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# ON A PROBLEM OF BJARNI JÓNSSON CONCERNING AUTOMORPHISMS OF A GENERAL ALGEBRA

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**1.** Let $G$ be a subgroup of the group $S_A$ of all permutations of a fixed set $A$. M. Armbrust and J. Schmidt in [1] and Bjarni Jónsson in [2] investigate the problem of existence of an algebra $\mathfrak{A} = \langle A, \boldsymbol{F}\rangle$ such that $G$ be the group $\mathrm{Aut}(\mathfrak{A})$ of all automorphisms of algebra $\mathfrak{A}$. Namely Jónsson formulates the following two properties of $G$:

$a_m(G)$: there exists an algebra $\mathfrak{A} = \langle A, \boldsymbol{F}\rangle$ such that every $F \epsilon \boldsymbol{F}$ is less than $m$-ary and $\mathrm{Aut}(\mathfrak{A}) = G$.

$\beta_m(G) \Leftrightarrow \bigwedge\limits_{\varphi \epsilon S_A} [( \bigwedge\limits_{\substack{X \subseteq A \\ |X| < m}} \bigvee\limits_{g \epsilon G} \varphi | X = g | X) \Rightarrow \varphi \epsilon G]$.

Jónsson has proved that $a_m(G) \Rightarrow \beta_{m+1}(G)$ and that, for $m \neq 2$, $\beta_m(G) \Rightarrow a_m(G)$.

The purpose of this paper is to modify the property $\beta_m(G)$ so as to obtain a property equivalent to $a_m(G)$ for all natural $m$.

Let us consider two other properties of $G$:

$$\gamma_m(G) \Leftrightarrow \bigwedge_{\varphi \epsilon S_A} \Big\{\Big[ \bigwedge_{\substack{X \subseteq A \\ |X| < m}} \bigvee_{g \epsilon G} \big(g | C_G(X)\big) = \big(\varphi | C_G(X)\big)\Big] \Rightarrow \varphi \epsilon G\Big\},$$

$$\gamma_2^*(G) \Leftrightarrow \bigwedge_{\varphi \epsilon S_A} \Big\{ \bigwedge_{x \epsilon A} \Big[ \bigvee_{g \epsilon G} \big(g | C_G(\{x\})\big) = \big(\varphi | C_G(\{x\})\big) \vee C_G(\{x\}) = \{x\} \wedge C_G(\{\varphi x\}) = \{\varphi x\}\Big] \Rightarrow \varphi \epsilon G\Big\},$$

where

$$C_G(X) = \{y | y \epsilon A \wedge \bigwedge_{g, g' \epsilon G} [g|X = g'|X \Rightarrow g(y) = g'(y)]\}. \tag{1}$$

This paper contains a proof of the equivalence $a_m(G) \Leftrightarrow \gamma_m(G)$ for natural $m > 2$, and of the equivalence $a_2(G) \Leftrightarrow \gamma_2^*(G)$.

I use some ideas of [1] and [2] in the proofs of lemmas 2, 3 and 4.

**2.** We shall assume that the fixed set $A$ has at least two elements. By $\boldsymbol{O}(A)$ we denote the set of all (finitary) operations in $A$. The arity of an operation $F$ will be denoted by $\varrho(F)$. For simplicity's sake, we shall often write $x$ instead of $\langle x_1, \ldots, x_m\rangle$ and, similarly, $\varphi x$ instead of $\langle \varphi x_1, \ldots, \varphi x_m\rangle$ for $\varphi \in \mathrm{S}_A$, when no confusion will arise. $Gx$ is the set of all images of the sequence $x$ under all $g \in G$.

Finally define $\boldsymbol{F}_{G,m}$ and $\mathfrak{A}_{G,m}$ by

$$\boldsymbol{F}_{G,m} = \{F | F \in \boldsymbol{O}(A) \wedge \varrho(F) < m \wedge \bigwedge_{g\in G} \bigwedge_{x \in A^{\varrho(F)}} F(gx) = gF(x)\} \tag{2}$$

and

$$\mathfrak{A}_{G,m} = \langle A, \boldsymbol{F}_{G,m}\rangle. \tag{3}$$

By $\mathrm{C}_{G,m}(X)$ we denote the subalgebra of the algebra $\mathfrak{A}_{G,m}$ generated by $X$. In the sequel, an essential part will be played by

LEMMA 1. *We have*

$$a_m(G) \Leftrightarrow \mathrm{Aut}(\mathfrak{A}_{G,m}) = G.$$

Proof. 1. The implication $\Leftarrow$ is obvious.

2. If $\mathfrak{A} = \langle A, \boldsymbol{F}\rangle$, $\mathrm{Aut}(\mathfrak{A}) = G$ and $\boldsymbol{F}$ is a family of at most $(m-1)$-ary operations, then $\boldsymbol{F} \subseteq \boldsymbol{F}_{G,m}$. Hence $\mathrm{Aut}(\mathfrak{A}_{G,m}) \subseteq \mathrm{Aut}(\mathfrak{A}) = G$. Since obviously $G \subseteq \mathrm{Aut}(\mathfrak{A}_{G,m})$, we have finally $\mathrm{Aut}(\mathfrak{A}_{G,m}) = G$.

**3.** LEMMA 2. *For every natural $m$ and every $X \subseteq A$, if $|X| < m$, then $\mathrm{C}_G(X) = \mathrm{C}_{G,m}(X)$.*

Proof. Since $\mathrm{C}_{G,m}(\emptyset)$ is the set of all algebraic constants of algebra $\mathfrak{A}_{G,m}$, the equation $\mathrm{C}_G(\emptyset) = \mathrm{C}_{G,m}(\emptyset)$ is a simple consequence of (1).

Now, let us suppose that $X = \{x_1, \ldots, x_k\}$, where $1 \leqslant k < m$. We denote by $x$ the sequence $\langle x_1, \ldots, x_k\rangle$. If $y \in \mathrm{C}_{G,m}(X)$, then $y = F(x)$ for a certain operation $F \in \boldsymbol{F}_{G,m}$. According to (1) and (2), if $g|X = g'|X$ for some $g, g' \in G$, then

$$gy = gF(x) = F(gx) = F(g'x) = g'F(x) = g'y.$$

Hence $\mathrm{C}_G(X) \supseteq \mathrm{C}_{G,m}(X)$.

To prove the converse relation, let us suppose that $y \in \mathrm{C}_G(X)$ and put

$$F(v) = \begin{cases} gy & \text{if} \quad v = gx \text{ and } g \in G, \\ v_1 & \text{if} \quad v \notin Gx \end{cases}$$

for $v \in A^k$. In view of (1), the element $F(v)$ does not depend on the choice of $g$ and so operation $F$ is well defined. Further, if $v = gx$, then $F(hv) = F(hgx) = hgy = hF(gx) = hF(v)$ for $h \in G$, and if $v \notin Gx$, then $hv \notin Gx$. Therefore $F(hv) = hv_1 = hF(v)$. Consequently $y \in \mathrm{C}_{G,m}(X)$, because $F \in \boldsymbol{F}_{G,m}$ and $F(x) = y$.

LEMMA 3. *If $m > 2$, then*

$$\varphi \in \mathrm{Aut}(\mathfrak{A}_{G,m}) \Leftrightarrow \bigwedge_{\substack{X \subseteq A \\ |X| < m}} \bigvee_{g \in G} \big(g|\mathrm{C}_G(X)\big) = \big(\varphi|\mathrm{C}_G(X)\big).$$

Proof. 1. $\Rightarrow$. First we show that if $2 \leqslant |X| < m$, then there exists a $g \in G$ such that $\varphi|X = g|X$. Contrary to this let us suppose that there exists a subset $X \subseteq A$ such that $\varphi|X \neq g|X$. Arranging the members of $X$ into a sequence $x = \langle x_1, \ldots, x_k \rangle$, where $2 \leqslant k < m$, we obtain

$$\varphi x \notin Gx. \tag{4}$$

For every $v = \langle v_1, \ldots, v_k \rangle \in A^k$, put

$$F(v) = \begin{cases} v_1 & \text{if} \quad v \in Gx, \\ v_2 & \text{if} \quad v \notin Gx. \end{cases} \tag{5}$$

Observe that $F$ commutes with every $g \in G$ and, consequently, it belongs to $\boldsymbol{F}_{G,m}$. From (4) and (5) and from the hypothesis that $\varphi \in \mathrm{Aut}(\mathfrak{A}_{G,m})$ we obtain

$$\varphi x_1 = F(\varphi x) = \varphi F(x) = \varphi x_2,$$

which contradicts the assumption that $x_1 \neq x_2$.

Now suppose that $X = \{x_1\}$. Let us put $Y = \{x_1, x_2\}$, where $x_1 \neq x_2$. In view of what we have already proved and because $m > 2$ and $|Y| = 2$ we have $\varphi|Y = g|Y$ for some $g \in G$, and, in particular, $\varphi|X = g|X$. We have thus proved that for every $X$ with $|X| < m$ there exists an element $g \in G$ which agrees with $\varphi$ on $X$. Moreover, $g, \varphi \in \mathrm{Aut}(\mathfrak{A}_{G,m})$ and, therefore, if $g|X = \varphi|X$, then $(g|\mathrm{C}_G(X)) = (\varphi|\mathrm{C}_G(X))$. By lemma 2 the implication $\Rightarrow$ is proved.

2. $\Leftarrow$. Let $x \in A^k$, $k < m$, $F \in \boldsymbol{F}_{G,m}$ and $\varrho(F) = k$. Let $X$ be the set of all terms of the sequence $x$. By hypothesis, $(g|\mathrm{C}_G(X)) = (\varphi|\mathrm{C}_G(X))$ holds for some $g \in G$. Since, from lemma 2,

$$\{F(x), x_1, \ldots, x_k\} \subseteq \mathrm{C}_{G,m}(X) = \mathrm{C}_G(X),$$

we have

$$F(\varphi x) = F(gx) = gF(x) = \varphi F(x).$$

The proof is complete.

LEMMA 4. *We have*

$$\varphi \in \mathrm{Aut}(\mathfrak{A}_{G,2})$$
$$\Leftrightarrow \bigwedge_{x \in A} \Big[ \bigvee_{g \in G} \big(\varphi|\mathrm{C}_G(\{x\})\big) = \big(g|\mathrm{C}_G(\{x\})\big) \vee \mathrm{C}_G(\{x\}) = \{x\} \wedge \mathrm{C}_G(\{\varphi x\}) = \{\varphi x\} \Big].$$

Proof. 1. $\Leftarrow$. Let $F \in \boldsymbol{F}_{G,2}$ and $x \in A$. If $\big(g|\mathrm{C}_G(\{x\})\big) = \big(\varphi|\mathrm{C}_G(\{x\})\big)$, then, in view of lemma 2,

$$F(\varphi x) = F(gx) = gF(x) = \varphi F(x).$$

If $\mathrm{C}_G(\{x\}) = \{x\}$ and $\mathrm{C}_G(\{\varphi x\}) = \{\varphi x\}$, then it follows from lemma 2 that $F(\varphi x) = \varphi x = \varphi F(x)$.

2. $\Rightarrow$. Let $\varphi$ be an automorphism of $\mathfrak{A}_{G,2}$ and let $x \in A$. We consider two cases (a) $\varphi x = gx$ for some $g \in G$, and (b) $\varphi x \notin Gx$.

(a) Since $\varphi$, $g$ are automorphisms of $\mathfrak{A}_{G,2}$, $\varphi$ and $g$ have to agree with one another on the subalgebra of the algebra $\mathfrak{A}_{G,2}$ generated by $\{x\}$. Thus, in view of lemma 2, we obtain $\big(g \,|\, \mathrm{C}_G(\{x\})\big) = \big(\varphi \,|\, \mathrm{C}_G(\{x\})\big)$.

(b) We shall prove that $\mathrm{C}_G(\{x\}) = \{x\}$. Contrary to this suppose that $x \neq y \in \mathrm{C}_G(\{x\})$. We put

$$F(v) = \begin{cases} gy & \text{if} \quad v = gx \text{ and } g \in G, \\ v & \text{if} \quad v \notin Gx. \end{cases}$$

Obviously $F(x) = y$ and $F(\varphi x) = \varphi F(x)$. Moreover, as it is easy to see, $F \in \boldsymbol{F}_{G,2}$. Hence we should have $\varphi y = \varphi F(x) = F(\varphi x) = \varphi x$. This, however, contradicts the assumption that $x \neq y$. Thus $\mathrm{C}_G(\{x\}) = \{x\}$. This means that, for an arbitrary operation $F \in \boldsymbol{F}_{G,2}$, we have $F(x) = x$ and $F(\varphi x) = \varphi x$, because $\varphi \in \mathrm{Aut}(\mathfrak{A}_{G,2})$. Thus $\mathrm{C}_G(\{\varphi x\}) = \{\varphi x\}$, which finishes the proof.

**4.** THEOREM. (i) *If* $m > 2$, *then* $a_m(G) \Leftrightarrow \gamma_m(G)$.

(ii) $a_2(G) \Leftrightarrow \gamma_2^*(G)$.

Proof. Lemmas 1 and 3 imply (i), lemmas 1 and 4 imply (ii).

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### *REFERENCES*

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