \in C \cap E^1_{i(1)}$. Likewise, $F_2(x) = T^2_{i(2)}$ for $x \in C \cap E^2_{i(1),i(2)}$ with $i(1), i(2)$ such that $T^1_{i(1)} \supset T^2_{i(2)}$. In general, set $F_n(x) = T^n_{i(n)}$ for

$x \in C \cap E^n_{i(1),\ldots,i(n)}$ with $i(1), \ldots, i(n)$ such that $T^1_{i(1)} \supset T^2_{i(2)} \supset \ldots \supset T^n_{i(n)}$. Let $p(x) = \bigcap_{n=1}^{\infty} F_n(x)$, a single point. By Fort's lemmas, p is continuous, a map of C onto M.

To show that p is a standard map: Suppose that $x \neq y$ with $p(x) = p(y)$. For each positive integer n, we wish tofi nd a point x_n of C within $1/n$ of x and a homeomorphism h_n of $G(p, M)$ for which $h_n(x_n)$ is within $1/n$ of y. Let k be a positive integer $> n$ such that $x \in C \cap E^k_{i(1),\ldots,i(k)}$, $y \in C \cap E^k_{j(1),\ldots,j(k)}$, $E^k_{i(1),\ldots,i(k)} \cap E^k_{j(1),\ldots,j(k)} = \emptyset$, and the diameter of each of $E^k_{i(1),\ldots,i(k)}$ and $E^k_{j(1),\ldots,j(k)}$ is less than $1/k$. Now, $p(C \cap E^k_{i(1),\ldots,i(k)}) = T^k_{i(k)}$ and $p(C \cap E^k_{j(1),\ldots,j(k)}) = T^k_{j(k)}$. We have $p(x) = p(y) \in T^k_{i(k)} \cap T^k_{j(k)} \neq \emptyset$, so that $T^k_{i(k)} \cap T^k_{j(k)}$ is the union of elements of the $(k+1)$st cover. At least one of these, call it simply T^{k+1}, contains $p(x) = p(y)$. To return to C, there is an $E^{k+1}_{i(1),\ldots,i(k),i} \subset E^k_{i(1),\ldots,i(k)}$ such that $p(C \cap E^{k+1}_{i(1),\ldots,i(k),i}) = T^{k+1}$. Let x_n be a point of $C \cap E^{k+1}_{i(1),\ldots,i(k),i}$ with p-image $p(x)$. Likewise, there is an $E^{k+1}_{j(1),\ldots,j(k),j} \subset E^k_{j(1),\ldots j(k)}$ such that $p(C \cap E^{k+1}_{j(1),\ldots,j(k),j}) = T^{k+1}$.

If we can find an $h_n \in G(p, M)$ which carries $C \cap E^{k+1}_{i(1),\ldots,i(k),i}$ onto $C \cap E^{k+1}_{j(1),\ldots,j(k),j}$, $h_n(x_n)$ will be within $1/n$ of y. On $C \setminus C \cap (E^{k+1}_{i(1),\ldots,i(k),i} \cup E^{k+1}_{j(1),\ldots,j(k),j})$, h_n is to be the identity. As before, we use a sequence of set-valued functions to define h_n on $C \cap (E^{k+1}_{i(1),\ldots,i(k),i} \cup E^{k+1}_{j(1),\ldots,j(k),j})$. We may shorten the notation by setting $E^{k+1}_{i(1),\ldots,i(k),i} = E_1^{k+1}$ and $E^{k+1}_{j(1),\ldots,j(k),j} = E_2^{k+1}$. Let $H_1(z) = C \cap E_2^{k+1}$ for $z \in C \cap E_1^{k+1}$ and $H_1(z) = C \cap E_1^{k+1}$ for $z \in C \cap E_2^{k+1}$. Now T^{k+1} is the union of subsets of the $(k+2)$nd cover, $\{T_j^{k+2}\}_{j=1}^{m(k+2)}$, possibly renumbered, and hence $C \cap E_i^{k+1} = C \cap (\bigcup_{j=1}^{m(k+2)} E^{k+2}_{i,j})$, $i = 1, 2$, where $p(C \cap E^{k+2}_{i,j}) = T_j^{k+2}$, $j = 1, \ldots, m(k+2)$. Define $H_2(z) = C \cap E^{k+2}_{2,j}$ for $z \in C \cap E^{k+2}_{1,j}$, and $H_2(z) = C \cap E^{k+2}_{1,j}$ for $z \in C \cap E^{k+2}_{2,j}$, $j = 1, \ldots, m(k+2)$. Generally, define $H_s(z) = C \cap E^{k+s}_{2,r(2),\ldots,r(s)}$ for $z \in C \cap E^{k+s}_{1,r(2),\ldots,r(s)}$, and $H_s(z) = C \cap E^{k+s}_{1,r(2),\ldots,r(s)}$ for $z \in C \cap E^{k+s}_{2,r(2),\ldots,r(s)}$, for those sequences $r(2), \ldots, r(s)$ for which $T^{k+1} \supset T^{k+2}_{r(2)} \supset \ldots \supset T^{k+s}_{r(s)}$. On $C \cap (E_1^{k+1} \cup E_2^{k+1})$, let $h_n(z) = \bigcap_{s=1}^{\infty} H_s(z)$, and observe that h_n on this set is, by Fort's lemmas, continuous. Since h_n merely interchanges, at each stage of its construction, correspondingly indexed sub-intervals—intersected with C—of $[0, 1]$, it is also 1-1, onto and a homeomorphism. Further, this switching of correspondingly indexed subintervals of E_1^{k+1} and E_2^{k+1}, assures us that $p(z) = ph_n(z)$ on $C \cap (E_1^{k+1} \cup E_2^{k+1})$. Outside this set, h_n was already given as the identity.

Remark. From the way in which the map p was constructed above we were able to choose the sequence $\{x_n\}_{n=1}^{\infty}$ such that $p(x) = p(x_n) = ph_n(x_n) = p(y)$, which implies that $p^{-1}(m)$, $m \in M$, is a perfect subset of C. This says that p may be very naturally chosen so that the closed set $p^{-1}(m)$, $m \in M$, is either a single point or a Cantor set.

The following simple theorem, while of secondary interest in the present development, is so nicely related to the Lemma as to suggest its inclusion.

THEOREM. *Let $f\colon M \to N$ be continuous and onto, with M and N compact metric. Then there exist standard maps $p\colon C \to M$ and $q\colon C \to N$ such that $G(p, M) \subset G(q, N)$.*

Proof. Let $\{T_i^1\}_{i=1}^{n(1)}, \{T_i^2\}_{i=1}^{n(2)}, \ldots$, be a sequence of finite closed covers of M with the properties:

1) Mesh of $\{T_i^k\}_{i=1}^{n(k)}$ and of $\{f(T_i^k)\}_{i=1}^{n(k)} < 1/k$, and

2) $T_i^k \cap T_j^k \neq 0$ (and hence $f(T_j^k \cap f(T_i^k) \neq \emptyset)$ is the union of two or more of the elements of $\{T_i^{k+1}\}_{i=1}^{n(k+1)}$ $(\{f(T_i^{k+1})\}_{i=1}^{n(k+1)})$.

One may now construct a standard map $p\colon C \to M$ exactly as in the Lemma and observe that $q = fp\colon C \to N$ will also be standard because each $f(T_i^k) \cap f(T_j^k) \neq \emptyset$ is the union of two or more elements of the cover $\{f(T_i^{k+1})\}_{i=1}^{n(k+1)}$.

For $h \in G(p, M)$, $p(x) = ph(x)$, $x \in C$, which implies $q(x) = fp(x) = fph(x) = qh(x)$, $x \in C$, and $h \in G(q, N)$.

3. Main theorem. *Let p be a standard map of the Cantor set C onto compact metric M. The compact metric space N is homeomorphic to M if and only if $G(p, M) = G(q, N)$ for some standard map q of C onto N.*

Proof. Doing the easy way first, suppose that $h\colon M \to N$ is a homeomorphism. Define $q\colon C \to N$ by $q = hp$; $G(q, N) = G(p, M)$ and q is a standard map.

In the other direction, suppose $G(p, M) = G(q, N)$ for standard maps p and q of C onto M and N, respectively. For $m \in M$, set $h(m) = q(x)$, where $x \in p^{-1}(m)$. Is this choice of $h(m)$ well-defined? Suppose $p(x) = p(y) = m$, then, since p is standard, there exist $\{x_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty} \subset G(p, M)$ such that $x_n \to x$ and $h_n(x_n) \to y$. Since $G(p, M) = G(q, N)$, $q(x_n) = q(h_n(x_n))$; $q(x_n) \to q(x)$ and $q(h_n(x_n)) \to q(y)$. Hence $q(x) = q(y)$. Clearly, the not-necessarily continuous function h is onto N.

To show continuity, let $A \subset N$ be closed; then $q^{-1}(A)$ is closed. Since $q^{-1}(A)$ is compact, $pq^{-1}(A)$ is compact and thus closed. However, $pq^{-1}(A)$ is just $h^{-1}(A)$, so h is continuous.

Finally, the continuous, onto function $h\colon M \to N$ is 1-1: Suppose $h(m_1) = h(m_2)$. Then there are x and y in C such that $p(x) = m_1$, $p(y) = m_2$ and $q(x) = q(y) = h(m_i)$, $i = 1, 2$. Since q is standard, let $\{x_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty} \subset G(q, N)$ be such that $x_n \to x$, $h_n(x_n) \to y$. Since $G(p, M) = G(q, N)$, $p(x_n) = p(h_n(x_n))$; $p(x_n) \to p(x) = m_1$, together with $p(h_n(x_n)) \to p(y) = m_2$, imply $m_1 = m_2$. Hence h is a homeomorphism.

It is natural to inquire whether the set equality of the theorem might be replaced by isomorphism. A simple example shows this is not possible: Let C be the Cantor middle third set on $[0, 1]$. Let M be a single point and $p\colon C \to M = \{m\}$; then $G(p, M) = H(C)$. Let N be the union of an isolated point and a copy of $C \cap [2/3, 1]$, with q carrying all of $C \cap [0, 1/3]$ onto the isolated point and $C \cap [2/3, 1]$ identically onto its copy in N. $G(q, N)$ is isomorphic to $H(C)$ but properly contained in it. Observe that $G(p, M)$ and $G(q, N)$ are not conjugate.

Conjugacy appears to be the more natural setting for algebraic comparisons—not surprising in view of [1]—and we get the following:

COROLLARY. *Let p and q be standard maps of C onto compact metric M and N, respectively. If, for some $h \in H(C)$, $G(p, M) = hG(q, N)h^{-1}$, M and N are homeomorphic.*

Proof. We observe that for any $h \in H(C)$, $G(ph, M) = h^{-1}G(p, M)h$: Let $f \in G(p, M)$; then, since $p(x) = pf(x)$, $x \in C$, $ph(x) = ph(h^{-1}fh)(x)$ for each $x \in C$. Thus, $G(ph, M) \supset h^{-1}G(p, M)h$. Likewise, if $g \in G(ph, M)$, then $ph(x) = phg(x)$ implies $phh^{-1}(x) = p(hgh^{-1})(x)$ for each $x \in C$. This says $hgh^{-1} \in G(p, M)$, and $G(ph, M) \subset h^{-1}G(p, M)h$. Finally, then, $G(ph, M) = h^{-1}G(p, M)h = G(q, N)$, and since $ph\colon C \to M$ is standard (because p is), M is homeomorphic to N by the theorem.

To summarize, each standard map of C onto a compact metric space determines a whole class of conjugate subgroups, each the group of some standard map onto the space, and two compact metric spaces are homeomorphic if and only if they determine, in this manner, precisely the same classes of conjugate subgroups of $H(C)$.

For p and q standard onto X, how, if at all, are $G(p, M)$ and $G(q, M)$ related? What algebraic properties of these groups are associated with what topological properties of their associated spaces? How might the $G(p, M)$'s be computed?

Added in proof. The following example is due simultaneously to Professors W. R. Alford and H. Cook: Let C' be the middle-third Cantor set on $[0, 1]$. Let $p'\colon C' \to [0, 1]$ be defined by identifying end points of deleted intervals of $[0, 1]$: 1/3 with 2/3, 1/9 with 2/9 and 7/9 with 8/9, etc. While p' is not standard, C' may be augmented to yied a Cantor set and a map, associated with p', which is standard. Consider the Cantor subset of

$$C' \times C'\colon\ C = [C' \times \{0\}] \cup [\{1/3, 2/3\} \times C'] \cup \big[\{1/9, 2/9, 7/9, 8/9\} \times [C' \cap [0, 1/3]]\big] \cup \ldots.$$

C is C' with matching Cantor sets stacked over the gap end points identified with each other by p'. Define $p\colon C \to [0, 1]$ by first projecting C onto $C' \times \{0\}$ and following this by p'; p is standard. The homeomorphisms of $G(p, [0, 1])$ are precisely those which only interchange points in the Cantor sets which are stacked over end points of the same gaps.

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Upper semi-continuous decompositions of irreducible continua

by

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Introduction. In 1935, B. Knaster showed [6] that there is a continuous collection of nondegenerate continua which is an arc with respect to its elements and such that the union of its elements is a compact irreducible continuum. In 1949, E. E. Moise [7] showed that there is no such collection with the additional property that each of its members is an arc. Moise's theorem was improved in 1952 by M. E. Hamstrom [4], and further improved in 1953 when E. Dyer showed [3] that there is no continuous collection of decomposable continua such that the union of the members is a compact irreducible continuum. It is known [6] that there is an upper semi-continuous collection of arcs which is an arc with respect to its elements and such that the union of its elements is a compact irreducible continuum. In this paper we consider an upper semi-continuous collection G having the following property.

(A) *If $g \in G$, each point of g is a limit point of the union of the members of each component of $G-g$.*

As a corollary to Theorem 4 of this paper we have that there is no upper semi-continuous collection G of arcs such that G has property (A), G is an arc with respect to its elements, and G^* is a compact irreducible metric continuum.

Continua with degenerate E-continua. If M is a compact, hereditarily decomposable, irreducible, metric continuum, M contains two continua such that each is the complement in M of a composant of M. These are called the *E-continua* of M by H. C. Miller [9]. We see from the following theorem that if M is chainable, M contains a continuum which has a degenerate E-continuum. The constructive proof of this theorem is essentially identical with that of G. W. Henderson for Theorem 13 of [5]. The theorem may also be compared with Theorem 6 of [1].

THEOREM 1. *If M is a hereditarily decomposable, compact, chainable metric continuum and P' and Q' are points of M and $\varepsilon > 0$, there is a sub-*