Equationally compact algebras II

by

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In this paper we collect miscellaneous theorems on equational compactness of algebras and the related notions of atomic and positive $\mathfrak{m}$-compactness of relational structures (see [18], [25], [26], and [27]). Our Theorem 1 is a slight improvement of the theorem on existence of saturated structures (see [6], Theorem 1 repeated as Theorems 3 and 5 in [18] and [8], Corollary 2.2) from which we have removed a restriction on the cardinality of the similarity type; also our proof is simpler than the previous proofs and is similar to some constructions in [9] and [11]. Our Theorem 2 is an easy consequence of Theorem A.2 of [7] (see also [10], Lemma 1 and [17], Theorems 2.4(e), 2.5(b) and 3.4 for related facts). Theorems 3 and 4 give new characterizations of atomic compactness and $\mathfrak{m}$-compactness (see [25] for related characterizations). Theorems 5 and 6 and the remarks following each of them give new examples of equationally compact algebras. Theorem 7 generalizes for arbitrary modules the known result (see [18] or [25] for references) that equationally $\aleph_0$-compact abelian groups are equationally compact. Theorems 8 and 9 give an algebraic generalization of the topological statement that every product of complex projective spaces is compact. Finally, we discuss some fundamental properties of the notions of compactness considered in the series of papers attempting to give a certain classification of properties of relational and algebraic structures which seems natural at the present stage of development of the theory of models.

We follow with little changes the terminology of part I [25]. As in [18] the prefix $\mathfrak{m}$- added to the word *compact* means that the sets of formulas in question which should have the compactness property are of cardinality at most $\mathfrak{m}$. $\mathfrak{m}$ denotes always an infinite cardinal.

THEOREM 1. *For every $\mathfrak{m}$ and every infinite algebraic structure $\mathfrak{A}$ there exists an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ (in symbols $\mathfrak{A} \prec \mathfrak{B}$) which is elementarily $\mathfrak{m}$-compact and $|\mathfrak{B}| = |\mathfrak{A}|^{\mathfrak{m}}$.*

Let us recall the well-known theorem of Vaught (see [7]) that if, moreover, $|\mathfrak{A}|^{\mathfrak{m}} = \mathfrak{m}^{+}$ and the cardinality of the similarity type is at most $\mathfrak{m}$, then the structure $\mathfrak{S}$ with the above properties is unique up to isomorphism.

To prove Theorem 1 we will give two lemmas. For every set U, every ultrafilter $\boldsymbol{F}$ of subsets of U and every relational structure $\mathfrak{S}$ we denote by $(\mathfrak{S}^U/\boldsymbol{F})^\bullet$ an isomorph of $\mathfrak{S}^U/\boldsymbol{F}$ by an isomorphism f such that if g is the natural isomorphism of $\mathfrak{S}$ into $\mathfrak{S}^U/\boldsymbol{F}$, then f restricted to $g(\mathfrak{S})$ equals g^{-1}. Thus $\mathfrak{S} \precsim (\mathfrak{S}^U/\boldsymbol{F})^\bullet$.

The proof of Lemma 2.2 in [25] (of C. Ryll-Nardzewski) immediately yields the following lemma.

LEMMA 1. *Let M be a set, $\mathfrak{S}$ an algebraic structure and Σ a set of formulas with constants in $\mathfrak{S}$* (¹) *such that $|\Sigma| \leqslant |M|$ and every finite subset of Σ is satisfiable in $\mathfrak{S}$. Let U be the set of all finite subsets of M and $\boldsymbol{F}$ an ultrafilter of subsets of U such that $\{X \in U\colon x \in X\} \in \boldsymbol{F}$ for every $x \in M$. Then Σ is satisfiable in $(\mathfrak{S}^U/\boldsymbol{F})^\bullet$.*

The proofs of Theorems 1.17 (due to Chang and Keisler) and 1.26 of [1] immediately yield the following lemma.

LEMMA 2. *Let $M, U, \boldsymbol{F}$ and $\mathfrak{S}$ satisfy the suppositions of Lemma 1 and $|M|$, $|\mathfrak{S}| \geqslant \aleph_0$. Then $|\mathfrak{S}^U/\boldsymbol{F}| = |\mathfrak{S}|^{|M|}$.*

Proof of Theorem 1. Let M, U and F be as in the lemmas, $|M| = \mathfrak{m}$ and let α be the initial ordinal of power $\mathfrak{m}^+$. We set

$$\mathfrak{S}_0 = \mathfrak{A} \quad \text{and} \quad \mathfrak{S}_\xi = \bigcup_{\eta<\xi} (\mathfrak{S}_\eta^U/\boldsymbol{F})^\bullet \quad \text{for every ordinal } \xi > 0 .$$

Now we show that the structure $\mathfrak{B} = \mathfrak{S}_\alpha$ satisfies the conclusions of Theorem 1. First, $\mathfrak{S}_\eta \precsim \mathfrak{S}_\zeta$ for every $\eta < \zeta$ (see [21]) and hence $\mathfrak{A} \precsim \mathfrak{B}$. Let Σ be a set of formulas with constants in $\mathfrak{B}$, $|\Sigma| \leqslant \mathfrak{m}$ and let all finite subsets of Σ be satisfiable in $\mathfrak{B}$. By the definition of α it is a regular ordinal and hence all the constants of Σ are in some $\mathfrak{S}_\xi$ with $\xi < \alpha$. Since $\mathfrak{S}_\xi \precsim \mathfrak{B}$, all finite subsets of Σ are satisfiable in $\mathfrak{S}_\xi$. Hence, by Lemma 1, Σ is satisfiable in $\mathfrak{S}_{\xi+1}$ and since $\mathfrak{S}_{\xi+1} \precsim \mathfrak{B}$, it is also satisfiable in $\mathfrak{B}$. Finally, since $|\alpha| \leqslant |\mathfrak{A}|^{\mathfrak{m}}$ and according to Lemma 2, we have $|\mathfrak{B}| = |\mathfrak{A}|^{\mathfrak{m}}$. Q.E.D.

An algebraic structure $\mathfrak{A}$ is called $\mathfrak{m}$-*universal* if every structure $\mathfrak{S}$ of the same similarity type, elementarily equivalent to $\mathfrak{A}$ and of power $\leqslant \mathfrak{m}$ is isomorphic to some elementary substructure of $\mathfrak{A}$. (²)

For every algebraic structure $\mathfrak{A}$ we denote by $(\mathfrak{A})_c$ an enrichment of $\mathfrak{A}$ by all elements of $\mathfrak{A}$ added as individual constants, i.e. 0-ary operations. Then, of course, the following conditions are equivalent.

(¹) We recall that the set of free variables is of arbitrary power, not necessarily denumerable, and that the satisfiability of a set of formulas with constants in $\mathfrak{S}$ means the satisfiability of all of them simultaneously by a system of elements of $\mathfrak{S}$ (see [18], [25]).

(²) See [4], [5] and the literature related to Theorem 2 quoted in the introduction for other studies related to this notion.

(i) $\mathfrak{A}$ is K—$\mathfrak{m}$-compact;

(ii) $(\mathfrak{A})_c$ is K—$\mathfrak{m}$-compact;

(iii) $(\mathfrak{A})_c$ is weakly K—$\mathfrak{m}$-compact.

An $\mathfrak{m}$-*reduct* of an algebraic structure $\mathfrak{A}$ is any structure obtained by removing all but $\mathfrak{m}$ relations and all but $\mathfrak{m}$ functions and individual constants of $\mathfrak{A}$. Clearly, conditions (i), (ii) and (iii) remain equivalent if "$\mathfrak{A}$" or "$(\mathfrak{A})_c$" is replaced by "every $\mathfrak{m}$-reduct of $\mathfrak{A}$" or "every $\mathfrak{m}$-reduct of $(\mathfrak{A})_c$", respectively.

THEOREM 2. *An algebraic structure $\mathfrak{A}$ is elementarily $\mathfrak{m}$-compact if and only if every $\mathfrak{m}$-reduct of $(\mathfrak{A})_c$ is $\mathfrak{m}$-universal.*

Proof. If $\mathfrak{A}$ is elementarily $\mathfrak{m}$-compact, then every $\mathfrak{m}$-reduct of $(\mathfrak{A})_c$ is $\mathfrak{m}$-universal (and even $\mathfrak{m}^+$-universal) by Theorem A.2 (the implication (ii) $\Rightarrow$ (i)) of [7].

Conversely, let every $\mathfrak{m}$-reduct of $(\mathfrak{A})_c$ be $\mathfrak{m}$-universal and let Σ be a set of formulas with constants in $\mathfrak{A}$ such that $|\Sigma| \leqslant \mathfrak{m}$ and every finite subset of Σ is satisfiable in $\mathfrak{A}$. Let Σ^* be a modification of Σ obtained by substituting every free variable x_s by a new constant symbol $\boldsymbol{c}_s$. Let $\mathfrak{A}_0$ be an $\mathfrak{m}$-reduct of $(\mathfrak{A})_c$, which contains all the relations, functions and constants corresponding to the predicates functors and constants occuring in Σ. Let Θ be the complete theory of $\mathfrak{A}_0$. Clearly, $\Sigma^* \cup \Theta$ is consistent and hence it has a model $\mathfrak{M}$ of power $\mathfrak{m}$ which is elementarily equivalent to $\mathfrak{A}_0$. Thus, $\mathfrak{A}_0$ being $\mathfrak{m}$-universal, there is an isomorphism $f\colon \mathfrak{M} \to \mathfrak{A}_0$ such that $f(\mathfrak{M}) \precsim \mathfrak{A}_0$. Therefore, c_s denoting the elements of $\mathfrak{M}$ interpreting the constants $\boldsymbol{c}_s$, the system $f(c_s)$ (s runs over the set of indices of the free variables of Σ) satisfies Σ in $\mathfrak{A}_0$ and in $\mathfrak{A}$. Q.E.D.

THEOREM 3. *For every algebraic structure $\mathfrak{A}$ and every $\mathfrak{m}$ the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is atomic $\mathfrak{m}$-compact.*

(ii) *Let Σ be any set of power $\leqslant \mathfrak{m}$ of formulas with constants in $\mathfrak{A}$ all having only one free variable x_0 and all of the form*

$$\exists x_1 \exists x_2 \ldots \exists x_m [\alpha_1 \wedge \ldots \wedge \alpha_n] ,$$

where m, n are any natural numbers and α_i are atomic formulas (with constants in $\mathfrak{A}$). If every finite subset of Σ is satisfiable in $\mathfrak{A}$, then Σ is satisfiable in $\mathfrak{A}$.

Proof. (i) $\Rightarrow$ (ii). Let Σ_0 be obtained from Σ by changing the bounded variables in such a way that none occurs simultaneously in two formulas and then removing all quantifiers. Clearly, if Σ is finitely satisfiable in $\mathfrak{A}$, then Σ_0 is also. Then, by (i), Σ_0 is satisfiable and Σ is also.

(ii) $\Rightarrow$ (i). Let Θ be a set of power $\leqslant \mathfrak{m}$ of atomic formulas with constants in $\mathfrak{A}$ with the free variables $x_0, x_1, \ldots, x_\xi, \ldots$ $(\xi < \alpha)$ such that every finite subset of Θ is satisfiable in $\mathfrak{A}$. Supposing (ii) we will construct by induction a sequence $a_0, a_1, \ldots, a_\xi, \ldots$ $(\xi < a)$ of elements of $\mathfrak{A}$ satisfying Θ in $\mathfrak{A}$. Let, for some $\tau < a$, the sequence $a_0, a_1, \ldots, a_\xi, \ldots$ $(\xi < \tau)$ be already constructed such that every finite subset of Θ_τ, where Θ_τ is obtained from Θ by substituting each x_ξ with $\xi < \tau$ by a_ξ, is satisfiable in $\mathfrak{A}$. We will find $a_\tau \in A$ such that $\Theta_{\tau+1}$ has the same property. Let Σ be the set of formulas with constants in $\mathfrak{A}$ obtained from Θ_τ by forming all finite conjunctions and adding at the front of each of them existential quantifiers bounding all free variables except x_τ. Clearly, Σ fulfils the suppositions of (ii) and hence some a_τ satisfies it in $\mathfrak{A}$. It is easy to see that a_τ gives the required properties of $\Theta_{\tau+1}$. This concludes the induction and (i) is proved.

It was proved in [25] that the atomic and positive compactness coincide. In the next theorem we give another proof of this fact and an alternative characterization of these notions.

THEOREM 4. *For every algebraic structure $\mathfrak{A}$ the following conditions are equivalent*:

(i) $\mathfrak{A}$ *is atomic* $|\mathfrak{A}|$-*compact*;

(ii) $\mathfrak{A}$ *is atomic compact*;

(iii) $\mathfrak{A}$ *is positively compact.*

Proof. (i) $\Rightarrow$ (ii). If a set of formulas with constants in $\mathfrak{A}$, with one free variable, is not satisfiable in $\mathfrak{A}$, then it has a subset of power $|\mathfrak{A}|$ which is not satisfiable in $\mathfrak{A}$. Hence the characterization of atomic $\mathfrak{m}$-compactness given in Theorem 3(ii) yields (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let Σ be a set of positive formulas with constants in $\mathfrak{A}$ every finite subset of which is satisfiable in $\mathfrak{A}$, and let Σ^* be obtained as in the proof of Theorem 2. Then, applying the compactness theorem to the union of Σ^* and the theory of $(\mathfrak{A})_c$, we get an elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ in which Σ is satisfiable. $\mathfrak{A}$ being an atomic compact elementary substructure of $\mathfrak{A}^*$ it is a retract of $\mathfrak{A}^*$ (3). Hence Σ is satisfiable in $\mathfrak{A}$.

(iii) $\Rightarrow$ (i) is obvious.

This concludes the proof of Theorem 4.

Theorem 4 has not been extended to the weak atomic or weak positive compactness. For the implication (ii) $\Rightarrow$ (iii) this is still an open question. For (i) $\Rightarrow$ (ii) this is not possible as can be seen on the following two examples.

E_1. A complete lattice $\mathfrak{S}$ such that $x \cap y = 0$ if $x \neq y \neq 1 \neq x$ and $x \cup y = 1$ if $x \neq y \neq 0 \neq x$, is weakly elementarily $|\mathfrak{S}|$-compact but is not weakly equationally $|\mathfrak{S}|^+$-compact (nor equationally $|\mathfrak{S}|$-compact). (4)

(3) See [25], Lemma 2.1.

(4) The assertion in brackets was proved in [27].

E_2. A ring which is an algebraically closed field of degree of transcendence $\mathfrak{m} \geqslant \aleph_0$ is weakly elementarily $\mathfrak{m}$-compact but is not weakly equationally $\mathfrak{m}^+$-compact (nor equationally $\mathfrak{m}$-compact). (5)

Still the following question is open:

Must every structure $\mathfrak{A}$ weakly atomic $|\mathfrak{A}|^+$-compact be weakly atomic compact?

A family of sets $\boldsymbol{F}$ is called *compact* if every subfamily of $\boldsymbol{F}$ every finite subset of which has a non-empty intersection has a non-empty intersection.

THEOREM 5. *Let $\mathfrak{A} = \langle A, g\rangle_{g \in G}$ be an algebra such that G is a group of transformations of A onto itself such that the family of fixed point sets, i.e. sets of the form $\{x \in A\colon g(x) = x\}$, where $g \in G$, is compact. Then $\mathfrak{A}$ is equationally compact.*

Proof. Each equation has the form (1) $g(x) = c$ or (2) $g_1(x) = g_2(x)$ or (3) $g_1(x) = g_2(y)$. Equations of type (1) or (3) allow us to remove some variables and hence every set of equations reduces to equations of type (2). Then the supposition of the theorem obviously implies its conclusion.

A class of algebras considered by Marczewski and Urbanik (see [14], [22] and [23]) also proves to contain only equationally compact algebras. This class is defined by the following property:

v. For every equation

$$(*) \qquad \varrho = \sigma,$$

whose free variables are $x_1, \ldots, x_n$ and which depends on x_1, i.e. $\exists x_0 \exists x_1 \ldots \ldots \exists x_n [\varrho(x_1/x_0) = \sigma(x_1/x_0) \wedge \varrho \neq \sigma]$ (6), there exists a term τ without the variable x_1 such that $(*)$ is equivalent to

$$x_1 = \tau.$$

It is easy to check that in these algebras each formula of the form considered in Theorem 3. (ii) either is not satisfiable or is identically true or is satisfiable by exactly one element. Thus equational compactness of such algebras follows from Theorem 3. In spite of their definition, apparently very general, algebras with property v constitute a quite narrow class satisfying a very concrete representation theorem proved by Urbanik in [22]. Using his result the fact that such algebras are equationally compact follows also from equational compactness of linear spaces (they have property v) which is known (see [25]) and our Theorem 5.

Let us consider another property of algebras weaker than v.

w. For every equation

$$(**) \qquad \varrho = \sigma,$$

(5) The assertion in brackets was proved in [18].

(6) (x_1/x_0) denotes the substitution of x_0 for x_1.

which is not identically true in the algebra, there exist a variable x_i and a term τ without the variable x_i such that (**) implies

$$x_i = \tau .$$

Examples are the locally absolutely free algebras of Malcev [13], i.e. algebras every finitely generated subalgebra of which is absolutely free (these algebras can also be characterized by the first order axioms, see [13], Theorem 1) and algebras of the form $(\mathfrak{A})_c$, where $\mathfrak{A}$ is locally absolutely free. Neither algebras with property w nor even absolutely free algebras need to be weakly equationally compact, as the example

$$\langle\{0, 1, 2, \ldots\}, x+1\rangle$$

shows. But property w implies the following one:

Every set of equations involving only finitely many variables every finite subset of which is satisfiable is satisfiable.

THEOREM 6. *If $\mathfrak{G}$ is a group of motions of the Euclidean space R^n which is generated by the group of all translations $\mathfrak{T}$ $(= R^n)$ and a finite group $\mathfrak{R}_0$ of rotations around one point, then $\mathfrak{G}$ is equationally compact.*

Proof. Let Σ be a set of equations with constants in $\mathfrak{G}$ finitely satisfiable in $\mathfrak{G}$, and let x_s $(s \in S)$ be the system of unknowns of Σ. Every motion of R^n is uniquely representable in the form tr, where t is a translation and r is a rotation. Σ being finitely satisfiable in G and since the topological power $\mathfrak{R}_0^S$ is compact, there exists a system $r_s^0 \in \mathfrak{R}_0$ $(s \in S)$ such that Σ together with the set of relations $x_s = t_s r_s^0 \wedge t_s \in T$ is finitely satisfiable. By [19] § 5, (9), this set of conditions reduces to a set of linear equations with constants in the linear space R^n. Hence, linear spaces being equationally compact (for a generalization of this fact see the remarks following Theorem 5), Σ is satisfiable in $\mathfrak{G}$. Q.E.D.

This theorem applies e.g. to a group of motions of R^3 which is generated by the group of all translations and the group of rotations of the icosahedron. Theorem 6 cannot be improved by removing the condition that $\mathfrak{R}_0$ is finite in spite of the fact that the group of all rotations is compact; indeed, the group of all motions of R^2 is not equationally compact. Notice the following simple example.

E_2. Let $\mathfrak{G}$ be a group which contains a subgroup isomorphic to the group $\mathfrak{G}_0$ of linear substitutions of the form

$$f_{k,r}(w) = 2^k w + r \quad (k \text{ any integer, } r \text{ any rational number}).$$

Consider the set of equations

$$(\overset{*}{**}) \qquad y_{s_1 s_2} x_{s_1} x_{s_2}^{-1} y_{s_1 s_2}^{-1} x_{s_2} x_{s_1}^{-1} = \varphi(f_{0,1}) \qquad (s_1 \neq s_2,\ s_1, s_2 \in S),$$

where x_s and $y_{s_1 s_2}$ are unknowns and φ is an isomorphism of $\mathfrak{G}_0$ into $\mathfrak{G}$. It is easy to check that each (at most denumerable) subset of this set of equations is satisfiable in $\varphi(\mathfrak{G}_0)$ (if $\mathfrak{G}_0$ were the group of all linear substitutions $aw+b$ $(a \neq 0)$, then each subset of power $\leqslant 2^{\aleph_0}$ would be satisfiable in $\varphi(\mathfrak{G}_0)$). But if $|S| > |\mathfrak{G}|$, the whole set is not satisfiable in $\mathfrak{G}$, because $(\overset{*}{**})$ implies $x_{s_1} \neq x_{s_2}$. (7)

As in [25] by an $\mathfrak{R}$-module we mean an algebra $\mathfrak{M} = \langle A, +, -, r\rangle_{r\in\mathfrak{R}}$, where $\langle A, +, -\rangle$ is an abelian group and $\mathfrak{R}$ is a ring of endomorphisms of this group which are unary operations of $\mathfrak{M}$.

THEOREM 7. *If an $\mathfrak{R}$-module $\mathfrak{M}$ is equationally $(|\mathfrak{R}| + \aleph_0)$-compact, then $\mathfrak{M}$ is equationally compact.*

Proof. The formulas a_i appearing in Theorem 3 (ii) are now linear equations of the form

$$r_{i0}x_0 + r_{i1}x_1 + \ldots + r_{im}x_m = a_i ,$$

and hence the set of elements satisfying any formula of the set Σ of Theorem 3 (ii) is a coset of the group $\langle A, +, -\rangle$ with respect to some subgroup completely determined by the system r_{ij} $(i = 1, \ldots, n, j = 1, \ldots, m)$. Since there is $|\mathfrak{R}| + \aleph_0$ such systems and no more such subgroups, and the intersection of two cosets with respect to the same subgroup is non-empty only if these cosets are equal, Theorem 7 follows.

The n-dimensional projective space $P_{\mathfrak{F}}^n$ over a field $\mathfrak{F}$ is the quotient $(\mathfrak{F}^{n+1} \setminus \{(0, \ldots, 0)\})/\sim$, where $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ means that there exists a $z \in \mathfrak{F}$ such that $(zx_0, \ldots, zx_n) = (y_0, \ldots, y_n)$. The set of all polynomials of $m(n+1)$ variables x_{ij} $(i = 1, \ldots, m, j = 0, 1, \ldots, n)$ homogeneous for each i with respect to the "point" $(x_{i0}, \ldots, x_{in})$ and with coefficients in $\mathfrak{F}$ will be denoted by W_{mn} and we set $W_n = W_{1n} \cup W_{1n} \cup \ldots$. For each $w \in W_{nm}$, let R_w be the subset of $(P_{\mathfrak{F}}^n)^m$ defined by the equation $w = 0$, and we put $\mathfrak{P}_{\mathfrak{F}}^n = \langle P_{\mathfrak{F}}^n, R_w\rangle_{w\in W_n}$. It is known that if $\mathfrak{F}$ is a dense locally compact topological field, then $P_{\mathfrak{F}}^n$ has a natural compact topology and all R_w are closed (see [20], V § 27, Example 48) and hence the relational structure $\mathfrak{R}_{\mathfrak{F}}^n$ is atomic compact. The problem of an algebraic characterization of fields $\mathfrak{F}$ for which the last assertion is true is still open but we will prove this assertion for algebraically closed fields. This result intersects with the above-mentioned topological result since such fields as the field of real numbers or p-adic numbers are locally compact and dense but not algebraically closed.

THEOREM 8. *If $\mathfrak{F}$ is an algebraically closed field then $\mathfrak{R}_{\mathfrak{F}}^n$ is atomic compact.*

Proof. First notice the following auxiliary statements.

(i) Let X_s $(s \in S)$ be a system of sets and for any finite set $F \subseteq S$ let T_F be a compact T_1-topology in the cartesian product $\mathbf{P}_{s\in F} X_s$ such

(7) This example yields a negative solution of the problem P483 of [18]. See also [18] for a related problem P482 concerning the group of integers, which is still open, and and example due to A. Ehrenfeucht concerning the ring of integers.

that each projection $\mathbf{P}_{s\in F} X_s \to \mathbf{P}_{s\in F_0} X_s$ ($F_0 \subseteq F$) is continuous and closed (i.e. images and reciprocal images of closed sets are closed) with respect to $\boldsymbol{T}_F$ and $\boldsymbol{T}_{F_0}$. Let $\boldsymbol{T}$ be a topology in $\mathbf{P}_{s\in S} X_s$ with a basis of open sets consisting of all cylinders over the open sets of the topologies $\boldsymbol{T}_F$. Then $\boldsymbol{T}$ is a compact T_1-topology.

This is a simple refinement of the Tychonoff product theorem.

Finite intersections of the sets R_w with $w \in W_{mn}$ are called *algberaic*. The following proposition is easy to check for any field $\mathfrak{F}$.

(ii) Singletons contained in $(P^n_{\mathfrak{F}})^m$ are algebraic sets, finite unions of algebraic sets in $(P^n_{\mathfrak{F}})^m$ are algebraic and cylinders of algebraic sets of $(P^n_{\mathfrak{F}})^m$ in $(P^n_{\mathfrak{F}})^{m+k}$ are algebraic.

A fundamental result of elimination theory (see e.g. [24] (Edition 1940), § 80 or [12], p. 45) states that

(iii) If $\mathfrak{F}$ is algebraically closed then the projections of algebraic sets of $(P^n_{\mathfrak{F}})^{m+k}$ into $(P^n_{\mathfrak{F}})^m$ are algebraic.

The Hilbert basis theorem implies the following statement.

(iv) A descending sequence of algebraic sets has finitely many different terms.

By (ii) and (iv) the algebraic sets of $(P^n_{\mathfrak{F}})^m$ is a family of all closed sets of a compact T_1-topology, the s.c. Zariski topology.

Now we conclude the proof of Theorem 8. Let Σ be a set of atomic formulas with constants in $\mathfrak{R}^n_{\mathfrak{F}}$, which is finitely satisfiable in $\mathfrak{R}^n_{\mathfrak{F}}$ and let S be the set of indices of free variables appearing in Σ. Let $X_s = P^n_{\mathfrak{F}}$ for every $s \in S$ and let the $\boldsymbol{T}_F$ of (i) be the Zariski topologies. By (ii), (iii) and (iv) the all suppositions of (i) are satisfied and hence (i) yields that Σ is satisfiable in $\mathfrak{R}^n_{\mathfrak{F}}$. Q.E.D. [8]

Theorem 8 can be generalized as follows. Consider the family $\boldsymbol{F}$ of all subsets of the Cartesian product $\mathbf{P}_{t\in T} P^{n(t)}_{\mathfrak{F}}$ ($1 \leqslant n(t) < \omega$) which are defined by means of equations $w = 0$, where w is any polynomial in the variables $x_{t,i}$ $\big(t \in T,\ i = 0, 1, \ldots, n(t)\big)$ homogeneous for each t with respect to the "point" $(x_{t,0}, \ldots, x_{t,n(t)})$ and with coefficients in $\mathfrak{F}$. Of course each $(x_{t,0}, \ldots, x_{t,n(t)})/\sim$ is considered as a variable running in the axis $P^{n(t)}_{\mathfrak{F}}$ with index t.

THEOREM 9. *If $\mathfrak{F}$ is algebraically closed, then $\boldsymbol{F}$ is a compact family of sets.*

Proof. Consider the mapping

$$f_m\colon (P^1_{\mathfrak{F}})^m \to P^m_{\mathfrak{F}}\,,$$

(8) We due to A. Białynicki-Birula some helpful discussions relating to this proof.

Note added in proof. Proposition (i) used in this proof may be interesting for itself, but is not essential here. Indeed, by (iii) the sets defined by the formulas appearing in Theorem 3(ii) are algebraic in $P^n_{\mathfrak{F}}$ and hence Theorem 3 and the compactness of Zariski's topology yield Theorem 8.

where the values of f_m,

$$f_m\big((u_1, v_1)/\sim, \ldots, (u_m, v_m)/\sim\big) = (a_0, \ldots, a_m)/\sim\,,$$

are defined by the relation

$$\prod_{i=1}^{m} (u_i X + v_i) = a_0 X^m + a_1 X^{m-1} + \ldots + a_{m-1} X + a_m\,.$$

Since $\mathfrak{F}$ is algebraically closed, f_m is onto and of course f_m is continuous in the Zariski topologies. To prove Theorem 9 it is enough to show that the family of sets $\big\{\bigcup_{p\in A} \mathbf{P}_{t\in T} f^{-1}_{n(t)}\big(p(t)\big)\colon A \in \boldsymbol{F}\big\}$ is compact. But this follows of course from the case $n = 1$ of Theorem 8. Q.E.D.

The following example shows that Theorems 8 and 9 fail for many fields which are not algebraically closed.

E_4. If $\mathfrak{F}$ is a proper subfield of the field of real numbers in which every positive number is a square, then $\mathfrak{R}^1_{\mathfrak{F}}$ is not atomic compact. Indeed, every closed interval of the projective line $P^1_{\mathfrak{F}}$ is of the form $\{x\colon \exists y [R_{w_0}(x, y)]\}$, where

$$w_0 = (ax_0 + bx_1) y_0^2 + (cx_0 + dx_1) y_1^2\,.$$

But $P^1_{\mathfrak{F}}$ is not topologically compact and hence taking a descending sequence of closed intervals of $P^1_{\mathfrak{F}}$ with an empty intersection we obtain a set of formulas of the form

$$R_{w_m}(x, y_m)\,, \quad m = 1, 2, \ldots$$

each finite subset of which is satisfiable in $\mathfrak{P}^1_{\mathfrak{F}}$ but the whole set is not.

We conclude this paper with some statements on the invariance of the various algebraic notions of compactness which we have considered with respect to some natural transformations of algebraic structures.

Given any function $f\colon X \to Y$, we put $f(x_1, \ldots, x_n) = \big(f(x_1), \ldots, f(x_n)\big)$ for every $x_1, \ldots, x_n \in X$, $f(U) = \{f(u)\colon u \in U\}$ for every $U \subseteq \bigcup_n X^n$ and $f(M) = \{f(U)\colon U \in M\}$ for every M such that $\bigcup M \subseteq \bigcup_n X^n$. For any class of formulas K and algebraic structure $\mathfrak{A}$ we denote by $K\mathfrak{A}$ the set of all relations defined in $\mathfrak{A}$ by the formulas of K. A structure $\mathfrak{A}$ is said to be *K-isomorphic* to a structure $\mathfrak{B}$ if there exists a function $f\colon A \to B$ which is one-to-one and onto and such that $f(K\mathfrak{A}) = K\mathfrak{B}$. In particular we get in this way the notions of atomic, positive and elementary isomorphisms of structures which may even differ by their similarity type. Finally, $\mathfrak{A}$ is said to be *K-c-isomorphic* to $\mathfrak{B}$ if $(\mathfrak{A})_c$ is K-isomorphic to $(\mathfrak{B})_c$.

The following propositions are obvious.

(i) Atomic isomorphism ($\mathfrak{c}$-isomorphism) implies positive isomorphism ($\mathfrak{c}$-isomorphism) and positive isomorphism ($\mathfrak{c}$-isomorphism) implies elementary isomorphism ($\mathfrak{c}$-isomorphism).

(ii) Atomic, positive or elementary isomorphism implies atomic, positive or elementary $\mathfrak{c}$-isomorphism respectively.

(iii) Weak K-$\mathfrak{m}$-compactness is an invariant of K-isomorphism.

(iv) Atomic (positive) [elementary] $\mathfrak{m}$-compactness is an invariant of atomic (positive) [elementary] $\mathfrak{c}$-isomorphism.

Moreover, by Theorem 4, we get

(v) Atomic compactness is an invariant of positive $\mathfrak{c}$-isomorphism.

Of course, (i) is derived from the fact that each of the classes of atomic, positive and elementary formulas is the closure of the previous class with respect to some operations. Thus (i) could still be generalized. The same is true for (ii) and (iv). Using Theorem 3, a part of (iv) can be generalized in the following way:

(vi) Atomic $\mathfrak{m}$-compactness is an invariant of K_0-isomorphism, where K_0 is the class of finite disjunctions of formulas of the form defined in Theorem 3 (ii).

Marczewski [15] introduces general algebras in such a way that they do not depend on the similarity type but only on the set of algebraic operations. In this way his concept of an algebra is already a type with respect to a certain isomorphism of classical algebras. Most of the literature quoted by him in [16] studies invariants of a related isomorphism called by him weak isomorphism (see especially [2] and [3]). Weak isomorphism is a relation smaller than atomic isomorphism.

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(⁹) Errata to [18]: On page 2, line 30 the word "such" should be removed. On page 7 in Theorem 5 "$2^{\aleph_\alpha}$" should be substituted by "$2^{\mathfrak{m}}$" and in footnote (⁵) "$\aleph_0$" by "$\aleph_1$".

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Reçu par la Rédaction le 13. 10. 1966