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On vanishing n-th ordered differences and Hamel bases

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Jensen's functional equation (over the reals R)

$$f\left\{\frac{x+y}{2}\right\} = \frac{f(x)+f(y)}{2} \tag{1}$$

has a general solution [1]

$$f(x) = A^0 + A^1(x), \qquad A^0 \text{ constant},$$

where $A^1(x)$ satisfies Cauchy's functional equation

$$A^1(x+y) = A^1(x) + A^1(y) . \tag{2}$$

If in (1) we set $y = x+2\nu$, then (1) may be written in the form

$$\Delta_\nu^2 f(x) \overset{\text{def}}{=} f(x+2\nu) - 2f(x+\nu) + f(x) \equiv 0 . \tag{1'}$$

The equation

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{3}$$

has a general solution [2]

$$g(x) = A^2(x) \tag{4}$$

where $A^2(x) = A_2(x, x)$ and $A_2(x, y)$ is a symmetric bi-additive function (satisfying (2) in each variable). If we set $y = \nu$ and replace x by $x+\nu$ in (3) we obtain

$$\Delta_\nu^2 g(x) = 2g(\nu) . \tag{3'}$$

Clearly, (3') implies

$$\Delta_\nu^3 g(x) \equiv 0 . \tag{5}$$

In this note we study the general solutions of the equation

$$\Delta_\nu^{n+1} f(x) \equiv 0 \tag{6}$$

and also $\Delta_\nu^n f(x) = g(\nu)$ when no regularity assumptions are imposed. It is known that if f satisfies (6) and is continuous at one point, or bounded on a set of positive measure, then $f(x)$ is continuous at all points and is therefore a polynomial of degree n, [3], [4], [5], [6].

Let A_p denote a symmetric multi-additive function on R^p to R, the reals: $A_p(x_1, x_2, \ldots, x_p)$ satisfies Cauchy's functional equation (2) in each argument, and $A_p(x_1, \ldots, x_p) = A_p(x_{i_1}, \ldots, x_{i_p})$ for permutations $(i_1, \ldots, i_p)$ of $(1, \ldots, p)$. By A^p is meant the function on R to R obtained by diagonalizing A_p, that is

$$A^p(x) = A_p(x, x, \ldots, x)\,.$$

The function A^p will play a role analogous to the power function $x \to x^p$.

LEMMA 1. *If $A^n(x)$ denotes the diagonalization at x of a symmetric multi-additive function $A_n(x_1, \ldots, x_n)$ of n arguments, then*

$$\Delta_\nu^p A^n(x) = \begin{cases} n!A^n(\nu) & \text{if} \quad p = n\,, \\ 0 & \text{if} \quad p > n\,. \end{cases} \tag{7}$$

Proof. Let $A_{n-r,r}(x; y)$ denote the value of $A_n(x_1, \ldots, x_n)$ for $x_i = x$, $i = 1, \ldots, n-r$ and $x_i = y$, $i = n-r+1, \ldots, n$. In particular $A_{0n}(y; x) = A_{n0}(x; y) = A^n(x)$. Then

$$A^n(x+y) = A_n(x+y, \ldots, x+y) = \sum_{\sigma=0}^{n} \binom{n}{\sigma} A_{n-\sigma,\sigma}(x; y)\,.$$

Further, from the additivity of A_n in each argument follows

$A_{n-\sigma,\sigma}(m_1x; m_2y) = m_1^{n-\sigma} m_2^{\sigma} A_{n-\sigma,\sigma}(x; y)$ for integer (or rational) m_1, m_2. Hence

$$\begin{aligned} \Delta_\nu^p A^n(x) &= \sum_{r=0}^{p} \binom{p}{r} (-1)^{p-r} A^n(x+r\nu) \\ &= \sum_{r=0}^{p} \sum_{\sigma=0}^{n} \binom{p}{r} \binom{n}{\sigma} (-1)^{p-r} A_{n-\sigma,\sigma}(x; r\nu) \\ &= \sum_{\sigma=0}^{n} \Big\{ \sum_{r=0}^{p} \binom{p}{r} (-1)^{p-r} r^\sigma \Big\} \binom{n}{\sigma} A_{n-\sigma,\sigma}(x; \nu)\,. \end{aligned}$$

But since [7]

$$\sum_{r=0}^{p} \binom{p}{r} (-1)^{p-r} r^\sigma = \begin{cases} 0 & \text{if} \quad \sigma < p\,, \\ p! & \text{if} \quad \sigma = p \end{cases}$$

the lemma follows.

COROLLARY 1. *If $f(x) = A^n(x)$, then $\Delta_\nu^n f(x) = n!f(\nu)$.*

COROLLARY 2. *If $f(x) = \sum_{r=0}^{n} A^r(x)$, where $A^0(x) = A^0$ is constant, then* $\Delta_\nu^{n+1} f(x) \equiv 0$.

THEOREM. *If $\underset{\nu}{\Delta}^{n+1}f(x) \equiv 0$ for all x and ν real, then there exists symmetric multi-additive functions A_p for $p = 1, 2, \ldots, n$ such that*

$$f(x) \equiv A^0 + A^1(x) + \ldots + A^n(x) \quad \text{for all } x\,, \tag{8}$$

where A^0 is constant. Conversely, any such $f(x)$ satisfies $\underset{\nu}{\Delta}^{n+1}f(x) \equiv 0$ for all x, ν real.

Proof. We may consider the elements b_a of a Hamel basis $\mathfrak{B}$ as a basis of the infinite dimensional vector space R over the rationals Q; if B is a finite subset of $\mathfrak{B}$, let R_B denote that subset of the reals R whose Hamel representation requires only elements of B:

$$R_B = \{x|\ x = \sum_{i=1}^{N} r_i b_{a_i} \cdot r_i \in Q \cdot b_{a_i} \in B\}\,.$$

If $[B]$ denotes the cardinality of the (finite) set B, we may introduce a bijective map φ_B: $R_B \to Q^{[B]}$, the space of rational $[B]$-tuples, defined by

$$\varphi_B(x) = \varphi_B\Big(\sum_{i=1}^{[B]} r_i b_{a_i}\Big) = (r_1, \ldots, r_{[B]}), \quad \text{for any } x \in R_B, \tag{9}$$

provided we specify the order in which the rational coefficients in $x = \sum r_i b_{a_i}$ are to be chosen. By Zermelo's theorem we may assume the Hamel basis $\mathfrak{B}$ to be transfinitely ordered, and that in (9), $a_i < a_j$ when $i < j$; the rational $[B]$-tuple associated with x by φ_B is then unique. We denote the inverse map by φ_B^{-1}.

If $Q^{[B]}$ is considered as a vector space over Q, clearly φ_B is Q-linear,

$$\varphi_B(x+y) = \varphi_B(x) + \varphi_B(y), \quad \varphi_B(ax) = a\varphi_B(x), \quad \text{for } x, y \in R_B,\ a \in Q\,. \tag{10}$$

Given any set of distinct real numbers $x_i \in R_B$ of the form

$$x_i = z + N_i \nu \tag{11}$$

for fixed $z, \nu \in R$, N_i integers; then $(x_i - x_j)/(x_i - x_k) = a_{ijk} \in Q$ for $i \neq j \neq k \neq i$.

By (10) it follows that $\varphi_B(x_i) - \varphi_B(x_j) = a_{ijk}[\varphi_B(x_i) - \varphi_B(x_k)]$ implying that the points $\varphi_B(x_j)$ are co-linear in $Q^{[B]}$. Conversely, a straight line in $Q^{[B]}$ is given parametrically by $r_i = \gamma_i t + \beta_i$ where $\gamma_i, \beta_i, t \in Q$, $i = 1, \ldots, [B]$ and co-linear points corresponding to the parameters t_j, $j = 1, \ldots, N$, have as images under φ_B^{-1} the real numbers

$$x_j = \sum_{s=1}^{[B]} (\gamma_s t_j + \beta_s) b_{a_s}, \quad j = 1, \ldots, N; \quad a_{s_1} < a_{s_2} \text{ when } s_1 < s_2\,.$$

But then

$$(x_i - x_j)/(x_i - x_k) = (t_i - t_j)/(t_i - t_k) \in Q \quad \text{for} \quad i \neq j \neq k \neq i\,,$$

and with

$$\frac{x_1 - x_k}{x_1 - x_2} = \frac{n_k}{m_k} \quad \text{for integers } n_k, m_k \text{ and } k = 2, \ldots, N$$

we may write

$$x_k = x_1 - \Big(n_k \prod_{\substack{i=1 \\ i \neq k}}^{N} m_i\Big) \frac{x_1 - x_2}{\Big\{\prod_{i=1}^{N} m_i\Big\}}$$

so that the x_i's are of the form (11). Hence, *a finite set of distinct real numbers x_i are of the form* (11) *if and only if their images in* $Q^{[B]}$ (under the map φ_B) *are co-linear.* Here B is any finite subset of $\mathfrak{B}$ relative to which all x_i's are expressible.

Let P^n_m denote some nth degree polynomial in m variables, $P^n_m : R^m \to R$. If f is a polynomial of the form $f(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \in R$, then

$$f\{\varphi_B^{-1}(r_1, \ldots, r_{[B]})\} = \sum_{i=0}^{n} a_i \Big(\sum_{j=1}^{[B]} r_j b_{\alpha_j}\Big)^i = P^n_{[B]}(r_1, \ldots, r_{[B]})\,. \tag{12}$$

Alternatively, if f coincides with a polynomial P^n_1 on R_B, then $f\varphi_B^{-1}$ coincides with a $P^n_{[B]}$ on $Q^{[B]}$. The converse however is not true, for consider $f\varphi_B^{-1}(r_1, \ldots, r_{[B]}) \equiv r_1$, that is, $f\big(\sum_{i=1}^{[B]} r_i b_{\alpha_i}\big) = r_1$, which is clearly not a polynomial in $x = \sum_{i=1}^{[B]} r_i b_{\alpha_i}$ (unless $[B] = 1$); f in this example is Q-linear on R_B however.

If f satisfies $\Delta_\nu^{n+1} f(x) \equiv 0$ for all $x, \nu \in R$ then clearly f coincides with a polynomial P^n_1 on any set of the form (11); alternatively $f\varphi_B^{-1}$ coincides with a polynomial $P^n_{[B]}$ on any line in $Q^{[B]}$. If we choose in particular the lines parallel to the co-ordinate axes in $Q^{[B]}$, we have that $f\varphi_B^{-1}$ is a polynomial in each variable, for every fixed value of the remaining variables, whence *$f\varphi_B^{-1}$ coincides with some $P^n_{[B]}$ on all $Q^{[B]}$*, and this for every B. As in the above example, it does not follow that f is itself a polynomial on R_B. However, we now show that f can be obtained by diagonalization of Q-multilinear functions.

It is well known [8] that any multinomial $P^n_{[B]}$, defined on $Q^{[B]}$, can be written uniquely in the form

$$P^n_{[B]} = \sum_{p=0}^{n} A^p_B \quad \text{where } A^0_B \text{ is a constant}\,, \tag{13}$$

and A_B^p is the diagonalization of a uniquely determined symmetric multilinear form $A_{B,p}$ in p variables, each ranging over $Q^{[B]}$. Since in (12) the $P_{[B]}^n$ depend on the subset $B \subset \mathfrak{B}$, so also the multilinear forms; the dependence is shown by the subscript B in $A_{B,p}$. Hence by (12) and (13)

$$(14) \qquad f\varphi_B^{-1} = \sum_{p=0}^{n} A_B^p \quad \text{on } Q^{[B]} \text{ for every } B$$

or alternatively

$$(14') \qquad f = \sum_{p=0}^{n} A_B^p \varphi_B \quad \text{on } R \text{ for every } B.$$

Each $A_B^p\varphi_B$ is itself the diagonalization of a multi-additive form on R_B^p since, by the linearity of φ_B, $A_{B,p}\{\varphi_B(x_1), \dots, \varphi_B(x_p)\}$ *is additive in each* $x_i \in R_B$.

We first treat the individual terms in (14′) and show that *if a function $g\colon R \to R$ reduces to $g = A_B^p\varphi_B$ on each R_B, then $g = \bar{A}^p$ on R where $\bar{A}^p$ is the diagonalization of a symmetric multilinear form $\bar{A}_p$ on R^p*. But one may simply define

$$(15) \qquad \bar{A}_p(x_1, \dots, x_p) = A_{B,p}\{\varphi_B(x_1), \dots, \varphi_B(x_p)\}$$

where B consists of elements of $\mathfrak{B}$ including those required in the Hamel representation of all x_i, $i = 1, \dots, p$. *We now prove that B, on the right hand side of* (15), *may be replaced by any $B' \supset B$, and hence that $\bar{A}_p$ is well defined by* (15) *on R^p*. Since each A_B^p determines a unique symmetric multilinear form $A_{B,p}$, it follows that two symmetric multilinear forms $A_{B,p}$ and $A'_{B,p}$ which differ at one point must give rise to different diagonalizations A_B^p and A'^p_B. Further a multilinear form $A_{B',p}(\vec{\xi}_1, \dots, \vec{\xi}_p)$ on p argument vectors $\vec{\xi}_i \in Q^{[B']}$ may be diagonalized ($\vec{\xi}_1 = \dots = \vec{\xi}_p = \vec{\xi}$) at some $\vec{\xi}$ belonging to a subspace $Q^{[B]}$ of $Q^{[B']}$; the same result is clearly obtained if the $\vec{\xi}_i$ are first restricted to the subspace, thereby defining a symmetric multilinear form $A'_{B,p}$ on $Q^{[B]}$, and then diagonalizing $A'_{B,p}$ at $\vec{\xi} \in Q^{[B]}$. Hence if $A_{B,p}$ does not coincide with $A'_{B,p}$, then $A_B^p \neq A'^p_B$ whence $A_B^p(\vec{\xi}) \neq A_{B'}^p(\vec{\xi})$ for some $\vec{\xi} \in Q^{[B]} \subset Q^{[B']}$. But by hypothesis

$$g(x) = A_B^p\varphi_B(x) = A_{B'}^p\varphi_{B'}(x) \quad \text{for} \quad x \in R_B \subset R_{B'}.$$

Hence $A_{B,p}\varphi_B$ must agree with $A_{B',p}\varphi_{B'}$ on all R_B as required.

It remains only to show that the individual terms in (14′) define a function $g\colon R \to R$. It is conceivable for example that a function $f\colon R \to R$, whose restriction to R_B is f_B, may be written in the form $(f_B - b_a) + b_a$ where b_a is the first element in the well ordered set B. Clearly $f_B - b_a$ and b_a do not define functions on R. That this is not the case in (14′) follows from Lemma 1 since $\Delta_\nu^n f(0) = n! A_B^n\varphi_B(\nu)$ for any $\nu \in R_B$. Hence,

$n!A_B^n\varphi_B$ on R_B defines the function $\underset{\nu}{\Delta}^n f(0)$ on R; the remaining terms therefore also define a function on R, satisfying $\underset{\nu}{\Delta}^n f(x) \equiv 0$. By repeating the above argument the desired result follows, proving the theorem.

COROLLARY 3. *A necessary and sufficient condition for the equation*

$$\underset{\nu}{\Delta}^n f(x) = g(\nu) \tag{16}$$

to have a solution is that $g(\nu) = n!A^n(\nu)$ *where* A^n *is the diagonalization of a multi-additive function* A_n *of* n *arguments. The general solution of* (16) *is then* $f(x) = A^n(x) + h(x)$ *where* $h(x)$ *is the general solution of* $\underset{\nu}{\Delta}^n h(x) \equiv 0$.

Proof. From (16) follows $\underset{\nu}{\Delta}^{n+1} f(x) \equiv 0$ whence f is given by (8). But by Lemma (1) follows $\underset{\nu}{\Delta}^n f(x) = n!A^n(\nu)$.

COROLLARY 4. *A necessary and sufficient condition for* $f(x)$ *to be of the form* $A^n(x)$ *is that* $\underset{\nu}{\Delta}^n f(x) = n!f(\nu)$.

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