Completeness theorems for some presupposition-free logics

by

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1. Introduction. Contemporary logic has often been faulted for requiring, first, that its individual variables have values, and, second, that its individual constants each designate a value of the individual variables. In recent years various modifications of QC=, the first-order quantificational calculus with identity, have been designed, that eschew one, or the other, or both of these presuppositions; (1) and some, though not all, of the calculi in question have received a semantical interpretation. (2) Our purpose in this paper is to furnish a systematic account—both from a syntactical point of view and a semantical one—of such like *presupposition-free logics*. We shall study ten of them in the body of the paper, and prove the (semantical) completeness of each, in Henkin's sense of the word 'completeness' and in Gödel's. (3) Further presupposition-free logics are discussed in footnotes and appendices; some may prove as interesting as the ten treated in the text, but for lack of space could not be accommodated therein.

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(1) See in particular [10], [5], and [7]. Modifications of QC—the first-order quantificational calculus without identity—that do not require 'x', 'y', 'z', etc., to have values, will be found in [2], [6], [13], and [14], and a like-minded one of QC= will be found in [15]. Those five calculi have no individual constants, though, and hence are in a class apart.

(2) One semantical account of Hintikka's modification of QC= in [5] is supplied in [16], another in [17]. To our knowledge no semantical account of Leblanc and Hailperin's modification of QC= in [10] has appeared so far.

(3) We take a calculus C to be (semantically) complete in Gödel's sense if every valid formula of C is derivable from $\emptyset$ in C, and to be (semantically) complete in Henkin's sense if every formula of C that is implied by (or, in Tarski's terminology, is a semantical consequence of) a set of formulas of C is derivable from that set in C. Completeness in Henkin's sense is referred to in [4] as *strong completeness*.

When an individual constant is not required to designate any value of 'x', 'y', 'z', etc., it may designate something not a value of a variable, a possibility often overlooked. It may also fail to designate at all, i.e., serve as a *non-designating* constant. Various options are open as regards the truth of statements containing a non-designating constant: (i) the statements in question may be denied any truth-value whatsoever; (ii) they may all be assigned some truth-value other than the classical T and F; (iii) some may be assigned T or F, and the rest assigned no truth-value; (iv) they may all be assigned the truth-value T; (v) they may all be assigned the truth-value F; and (vi) some (but not all) may be assigned T, and the rest F. Perhaps the most interesting alternative under (vi) is the one, say, (vi′), in which all atomic statements containing a non-designating constant are arbitrarily assigned one of the two truth-values T and F, and the truth-value of non-atomic ones is determined by the standard semantical rules of truth. For lack of space we relegate treatment of (i) to footnote 22, pass over (ii), and refer the reader to [17] for treatment of the major alternative under (iii), thus restricting ourselves in the main text to (iv), (v), and (vi′).

Among our ten modifications of $QC{=}$, five (namely, $QC\overset{2}{=}$, $QC\overset{4}{=}$, $QC\overset{6}{=}$, $QC\overset{8}{=}$, and $QC\overset{10}{=}$), lift the classical restriction that variables must have values by acknowledging the empty set $\emptyset$ as a possible domain. The others do not. $QC\overset{3}{=}$, $QC\overset{4}{=}$, $QC\overset{7}{=}$, and $QC\overset{8}{=}$ allow for non-designating constants, yet require that designating ones each designate a value of a variable. $QC\overset{3}{=}$ and $QC\overset{4}{=}$ handle formulas that contain a non-designating constant in the spirit of (iv) above; $QC\overset{7}{=}$ and $QC\overset{8}{=}$ in the spirit of (v). $QC\overset{5}{=}$ and $QC\overset{6}{=}$ are like $QC\overset{3}{=}$ and $QC\overset{4}{=}$, respectively ($QC\overset{9}{=}$ and $QC\overset{10}{=}$ like $QC\overset{7}{=}$ and $QC\overset{8}{=}$, respectively), except for allowing designating constants to designate something not a value of a variable. This is accomplished by introducing, besides the domain D (or *inner* domain) serving as the range of values of the individual variables, an *outer* domain D' that is disjoint from D and whose members may also be assigned to the individual constants of $QC\overset{5}{=}$, $QC\overset{6}{=}$, $QC\overset{9}{=}$, and $QC\overset{10}{=}$. (4)

$QC\overset{1}{=}$ and $QC\overset{2}{=}$, possibly the most interesting of our ten calculi, are susceptible of various semantical accounts. The most general of these, call it *Account One*, (a) uses an outer as well as an inner domain, thus permitting an individual constant to designate something not the value of a variable, and also (b) allows for non-designating constants, formulas that contain such being handled in the spirit of (vi′) above. It turns out,

(4) The notion of an outer domain was mentioned to Professor Leblanc by Professor Joseph S. Ullian in the spring of 1962.

though, that as regards soundness and completeness (5), (b) is dispensable in the presence of (a), and likewise (a) dispensable in the presence of (b). (6) Thus two further accounts yield completeness theorems for $QC\overset{1}{=}$ and $QC\overset{2}{=}$: one requiring every individual constant of $QC\overset{1}{=}$ and $QC\overset{2}{=}$ to designate (*Account Two*), the other requiring every such constant—when it designates—to designate a value of 'x', 'y', 'z', etc. (*Account Three*). For convenience (and novelty, since Account Three is close to van Fraassen's in [16]), we employ Account Two in the main text, and relegate Accounts One and Three to Appendix II.

Anticipations of one of our results, Theorem T7(b) in Section IV, will be found in [14] for the case where A is a closed formula of $QC\overset{2}{=}$ that contains no identity sign nor any individual constant, in [15] for the case where A is a formula of $QC\overset{2}{=}$ that contains no individual constant, and in [16] for the case where A is a closed formula of $QC\overset{2}{=}$ that contains no sentence variable. (7) To our knowledge the remaining results, though, are new. (8)

The argument whereby we deduce from the completeness in Henkin's sense of $QC\overset{5}{=}$ and $QC\overset{6}{=}$ that of $QC\overset{3}{=}$ and $QC\overset{4}{=}$, can be put to further use: deducing from the completeness in Henkin's sense of $QC\overset{1}{=}$ that of $QC{=}$ or, as we shall label that calculus here, $QC\overset{0}{=}$. The result is a familiar one, but detailed and correct proof of it is somewhat of a rarity. (9)

(5) We take a calculus C to be (semantically) sound in Gödel's sense if every formula of C that is derivable from $\emptyset$ in C is valid, and to be (semantically) sound in Henkin's sense if every formula of C that is derivable in C from a set of formulas of C is implied by that set.

(6) An individual constant 'a' of $QC\overset{1}{=}$ or $QC\overset{2}{=}$ that does not designate at all can always be made to designate something not a value of a variable: adding to D' (the outer domain) some *ad hoc* thing d' not in D (the inner domain), assigning d' to 'a', and adding suitable m-tuples ($m = 1, 2, \ldots$) to the subsets of $(D \cup D')^m$ already assigned to the predicate variables of $QC\overset{1}{=}$ or $QC\overset{2}{=}$, will do it. And one that designates something not a value of a variable can always be made not to designate at all: assigning suitable truth-values to the atomic formulas of $QC\overset{1}{=}$ or $QC\overset{2}{=}$ that contain 'a', is all that is needed. Hence Accounts Two and Three are sure to fit $QC\overset{1}{=}$ and $QC\overset{2}{=}$ if Account One does.

(7) [14] borrows heavily from [2], and so to a lesser extent does [15].

(8) Professor van Fraassen has recently obtained a proof of T7(a) for the case where S is a set of closed formulas of $QC\overset{2}{=}$ that contain no sentence variable and A is a closed formula of $QC\overset{2}{=}$ that contains no sentence variable either. The proof has not reached print yet.

(9) Proof of the completeness in Gödel's sense of $QC\overset{0}{=}$ goes back of course to Gödel in [1]. Proof of its completeness in Henkin's sense can be retrieved from [3], once the definition of a formal deduction from assumptions is amended to read as in [12]. For further details on this last matter, see footnote (13).

2. Syntax. We attend in this section to the syntax of $\mathrm{QC}\overset{0}{=}$-$\mathrm{QC}\overset{10}{=}$ (calculi which of course all have the same signs and hence the same formulas), and that of $\mathrm{QC}\overset{\infty}{=}$, an auxiliary calculus that is like $\mathrm{QC}\overset{1}{=}$ except for having $\aleph_0$ extra individual constants of its own and hence extra formulas of its own. In the definitions and remarks that follow, the index i is meant—unless otherwise indicated—to run from 0 to 10.

D1. The signs of $\mathrm{QC}\overset{i}{=}$ are the two connectives '$\sim$' and '$\supset$', the one quantifier letter '$\forall$', the identity sign '$=$', the two parentheses '(' and ')', one or more sentence variables (among them 'p'), one or more monadic predicate variables (among them 'f'), for each m from 2 on zero or more m-adic predicate variables, $\aleph_0$ individual variables (among them 'x' and 'y'), and one or more individual constants.

D1$^\infty$. The signs of $\mathrm{QC}\overset{\infty}{=}$ are those of $\mathrm{QC}\overset{1}{=}$, together with $\aleph_0$ extra individual constants.

Remarks. The individual variables and individual constants of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] will be collectively known as the *individual terms* of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$], and be presumed to be arranged in some fixed alphabetical order. We shall refer to the sentence variables of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] by means of '$P$', to its predicate variables by means of 'F', and to its individual terms by means of 'X', 'Y', and 'Z'.

D2. Let A be a finite sequence of signs of $\mathrm{QC}\overset{i}{=}$.

(a) If A is a sentence variable of $\mathrm{QC}=$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(b) If A is of the kind $F(X_1, X_2, \ldots, X_m)$, where F is an m-adic ($m \geqslant 1$) predicate variable of $\mathrm{QC}\overset{i}{=}$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(c) If A is of the kind $(X = Y)$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(d) If A is of the kind $\sim B$ and B counts as a formula of $\mathrm{QC}\overset{i}{=}$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(e) If A is of the kind $(B \supset C)$ and both B and C count as formulas of $\mathrm{QC}\overset{i}{=}$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(f) If A is of the kind $(\forall X)B$, where X is an individual variable of $\mathrm{QC}\overset{i}{=}$, and B counts as a formula of $\mathrm{QC}\overset{i}{=}$, then A counts as a formula of $\mathrm{QC}\overset{i}{=}$;

(g) A counts as a formula of $\mathrm{QC}\overset{i}{=}$ pursuant only to one or another of (a)-(f).

D2$^\infty$. Like D2, but with '$\mathrm{QC}\overset{\infty}{=}$' in place of '$\mathrm{QC}\overset{i}{=}$'.

Remarks. Except D6-D8 and D10, every further definition in this section will be understood to carry along its analogue for $\mathrm{QC}\overset{\infty}{=}$. We shall refer to the formulas of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] by means of '$A$', '$B$', and '$C$', and to sets of such by means of 'S'. To abridge matters, we shall usually write '$(A \& B)$' for '$\sim(A \supset \sim B)$', '$(A \vee B)$' for '$(\sim A \supset B)$', '$(A \equiv B)$' for '$((A \supset B) \& (B \supset A))$', '$(\exists X)A$' for '$\sim(\forall X)\sim A$', and '$\mathrm{E}!X$' for '$(\exists x)(x = X)$' when X is distinct from 'x', otherwise for '$(\exists y)(y = X)$'; and we shall usually omit outer parentheses. Lastly, we shall assume that the formulas of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] are arranged in some fixed alphabetical order.

D3. Let A and B be formulas of $\mathrm{QC}\overset{i}{=}$.

(a) If A and B are the same, then B counts as a component of A;

(b) If A is of the kind $\sim C$ and B counts as a component of C, then B counts as a component of A;

(c) If A is of the kind $C \supset C'$ and B counts as a component of C or of C', then B counts as a component of A;

(d) If A is of the kind $(\forall X)C$ and B counts as a component of C, then B counts as a component of A;

(e) B counts as a component of A pursuant only to one another of (a)-(d).

D4. Let X be an individual term of $\mathrm{QC}\overset{i}{=}$, and A be a formula of $\mathrm{QC}\overset{i}{=}$.

(a) An occurrence of X in A is said to be bound if it is in a component of A of the kind $(\forall X)B$, otherwise to be free.

(b) X is said to occur bound *in* A if at least one occurrence of X in A is bound, to occur free in A if at least one occurrence of X in A is free.

Remark. It follows from D2-D4 that every occurrence of an individual constant of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] in a formula of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] is free.

Convention. Let A be a formula of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$], and let X be an individual term of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$].

(1) Let Y be an individual variable of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$]. If X does not occur free in A or X occurs free in at least one component of A of the kind $(\forall Y)B$, we shall take both $A(Y/X)$ and $A(Y//X)$ to be A; otherwise, we shall take $A(Y/X)$ to be the result of replacing *every* free occurrence of X in A by an occurrence of Y, and $A(Y//X)$ to be any result of replacing *zero or more* free occurrences of X in A by an occurrence of Y.

(2) Let Y be an individual constant of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$]. If X does not occur free in A, we shall take both $A(Y/X)$ and $A(Y//X)$ to be A. If X occurs free in A, we shall take $A(Y/X)$ to be the result of replacing every free occurrence of X in A by an occurrence of Y, and $A(Y//X)$ to be any result of replacing zero or more free occurrences of X in A by an occurrence of Y.

D5. Let A be a formula of $\mathrm{QC}\overset{i}{=}$.

(a) If A is as in D2(a)-(c), then A is said to be atomic.

(b) If no individual variable of $\mathrm{QC}\overset{i}{=}$ occurs free in A, then A is said to be closed; otherwise, to be open.

(c) If no individual constant of $\mathrm{QC}\overset{i}{=}$ occurs in A, then A is said to be constant-free.

(d) If '$\forall$' does not occur in A, then A is said to be quantifier-free.

Remark. In view of D5 a formula A of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] is closed and quantifier-free if no individual variable of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] occurs at all in A.

D6. Let A be a formula of $\mathrm{QC}\overset{i}{=}$, where $i = 2$ or 6.

Case 1: A is open. Then '$p \supset p$' counts as the $\varnothing$-associate of A.

Case 2: A is closed.

(a) If A is atomic, then A counts as the $\varnothing$-associate of A;

(b) If A is of the kind $\sim B$ and B' is the $\varnothing$-associate of B, then $\sim B'$ counts as the $\varnothing$-associate of A;

(c) If A is of the kind $B \supset C$, and B' and C' are the $\varnothing$-associates of B and C, respectively, then $B' \supset C'$ counts as the $\varnothing$-associate of A;

(d) If A is of the kind $(\forall X)B$, then '$p \supset p$' counts as the $\varnothing$-associate of A. (10)

Remark. It follows from the above definitions that the $\varnothing$-associate of a closed formula of $\mathrm{QC}\overset{i}{=}$ ($i = 2$ or 6) is closed and quantifier-free.

D7. (a) A formula of $\mathrm{QC}\overset{i}{=}$ counts as an axiom of $\mathrm{QC}\overset{0}{=}$, $\mathrm{QC}\overset{3}{=}$, and $\mathrm{QC}\overset{7}{=}$ if it is of one of the following eight kinds, *where in the fourth case X is understood not to occur free in A*:

$$A \supset (B \supset A),$$

$$\big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big),$$

$$(\sim A \supset \sim B) \supset (B \supset A),$$

$$A \supset (\forall X)A,$$

$$(\forall X)(A \supset B) \supset \big((\forall X)A \supset (\forall X)B\big),$$

$$(\forall X)A \supset A(Y/X),$$

$$X = X,$$

and

$$X = Y \supset \big(A \supset A(Y//X)\big).$$

(b) A formula of $\mathrm{QC}\overset{i}{=}$ counts as an axiom of $\mathrm{QC}\overset{1}{=}$, $\mathrm{QC}\overset{5}{=}$, and $\mathrm{QC}\overset{9}{=}$ if it is of one of the eight kinds listed under (a), *but with Y understood to be an individual variable (rather than an individual variable or an individual constant) of $\mathrm{QC}\overset{i}{=}$ when the formula is of the kind $(\forall X)A \supset A(Y/X)$.*

(c) A formula of $\mathrm{QC}\overset{i}{=}$ counts as an axiom of $\mathrm{QC}\overset{2}{=}$, $\mathrm{QC}\overset{6}{=}$, and $\mathrm{QC}\overset{10}{=}$ if it is of one of the eight kinds listed under (a), *but with Y understood to be an individual variable of $\mathrm{QC}\overset{i}{=}$, and with X understood to occur free in A, when the formula is of the kind $(\forall X)A \supset A(Y/X)$.*

(d) A formula of $\mathrm{QC}\overset{i}{=}$ counts as an axiom of $\mathrm{QC}\overset{4}{=}$ and $\mathrm{QC}\overset{8}{=}$ if it is of one of the eight kinds listed under (a), *but with X understood to occur free in A when the formula is of the kind $(\forall X)A \supset A(Y/X)$.*

D7$^{\infty}$. A formula of $\mathrm{QC}\overset{\infty}{=}$ counts as an axiom of $\mathrm{QC}\overset{\infty}{=}$ if it is of one of the eight kinds listed under (a), *but with Y understood to be an individual variable of $\mathrm{QC}\overset{\infty}{=}$ when the formula is of the kind $(\forall X)A \supset A(Y/X)$.*

D8. (a) Let A and B be formulas of $\mathrm{QC}\overset{i}{=}$.

(a1) B is said to follow from A and $A \supset B$ by means of rule R1.

(a2) If A is closed or B is open, then B is said to follow from A and $A \supset B$ by means of rule R2. (11)

(a3) If every individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A also occurs in B, then B is said to follow from A and $A \supset B$ by means of rule R3.

(a4) If A is closed or B is open, and every individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A also occurs in B, then B is said to follow from A and $A \supset B$ by means of rule R4.

(a5) B is said to follow from A and $\sim A$ by means of rule R5.

(a6) If A is closed or B is open, then B is said to follow from A and $\sim A$ by means of rule R6.

(b) Let $(\forall X)A$ be a formula of $\mathrm{QC}\overset{i}{=}$, and Y be an individual variable of $\mathrm{QC}\overset{i}{=}$ that does not occur free in $(\forall X)A$. Then $(\forall X)A$ is said to follow from $A(Y/X)$ by means of rule R7.

D8$^{\infty}$. (a) Let A and B be formulas of $\mathrm{QC}\overset{\infty}{=}$. Then B is said to follow from A and $A \supset B$ by means of rule R1.

(b) Like D8(b), but with '$\mathrm{QC}\overset{\infty}{=}$' in place of '$\mathrm{QC}\overset{i}{=}$'.

D9. Let K be a finite column of formulas of $\mathrm{QC}\overset{i}{=}$; let $(\forall X)A$ be any entry in K that follows from a previous one by means of rule R7; and let $A(Y/X)$ be the earliest entry in K that is previous to $(\forall X)A$ and from which $(\forall X)A$ follows by means of rule R7. If X and Y are the same, then Y is said to be universalized upon in K; otherwise, Y is said to be quasi-universalized upon in K.

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(10) In [13] Mostowski took a vacuous quantification like '$(\forall x)p$' to amount to 'p' when 'x' has no values, and hence to be true—so to speak—only if 'p' is true. Soon afterwards Hailperin urged in [2] that '$(\forall x)p$' be invariably held true when 'x' has no values. We abide in the main text by Hailperin's recommendation, but list in Appendix I the various changes that Mostowski's handling of '$(\forall x)p$' (and its congeners) calls for in Sections 2-3. Case 2 of D6 will be one of the items affected.

(11) We borrow R2 from Schneider's [15]. Mostowski's rule in [13]: "If every individual variable [of $\mathrm{QC}\overset{i}{=}$] that occurs free in A occurs free in B, then B follows from A and $A \supset B$," could do duty for R2; it would, however, occasionally make for slightly longer proofs on pp. 138-143.

D10. Let A be a formula of $\mathrm{QC}^{i}_{=}$, S be a finite set of formulas of $\mathrm{QC}^{i}_{=}$, and K be a finite column of formulas of $\mathrm{QC}^{i}_{=}$ such that (i) K closes with A and (ii) no individual variable of $\mathrm{QC}^{i}_{=}$ that is universalized or quasi-universalized upon in K occurs free in any member of S.

Case 1: $i = 0$ or 1. If every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows from two previous entries by means of rule R1, or follows from a previous entry by means of rule R7, then K counts as a derivation of A from S in $\mathrm{QC}^{i}_{=}$. (12)

Case 2: $i = 2$. If every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows from two previous entries by means of rule R2, or follows from a previous entry by means of rule R7, then K counts as derivation of A from S in $\mathrm{QC}^{i}_{=}$.

Case 3: $i = 3$ or 5. If every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows from two previous entries by means of rule R3, or follows from a previous entry by means of rule R7, then K counts as a derivation of A from S in $\mathrm{QC}^{i}_{=}$.

Case 4: $i = 4$ or 6. If every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows two previous entries by means of rule R4, or follows from a previous entry by means of rule R7, then K counts as a derivation of A from S in $\mathrm{QC}^{i}_{=}$.

Case 5: $i = 7$ or 9. If (i) every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows from two previous entries by means of one of rules R1 and R5, or follows from a previous entry by means of rule R7, and (ii) every individual constant of $\mathrm{QC}^{i}_{=}$ occurring in any entry in K that counts as an axiom of $\mathrm{QC}^{i}_{=}$ occurs in some member or other of S, then K counts as a derivation of A from S in $\mathrm{QC}^{i}_{=}$.

(12) In most of the literature on $\mathrm{QC}^{0}_{=}$, the following rule, call it R7′, does duty for R7: "$(\forall X)A$ follows from A". An individual variable X of $\mathrm{QC}^{0}_{=}$ is then said to be *universalized upon* in a finite column of formulas of $\mathrm{QC}^{0}_{=}$ if at least one entry in the column is of the kind $(\forall X)A$ and a previous one is of the kind A; and a finite column of formulas of $\mathrm{QC}^{0}_{=}$ that closes with a formula A of $\mathrm{QC}^{0}_{=}$ and every one of whose entries belongs to a set S of formulas of $\mathrm{QC}^{0}_{=}$, counts as an axiom of $\mathrm{QC}^{0}_{=}$, follows from two previous entries by means of rule R1, or follows from a previous entry by means of rule R7′, is said to *constitute a derivation* of A from S in $\mathrm{QC}^{0}_{=}$ if no individual variable of $\mathrm{QC}^{0}_{=}$ that is universalized upon in the column occurs free in a member of S. As Montague and Henkin have shown in [12], however, this account of things blocks proof of L3(c) on p. 138, and hence blocks proof of T10(a) for the case where $i = 0$. Our repair here is reminiscent of the one in [8]. Montague and Henkin offer a different one, which incidentally makes do with R7. For a survey of yet other repairs, see [9], where line 9 on p. 34 should read 'nor in $(\forall X)A$', and the end of line 13 read 'nor in $(\exists X)A$, nor in B'.

Case 6: $i = 8$ or 10. If (i) every entry in K belongs to S, counts as an axiom of $\mathrm{QC}^{i}_{=}$, follows from two previous entries by means of one of rules R2 and R6, or follows from a previous entry by means of rule R7, and (ii) every individual constant of $\mathrm{QC}^{i}_{=}$ occurring in any entry in K that counts as an axiom of $\mathrm{QC}^{i}_{=}$ occurs in some member or other of S, then K counts as a derivation of A from S in $\mathrm{QC}^{i}_{=}$.

D10$^{\infty}$. Like D10, Case 1, but with '$\mathrm{QC}^{\infty}_{=}$' in place of '$\mathrm{QC}^{i}_{=}$'.

Remark. The restrictions placed in $\mathrm{QC}^{0}_{=}$-$\mathrm{QC}^{10}_{=}$ upon $(\forall X)A \supset A(Y/X)$, R1, and (when applicable) R5, can be tabulated as follows:

TABLE I (13)

	$(\forall X)A \supset A(Y/X)$	R1	R5
0	no restriction	no restriction	
1	Y a variable	no restriction	
2	X free in A & Y a variable	A closed or B open (= R2)	
3	no restriction	every constant in A occurring in B (= R3)	
4	X free in A	A closed or B open & every constant in A occurring in B (= R4)	
5	Y a variable	every constant in A occurring in B (= R3)	
6	X free in A & Y a variable	A closed or B open & every constant in A occurring in B(= R4)	
7	no restriction	no restriction	no restriction
8	X free in A	A closed or B open (= R2)	A closed or B open (=R6)
9	Y a variable	no restriction	no restriction
10	X free in A & Y a variable	A closed or B open (= R2)	A closed or B open (= R6)

(13) Seemingly missing from our roster of calculi is a modification of $\mathrm{QC}^{0}_{=}$, call it $\mathrm{QC}^{0'}_{=}$, that would stand to $\mathrm{QC}^{0}_{=}$ as $\mathrm{QC}^{4}_{=}$ stands to $\mathrm{QC}^{3}_{=}$ (and $\mathrm{QC}^{8}_{=}$ to $\mathrm{QC}^{7}_{=}$), and in which (i) X should occur free in A if $(\forall X)A \supset A(Y/X)$ is to count as an axiom and (ii) A should be closed or B open if B is to follow from A and $A \supset B$. When $\mathrm{QC}^{0}_{=}$ and $\mathrm{QC}^{0'}_{=}$ have no individual constants (as is the case in [15]), $\mathrm{QC}^{0}_{=}$ and $\mathrm{QC}^{0'}_{=}$ do differ; under the present circumstances, though, they amount to the same. By the same reasoning as in the proof of L5(b) below, A is derivable from $S \cup \{(\exists x)(f(x) \vee \sim f(x))\}$ in $\mathrm{QC}^{0'}_{=}$ if A is derivable from S in $\mathrm{QC}^{0}_{=}$. But, since '$f(a)$ & $\sim f(a)$', where 'a' is an individual constant of $\mathrm{QC}^{0'}_{=}$, follows from '$(\forall x)(f(x)$ & $\sim f(x))$' and '$(\forall x)(f(x)$ & $\sim f(x)) \supset (f(a)$ & $\sim f(a))$' by means of rule R2, '$(\exists x)(f(x) \vee \sim f(x))$' is derivable from $\emptyset$ in $\mathrm{QC}^{0'}_{=}$. Hence A is derivable from S in $\mathrm{QC}^{0'}_{=}$ if is derivable from S in $\mathrm{QC}^{0}_{=}$.

D11. Let A be a formula of $\mathrm{QC}\overset{i}{=}$, and S be a set of formulas of $\mathrm{QC}\overset{i}{=}$.

Case 1: S is finite. Then A is said to be derivable from S in $\mathrm{QC}\overset{i}{=}$ if there is a derivation of A from S in $\mathrm{QC}\overset{i}{=}$.

Case 2: S is infinite. Then A is said to be derivable from S in $\mathrm{QC}\overset{i}{=}$ if there is a finite subset S' of S such that A is derivable from S' in $\mathrm{QC}\overset{i}{=}$. (14)

Remarks: It follows from D10-D11 that a formula A of $\mathrm{QC}\overset{i}{=}$ is not derivable from $\emptyset$ in $\mathrm{QC}\overset{i}{=}$ for any i from 7 to 10 unless A is constant-free. To abridge matters, we shall write, e.g., '$S \vdash_1 A$' for 'A is derivable from S in $\mathrm{QC}\overset{1}{=}$', and '$S \vdash^\infty A$' for '$A$ is derivable from S in $QC\overset{\infty}{=}$'; we shall further write, e.g., '$\vdash_1 A$' for '$\emptyset \vdash_1 A$', and '$\vdash^\infty A$' for '$\emptyset \vdash^\infty A$'; and we shall call a derivation in $\mathrm{QC}\overset{i}{=}$ of a formula A of $\mathrm{QC}\overset{i}{=}$ from a set S of formulas of $\mathrm{QC}\overset{i}{=}$ constant-free if every entry in the derivation is constant-free.

D12. A set S of formulas of $\mathrm{QC}\overset{i}{=}$ is said to be inconsistent in $\mathrm{QC}\overset{i}{=}$ if $S \vdash_i p \,\&\, \sim p$; otherwise to be consistent in $\mathrm{QC}\overset{i}{=}$.

The following consequences of D1-D12 merit separate recording as lemmas.

L1. *Let S be a set of formulas of* $\mathrm{QC}\overset{\infty}{=}$; *$A$ and B be formulas of* $\mathrm{QC}\overset{\infty}{=}$; *$X$ and Y be individual variables of* $\mathrm{QC}\overset{\infty}{=}$; *and Z, Z', and Z'' be individual terms of* $\mathrm{QC}\overset{\infty}{=}$.

(a) *If $S \vdash^\infty A$, then there is a finite subset S' of S such that $S' \vdash^\infty A$.*

(b) *If A belongs to S or is an axiom of* $\mathrm{QC}\overset{\infty}{=}$, *then $S \vdash^\infty A$.*

(c) *If $S \vdash^\infty A$, then $S \cup S' \vdash^\infty A$ for every set S' of formulas of* $\mathrm{QC}\overset{\infty}{=}$.

(d) *Let every member of S be a formula of* $\mathrm{QC}\overset{1}{=}$ *and A be a formula of* $\mathrm{QC}\overset{1}{=}$. *Then $S \vdash^\infty A$ if and only if $S \vdash_1 A$.*

(e) *S is inconsistent in* $\mathrm{QC}\overset{\infty}{=}$ *if and only if some finite subset of S is inconsistent in* $\mathrm{QC}\overset{\infty}{=}$.

(f) *If $S \vdash^\infty A'$ and $S \vdash^\infty \sim A'$ for some formula A' of* $\mathrm{QC}\overset{\infty}{=}$, *then S is inconsistent in* $\mathrm{QC}\overset{\infty}{=}$.

(g) *If $S \cup \{A\}$ is inconsistent in* $\mathrm{QC}\overset{\infty}{=}$, *then $S \vdash^\infty \sim A$.*

(h) *If $S \cup \{A\} \vdash^\infty B$, then $S \vdash^\infty A \supset B$.*

(i) *If $S \vdash^\infty A$ and $S \vdash^\infty A \supset B$, then $S \vdash^\infty B$.*

(j) *If $S \vdash^\infty A$ or $S \vdash^\infty \sim A$, then $S \vdash^\infty A \supset B$ if and only if $S \vdash^\infty B$ or it is not the case that $S \vdash^\infty A$,*

(14) Proof is easily obtained that, where S is finite, A is derivable from S in $\mathrm{QC}\overset{1}{=}$ if and only if A is derivable from S in Leblanc and Hailperin's variant of $\mathrm{QC}=$ in [10]; and that A is derivable from $\emptyset$ in $\mathrm{QC}\overset{2}{=}$ if and only if A is provable in Hintikka's variant of $\mathrm{QC}=$ in [5]. Details are left to the reader.

(k) *If $S \vdash^\infty A(Y/X)$, then $S \vdash^\infty (\forall X)A$, so long as Y does not occur free in any member of S nor in $(\forall X)A$.*

(l) *If $S \vdash^\infty (\exists X)A$ and $S \cup \{A(Y/X)\} \vdash^\infty B$, then $S \vdash^\infty B$, so long as Y does not occur free in any member of S, nor in $(\exists X)A$, nor in B.*

(m) *If $S \vdash^\infty \sim(\forall X)A$, then $S \vdash^\infty (\exists X)\sim A$.*

(n) *$S \vdash^\infty \mathrm{E}!X$.*

(o) *If $S \vdash^\infty (\exists X)A$ and $S \cup \{A(Z/X), \mathrm{E}!Z\} \vdash^\infty B$, then $S \vdash^\infty B$, so long as Z does not occur free in any member of S, nor in $(\exists X)A$, nor in B.*

(p) *$S \vdash^\infty \mathrm{E}!Z$ if and only if $S \vdash^\infty (\exists X)(X = Z)$.*

(q) *If $S \vdash^\infty (\forall X)A$, then $S \vdash^\infty \mathrm{E}!Z \supset A(Z/X)$.*

(r) *$S \vdash^\infty (\exists X)\big((\exists X)A \supset A\big)$.*

(s) *$S \vdash^\infty Z = Z$.*

(t) *It $S \vdash^\infty Z = Z'$, then $S \vdash^\infty Z' = Z$.*

(u) *If $S \vdash^\infty Z = Z'$ and $S \vdash^\infty Z' = Z''$, then $S \vdash^\infty Z = Z''$.*

(v) *If $S \vdash^\infty Z = Z'$, then $S \vdash^\infty A$ if and only if $S \vdash^\infty A(Z'//Z)$.*

L2. *Let S be a set of closed formulas of* $\mathrm{QC}\overset{2}{=}$; *$A$ and B be closed formulas of* $\mathrm{QC}\overset{2}{=}$; *and X, Y, and Z be individual constants of* $\mathrm{QC}\overset{2}{=}$.

(a) *S is inconsistent in* $\mathrm{QC}\overset{2}{=}$ *if and only if some finite subset of S is inconsistent in* $\mathrm{QC}\overset{2}{=}$.

(b) *If $S \vdash_2 A'$ and $S \vdash_2 \sim A'$ for some closed formula A' of* $\mathrm{QC}\overset{2}{=}$, *then S is inconsistent in* $\mathrm{QC}\overset{2}{=}$.

(c) *If $S \cup \{A\}$ is inconsistent in* $\mathrm{QC}=$, *then $S \vdash_2 \sim A$.*

(d) *If $S \vdash_2 A$ or $S \vdash_2 \sim A$, then $S \vdash_2 A \supset B$ if and only if $S \vdash_2 B$ or it is not the case that $S \vdash_2 A$.*

(e) *$S \vdash_2 X = X$.*

(f) *If $S \vdash_2 X = Y$, then $S \vdash_2 Y = X$.*

(g) *If $S \vdash_2 X = Y$ and $S \vdash_2 Y = Z$, then $S \vdash_2 X = Z$.*

(h) *If $S \vdash_2 X = Y$, then $S \vdash_2 A$ if and only if $S \vdash_2 A(Y//X)$.*

L3. *Let S be a set of formulas of* $\mathrm{QC}\overset{i}{=}$; *$A$, B, and C be formulas of* $\mathrm{QC}\overset{i}{=}$; *$X$ and Y be individual variables of* $\mathrm{QC}\overset{i}{=}$; *and $0 \leqslant i \leqslant 6$.*

(a) *If $S \vdash_i A$, then there is a finite subset S' of S such that $S' \vdash_i A$.*

(b) *If A belongs to S or is an axiom of* $\mathrm{QC}\overset{i}{=}$, *then $S \vdash_i A$.*

(c) *If $S \vdash_i A$, then $S \cup S' \vdash_i A$ for every set S' of formulas of* $\mathrm{QC}\overset{i}{=}$.

(d) *If $S \cup \{\sim A\}$ is inconsistent in* $\mathrm{QC}\overset{i}{=}$, *then $S \vdash_i A$.*

(e) *If $S \cup \{A\} \vdash_i B$ and $S \cup \{\sim A\} \vdash_i B$, then $S \vdash_i B$, so long as no individual term of* $\mathrm{QC}\overset{i}{=}$ *occurs free in A.*

(f) *If $S \cup \{A\} \vdash_i B$, then $S \vdash_i A \supset B$.*

(g) *If $S \vdash_i A$ and $S \vdash_i A \supset B$, then $S \vdash_i B$, so long as (1) in case $i = 2$, 4, or 6, A is closed or B is open, and (2) in case $3 \leqslant i \leqslant 6$, every individual constant of* $\mathrm{QC}\overset{i}{=}$ *that occurs in A also occurs in B.*

(h) *If $S \vdash_i A$, then $S \vdash_i B \supset A$.*

(i) *If* $S \vdash_i A \supset (B \supset C)$, *then* $S \vdash_i (A \supset B) \supset (A \supset C)$.

(j) *If* $S \vdash_i \sim A \supset \sim B$, *then* $S \vdash_i B \supset A$.

(k) *If* $S \vdash_i A \supset B$, *then* $S \vdash_i \sim B \supset \sim A$.

(l) *If* $S \vdash_i A$ *and* $S \vdash_i B$, *then* $S \vdash_i A \,\&\, B$.

(m) *If* $S \vdash_i A(Y/X)$, *then* $S \vdash_i (\forall X)A$, *so long as* Y *does not occur free in any member of* S *nor in* $(\forall X)A$.

(n) *If* $S \vdash_i (\exists X)A$, *then* $S \vdash_i A$, *so long as* X *does not occur free in* A.

(o) *If* $S \vdash_i A \supset B(Y/X)$, *then* $S \vdash_i A \supset (\forall X)B$, *so long as* Y *does not occur free in any member of* S *nor in* $A \supset (\forall X)B$.

(p) *If* $S \vdash_i (\forall X)(A \supset B)$. *then* $S \vdash_i (\forall X)A \supset (\forall X)B$.

(q) *If* $S \vdash_i (\exists x)\big(f(x) \vee \sim f(x)\big)$, *then* $S \vdash_i (\exists X)\big(f(X) \vee \sim f(X)\big)$.

L4. *Let* $i = 2$ *or* 6.

(a) *Let* S' *consist of the* Ø*-associates of the members of* S, *and let* A' *be the* Ø*-associate of* A. *If* $S' \vdash_i A'$, *then* $S \cup \{\sim(\exists x)\big(f(x) \vee \sim f(x)\big)\} \vdash_i A$.

(b) *Let* A *be an open formula of* $\mathrm{QC}\overset{i}{=}$. *Then* $\{\sim(\exists x)\big(f(x) \vee \sim f(x)\big)\} \vdash_i A$.

L5. (a) *Let* X *be an individual variable of* $\mathrm{QC}\overset{i}{=}$ *that does not occur free in* A, *and* $i = 2$ *or* 6. *Then* $\vdash_i (\exists x)\big(f(x) \vee \sim f(x)\big) \supset \big((\forall X)A \supset A\big)$.

(b) *If* $S \vdash_1 A$, *then* $S \cup \{(\exists x)\big(f(x) \vee \sim f(x)\big)\} \vdash_2 A$.

(c) *If* $S \vdash_5 A$, *then* $S \cup \{(\exists x)\big(f(x) \vee \sim f(x)\big)\} \vdash_6 A$.

L6. *Let* S *be a finite set of formulas of* $\mathrm{QC}\overset{2}{=}$.

(a) *Let* A *and* B *be constant-free formulas of* $\mathrm{QC}\overset{2}{=}$. *Then there is a constant-free derivation of* $A \supset (B \equiv B)$ *from* S *in* $\mathrm{QC}\overset{2}{=}$.

(b) *Let* A *be an atomic formula of* $\mathrm{QC}\overset{2}{=}$ *that is constant-free, and* Y *be an individual variable of* $\mathrm{QC}\overset{2}{=}$. *Then there is a constant-free derivation from* S *in* $\mathrm{QC}\overset{2}{=}$ *of*

$$(\forall X)(X = Y) \supset \big((\forall X)A \equiv A(Y/X)\big)\,,$$

$$(\forall X)(X = Y) \supset \big((\forall X)A \equiv (\forall X)A(Y//X)\big)\,,$$

$$(\forall X)(Y = X) \supset \big((\forall X)A \equiv A(X/Y)\big)\,,$$

and

$$(\forall X)(X = X) \supset \big((\forall X)A \equiv (\forall X)A(X//Y)\big)\,.$$

(c) *If there is a constant-free derivation of* $A \supset (B \equiv C)$ *from* S *in* $\mathrm{QC}\overset{2}{=}$, *then there is one of* $A \supset (\sim B \equiv \sim C)$.

(d) *It there is a constant-free derivation of* $A \supset (B \equiv B')$ *from* S *in* $\mathrm{QC}\overset{2}{=}$, *and one of* $A \supset (C \equiv C')$, *then there is one of* $A \supset \big((B \supset C) \equiv (B' \supset C')\big)$.

(e) *If there is a constant-free derivation of* $A \supset (B \equiv C)$ *from* S *in* $\mathrm{QC}\overset{2}{=}$, *then there is one of* $A \supset \big((\forall X)B \equiv (\forall X)C\big)$, *so long as* X *does not occur free in* A.

(f) A *being a formula of* $\mathrm{QC}\overset{2}{=}$, B *being one of the kind* $A(Y//X)$, *where exactly one of* X *and* Y *is an individual constant of* $\mathrm{QC}\overset{2}{=}$, *and* Z *being an individual variable of* $\mathrm{QC}\overset{2}{=}$ *that does not occur in* A *and is distinct from* Y, *let* A' *and* B' *be the results of replacing by an occurrence of* Z *any occurrence in* A *and* B, *respectively, of any individual constant of* $\mathrm{QC}\overset{2}{=}$, *and let* A'' *and* B'' *be the results of replacing by an occurrence of* $(\forall Z)C$ *any occurrence in* A' *and* B', *respectively, of any atomic formula* C *of* $\mathrm{QC}\overset{2}{=}$ *that contains* Z. *Then there is a constant-free derivation of* $(\forall Z)(Z = Y) \supset (A'' \supset B'')$ *from* S *in* $\mathrm{QC}\overset{2}{=}$ *if* X *an individual constant of* $\mathrm{QC}\overset{2}{=}$, *otherwise one of* $(\forall Z)(X = Z) \supset (A'' \supset B'')$.

(g) *If there is a constant-free derivation of* $A \supset (B \equiv C)$ *from* S *in* $\mathrm{QC}\overset{2}{=}$, *then there is one of* $A \supset (B \supset C)$.

L7. *Let* S *be a finite set of constant-free formulas of* $\mathrm{QC}\overset{i}{=}$, A *be a constant-free formula of* $\mathrm{QC}\overset{i}{=}$, *and* $i = 1$ *or* 2. *If* $S \vdash_i A$, *then there is a constant-free derivation of* A *from* S *in* $\mathrm{QC}\overset{i}{=}$.

L8. (a) *If* $\vdash_1 A$, *then* $\vdash_5 A$.

(b) *If* $\vdash_2 A$, *then* $\vdash_6 A$. (15)

L9. *Let every individual constant of* $\mathrm{QC}\overset{1}{=}$ *(hence,* $\mathrm{QC}\overset{2}{=}$, $\mathrm{QC}\overset{5}{=}$, *and* $\mathrm{QC}\overset{6}{=}$*) that occurs in one or more members of* S *occurs in* A.

(a) *If* $S \vdash_1 A$, *then* $S \vdash_5 A$.

(b) *If* $S \vdash_2 A$, *then* $S \vdash_6 A$, *so long as every member of* S *is closed or* A *is open.*

L10. *Let* S' *consist of every closed formula of* $\mathrm{QC}\overset{1}{=}$ *(hence,* $\mathrm{QC}\overset{5}{=}$ *and* $\mathrm{QC}\overset{6}{=}$*) of the kind* $\mathrm{E}!X$. (16)

(a) *If* $S \cup S' \vdash_1 A$, *then* $S \vdash_0 A$.

(b) *If* $S \cup S' \vdash_5 A$, *then* $S \vdash_3 A$.

(c) *If* $S \cup S' \vdash_6 A$, *then* $S \vdash_4 A$.

L11. *Let* $7 \leqslant i \leqslant 10$.

(a) *If* $S \vdash_i A$, *then* $S \cup S' \vdash_i A$ *for every set* S' *of formulas of* $\mathrm{QC}\overset{i}{=}$.

(b) *If* $S \vdash_i p \,\&\, \sim p$, *then* $S \vdash_i A$.

L12. *Let every individual constant of* $\mathrm{QC}\overset{1}{=}$ *(hence,* $\mathrm{QC}\overset{2}{=}$, $\mathrm{QC}\overset{9}{=}$, *and* $\mathrm{QC}\overset{10}{=}$*) that occurs in* A *occur in one or more members of* S.

(a) *If* $S \vdash_1 A$, *then* $S \vdash_9 A$.

(b) *If* $S \vdash_2 A$, *then* $S \vdash_{10} A$.

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(15) Note also that if $\vdash_0 A$, then $\vdash_3 A$. Hence, so far as derivability from Ø (*though not from arbitrary* S) goes, $\mathrm{QC}\overset{0}{=}$ and $\mathrm{QC}\overset{3}{=}$ coincide, as do $\mathrm{QC}\overset{1}{=}$ and $\mathrm{QC}\overset{5}{=}$, and $\mathrm{QC}\overset{2}{=}$ and $\mathrm{QC}\overset{6}{=}$, which leaves us with only three variants of $\mathrm{QC}=$: $\mathrm{QC}\overset{1}{=}$, $\mathrm{QC}\overset{3}{=}$, and $\mathrm{QC}\overset{4}{=}$.

(16) I.e., let S' consist of every formula of $\mathrm{QC}\overset{1}{=}$ of the kind $\mathrm{E}!X$, where X is an individual constant of $\mathrm{QC}\overset{1}{=}$.

L13. *Let* S' *consist of every closed formula of* $\mathrm{QC}\overset{9}{=}$ (*hence*, $\mathrm{QC}\overset{10}{=}$) *of the kind* $\mathrm{E}!X$, *where* X *occurs in one or more members of* S.

(a) *If* $S \cup S' \vdash_9 A$, *then* $S \vdash_7 A$.

(b) *If* $S \cup S' \vdash_{10} A$, *then* $S \vdash_8 A$.

Proof of the above lemmas is a routine matter, except possibly for L1(c), L1(h), L1(o), L1(q), L3(c), L3(f), L4(a), L5, L6(f), L7, L8, L9, L11(a), and L12. Since proof of L1(c), L1(h), L3(c), L3(f), and L11(a) can be retrieved from [8], [17] proof of L4(a) can be retrieved from [15], proof of L5(c) is like that of L5(b), proof of L8(b) is like that of L8(a), proof of L9(b) is like that of L9(a), and proof of L12(b) is like that of L12(a), we shall restrict ourselves here to L1(o), L1(q), L5(a)-(b), L6(f), L7, L8(a), L9(a), and L12(a).

For proof of L1(o). Let $S \vdash^\infty (\exists X)A$ and $S \cup \{A(Z/X), \mathrm{E}!Z\} \vdash^\infty B$, where Z is as in L1(o). Then in view of L1(a) and L1(c) there is a finite subset S' of S such that $S' \vdash^\infty (\exists X)A$ and $S' \cup \{A(Z/X), \mathrm{E}!Z\} \vdash^\infty B$, and hence, in particular, a finite column of formulas of $\mathrm{QC}\overset{\infty}{=}$, say, the formulas $C_1, C_2, \ldots, C_p$, that counts as a derivation of B from $S' \cup \{A(Z/X), \mathrm{E}!Z\}$ in $\mathrm{QC}\overset{\infty}{=}$. Now let Y be the alphabetically earliest individual variable of $\mathrm{QC}\overset{\infty}{=}$ that does not occur in any member of $S' \cup \{A(Z/X), \mathrm{E}!Z\}$ nor in any one of $C_1, C_2, \ldots, C_p$; and for each i from 1 to p let C'_i be $C_i(Y/Z)$. In view of the restrictions placed upon Z, C'_i $(i = 1, 2, \ldots, p)$ is sure to be C_i if C_i belongs to S', C_i again if C_i is B, and $A(Y/X)$ if C_i is $A(Z/X)$. But, if so, then the column made up of $C'_1, C'_2, \ldots,$ and C'_p is sure to count as derivation of B from $S' \cup \{A(Y/X), \mathrm{E}!X\}$ in $\mathrm{QC}\overset{\infty}{=}$. Hence in view of L1(h) $S' \cup \{A(Y/X)\} \vdash^\infty \mathrm{E}!Y \supset B$, and hence in view of L1(n) and L1(i) $S' \cup \{A(Y/X)\} \vdash^\infty B$. But Y is sure not to occur free in $(\exists X)A$. Hence, since $S' \vdash^\infty (\exists X)A$, then $S' \vdash^\infty B$ in view of L1(l), and hence $S \vdash^\infty B$ in view of L1(c).

For proof of L1(q). [18] Let $S \vdash^\infty (\forall X)A$. Then in view of L1(a) there is a finite subset S' of S such that $S' \vdash^\infty (\forall X)A$. Hence in view of L1(c) $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty (\forall X)A$, where Y is the alphabetically earliest individual variable of $\mathrm{QC}\overset{\infty}{=}$ that does not occur in any member of $S' \cup \{\mathrm{E}!Z\}$ not in $(\forall X)A$. But in view of L1(b) $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty (\forall X)A \supset A(Y/X)$. Hence in view of L1(i) $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty A(Y/X)$. But in view of L1(b) $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty Y = Z$, and $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty Y = Z \supset (A(Y/X) \supset A(Z/X))$. Hence in view of L1(i) $S' \cup \{\mathrm{E}!Z, Y = Z\} \vdash^\infty A(Z/X)$. But in view of L1(b) and L1(p) $S' \cup \{\mathrm{E}!Z\} \vdash^\infty (\exists Y)(Y = Z)$. Hence in view of L1(l) $S' \cup \{\mathrm{E}!Z\} \vdash^\infty A(Z/X)$, hence in view of L1(h) $S' \vdash^\infty \mathrm{E}!Z \supset A(Z/X)$, and hence in view of L1(c) $S \vdash^\infty \mathrm{E}!Z \supset A(Z/X)$.

For proof of L5(a). Let X and i be as in L5(a). The following column of formulas of $\mathrm{QC}\overset{i}{=}$,

(1)	$\sim A$	
(2)	$1 \supset \big(\sim\sim(f(X) \vee \sim f(X)) \supset \sim A\big)$	(Axiom)
(3)	$\sim\sim(f(X) \vee \sim f(X)) \supset \sim A$	(R2 or R4, 1, 2)
(4)	$3 \supset \big(A \supset \sim(f(X) \vee \sim f(X))\big)$	(Axiom)
(5)	$A \supset \sim(f(X) \vee \sim f(X))$	(R2 or R4, 3, 4)
(6)	$(\forall X)5$	(R7, 5)
(7)	$6 \supset \big((\forall X)A \supset (\forall X)\sim(f(X) \vee \sim f(X))\big)$	(Axiom)
(8)	$(\forall X)A \supset (\forall X)\sim(f(X) \vee \sim f(X))$	(R2 or R4, 6, 7),

counts as a derivation of $(\forall X)A \supset (\forall X)\sim(f(X) \vee \sim f(X))$ from $\{(\exists x)(f(x) \vee \sim f(x)), \sim A\}$ in $\mathrm{QC}\overset{i}{=}$. Hence in view of L3(k) $\{(\exists x)(f(x) \vee \sim f(x)), \sim A\} \vdash_i (\exists X)(f(X) \vee \sim f(X)) \supset \sim(\forall X)A$. But in view of L3(b) and L3(q) $\{(\exists x)(f(x) \vee \sim f(x)), \sim A\} \vdash_i (\exists X)(f(X) \vee \sim f(X))$. Hence in view of L3(g) $\{(\exists x)(f(x) \vee \sim f(x)), \sim A\} \vdash_i \sim(\forall X)A$, hence in view of L3(f) $\{(\exists x)(f(x) \vee \sim f(x))\} \vdash_i \sim A \supset \sim(\forall X)A$, hence in view of L3(j) $\{(\exists x)(f(x) \vee \sim f(x))\} \vdash_i (\forall X)A \supset A$, and hence in view of L3(f) $\vdash_i (\exists x)(f(x) \vee \sim f(x)) \supset ((\forall X)A \supset A)$.

For proof of L5(b). Let $S \vdash_1 A$. Then in view of L3(a) and L3(c) there is a finite subset S' of S such that $S' \cup \{(\exists x)(f(x) \vee \sim f(x))\} \vdash_1 A$, and hence there is a finite column of formulas of QC=, say, the formulas $B_1, B_2, \ldots,$ and B_p $(p > 0)$, that counts as derivation of A from $S' \cup \{(\exists x)(f(x) \vee \sim f(x))\}$ in $\mathrm{QC}\overset{1}{=}$. But, if so, then $S' \cup \{(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 B_j$— $S'' \vdash_2 B_j$, for short—for each j from 1 to p, as can be shown by mathematical induction on j. For suppose that B_j belongs to S'' or is an axiom of $\mathrm{QC}\overset{i}{=}$ that counts as an axiom of $\mathrm{QC}\overset{2}{=}$. Then in view of L3(b) $S'' \vdash_2 B_j$. Or suppose that B_j is an axiom of $\mathrm{QC}\overset{1}{=}$ that does not count as an axiom of $\mathrm{QC}\overset{2}{=}$. In view of L5(a) and L3(c) $S'' \vdash_2 (\exists x)(f(x) \vee \sim f(x)) \supset B_j$. Hence in view of L3(b) and L3(g) $S'' \vdash_2 B_j$. Or suppose that B_j follows from two previous entries, say, B_g and $B_g \supset B_j$ $(= B_h)$, by means of rule R1; suppose $S'' \vdash_2 B_g$ and $S'' \vdash_2 B_h$; and suppose that B_g is closed or B_j is open. Then in view of L3(g) $S'' \vdash_2 B_j$. Suppose then

[17] [8] uses in place of R7 a rule that reads: "$(\forall Y)A(Y/X)$ follows from A", and a slightly different notion of derivation (in $\mathrm{QC}\overset{0}{=}$). Nonetheless, the proof of MT2.4.7 on pp. 132-133 readily converts into one of L1(c), L3(c), and L11(a), and the proof of MT2.4.9 on pp. 133-135 into one of L1(h) and L3(f).

[18] Proof of L1(q), for the case where S is a finite set of formulas of $\mathrm{QC}\overset{1}{=}$, first appeared in [10]. Proof of the kindred result: "If $\vdash_2 (\forall X)A$, then $\vdash_2 \mathrm{E}!Z \supset A(Z/X)$", appeared simultaneously in [5].

that B_g is open and B_j is closed, and suppose that X is the alphabetically earliest individual variable of $\mathrm{QC}\overset{2}{=}$ not to occur in any member of S'' nor in B_j. Since $S'' \vdash_2 B_h$, then in view of L3(h) $S'' \vdash_2 \big(f(X) \vee \sim f(X)\big) \supset B_h$, and hence in view of L3(i)

$$S'' \vdash_2 \Big(\big(f(X) \vee \sim f(X)\big) \supset B_g\Big) \supset \Big(\big(f(X) \vee \sim f(X)\big) \supset B_j\Big) .$$

But since $S'' \vdash_2 B_g$, then in view of L3(h) $S'' \vdash_2 \big(f(X) \vee \sim f(X)\big) \supset B_g$. Hence in view of L3(g) $S'' \vdash_2 \big(f(X) \vee \sim f(X)\big) \supset B_j$, hence in view of L3(k) $S'' \vdash_2 \sim B_j \supset \sim \big(f(X) \vee \sim f(X)\big)$, hence in view of L3(m) $S'' \vdash_2 (\forall X)\big(\sim B \supset \sim\big(f(X) \vee \sim f(X)\big)\big)$, hence in view of L3(p) $S'' \vdash_2 (\forall X) \sim B \supset (\forall X) \sim\big(f(X) \vee \sim f(X)\big)$, and hence in view of L3(k) $S'' \vdash_2 (\exists X)\big(f(X) \vee \sim f(X)\big) \supset (\exists X) B_j$. But in view of L3(a) and L3(q) $S'' \vdash_2 (\exists X)\big(f(X) \vee \sim f(X)\big)$. Hence in view of L3(g) $S'' \vdash_2 (\exists X) B_j$, and hence in view of L3(n) $S'' \vdash_2 B_j$. Or suppose that B_j follows from a previous entry, say, B_h, by means of rule R7, and suppose $S'' \vdash_2 B_h$. Then in view of L3(m) $S'' \vdash_2 B_j$, which completes the induction. But, if $S'' \vdash_2 B_j$ for each j from 1 to p, then $S'' \vdash_2 B_p$ $(= A)$. Hence in view of L3(a)

$$S \cup \big\{(\exists x)\big(f(x) \vee \sim f(x)\big)\big\} \vdash_2 A .$$

For proof of L6(f). Let S be as in L6; and let A, B, X, Y, Z, A', B', A'', and B'' be as in L6(f).

Case 1: X is an individual constant of $\mathrm{QC}\overset{2}{=}$.

Subcase 1.1: Y does not occur free in B, and hence B'' is the same as A''. Then in view of L6(a) there is a constant-free derivation of $(\forall Z)(Z = Y) \supset (A'' \equiv B'')$ from S in $\mathrm{QC}\overset{2}{=}$.

Subcase 1.2: Y occurs free in B. In view of L6(b)-(e) it readily follows by mathematical induction on the number of occurrences of '$\sim$', '$\supset$', and '$\forall$' in A that there is a constant-free derivation of $(\forall Z)(Z = Y) \supset (A'' \equiv B'')$ from S in $\mathrm{QC}\overset{2}{=}$. For suppose in particular that A is of the kind $(\forall X_1)A_1$, and hence B of the kind $(\forall X_1)A_1(Y//X_1)$; suppose A_1' and B_1' respectively stand to A_1 and B_1 as A' and B' respectively stand to A and B; and suppose that A_1'' and B_1'' respectively stand to A_1' and B_1' as A'' and B'' respectively stand to A' and B'. Since Y is presumed to occur free in B and hence in $(\forall X_1)B_1''$, then X_1 is sure to be distinct from Y. Hence, if there is a constant-free derivation of $(\forall Z)(Z = Y) \supset (A_1'' \equiv B_1'')$ from S in $\mathrm{QC}\overset{2}{=}$, then in view of L6(e) there is one of $(\forall Z)(Z = Y) \supset \big((\forall X_1)A_1'' \equiv (\forall X_1)B_1''\big)$.

Case 2: Y is an individual constant of $\mathrm{QC}\overset{2}{=}$. Proof like that of Case 1.

For proof of L7. Let $S \vdash_i A$, where S, A, and i are as in L7. Then there is a finite column of formulas of $\mathrm{QC}\overset{i}{=}$, say, the formulas $B_1, B_2, \ldots, B_p$, that counts as a derivation of A from S in $\mathrm{QC}\overset{i}{=}$. Let X be the alphabetically earliest individual variable of $\mathrm{QC}\overset{i}{=}$ that does not occur in any one of $B_1, B_2, \ldots, B_p$, and for each j from 1 to p let B_p be the result of replacing by an occurrence of X any occurrence in B_j of any individual constant of $\mathrm{QC}\overset{i}{=}$.

Case 1: $i = 1$. Then the column made up of $B_1', B_2', \ldots$, and B_p' counts as a constant-free derivation of B_p' $(= A)$ from S in $\mathrm{QC}\overset{i}{=}$.

Case 2: $i = 2$. For each j from 1 to p, let B_j'' be the result of replacing by an occurrence of $(\forall Y)C$ any occurrence in B_j' of any atomic formula C of $\mathrm{QC}\overset{i}{=}$ that contains X. It is easily shown by mathematical induction on j that for each j from 1 to p there is a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$. For suppose that B_j belongs to S or is an axiom of $\mathrm{QC}\overset{i}{=}$, and suppose in the latter case that B_j'' is an axiom of $\mathrm{QC}\overset{i}{=}$. Then B_j'' counts as a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$. Or suppose that B_j is of the kind $Y = Y$, where Y is an individual constant of $\mathrm{QC}\overset{i}{=}$. Then the column made up of $X = X$ and $(\forall X)(X = X)$ counts as a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$. Or suppose that B_j is of the kind $Y = Z \supset \big(C' \supset C'(Z//Y)\big)$, where exactly one of Y and Z is an individual constant of $\mathrm{QC}\overset{i}{=}$. Then in view of L6(f)-(g) there is a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$. [19] Or suppose that B_j follows from two previous entries, say, B_g and $B_g \supset B_j$ $(= B_h)$, by means of rule R2, and suppose that there is a constant-free derivation of B_g'' and one of B_h'' from S in $\mathrm{QC}\overset{i}{=}$. Since B_h'' is $B_g'' \supset B_j''$ and B_g'' is closed or B_j'' is open, then the column made up of the derivation of B_g'' (from S in $\mathrm{QC}\overset{i}{=}$), that of $B_g'' \supset B_j''$ and B_j'' counts as a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$. Or suppose that B_j follows from a previous entry, say, B_h, by means of rule R7, and suppose that there is a constant-free derivation of B_h'' from S in $\mathrm{QC}\overset{i}{=}$. Then the column made up of the derivation of B_h'' (from S in $\mathrm{QC}\overset{i}{=}$) and B_j'' counts as a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$, which completes the induction. But, if so, then there is a constant-free derivation of B_p'' $(= A)$ from S in $\mathrm{QC}\overset{i}{=}$.

For proof of L8(a). Let $\vdash_1 A$.

Case 1: No individual constant of $\mathrm{QC}\overset{1}{=}$ occurs in A. Then in view of L7 there is a constant-free derivation of A from $\emptyset$ in $\mathrm{QC}\overset{1}{=}$, and hence there is a derivation of A from $\emptyset$ in $\mathrm{QC}\overset{5}{=}$.

[19] Note that when B_j is of the kind $Y = Z \supset \big(C' \supset C'(Z//Y)\big)$, where both Y and Z are individual constants of $\mathrm{QC}\overset{i}{=}$, then B_j'' counts as an axiom of $\mathrm{QC}\overset{i}{=}$ and hence as a constant-free derivation of B_j'' from S in $\mathrm{QC}\overset{i}{=}$.

Case 2: At least one individual constant of $\mathrm{QC}\overset{1}{=}$ occurs in A. Since $\vdash_1 A$, there is a finite column of formulas of $\mathrm{QC}\overset{1}{=}$, say, the formulas $B_1, B_2, \ldots, B_p$, that counts as a derivation of A from $\emptyset$ in $\mathrm{QC}\overset{1}{=}$. Let $X_1, X_2, \ldots, X_n$ be in alphabetical order all the individual constants of $\mathrm{QC}\overset{1}{=}$ that occur in A; for each j from 1 to p let B'_j be the result of replacing by an occurrence of X_1 any occurrence in B_j of any individual constant of $\mathrm{QC}\overset{1}{=}$ that does occur in A; and for each j from 1 to p, let B''_p be

$$\big((\ldots(X_1 = X_1 \,\&\, X_2 = X_2) \,\&\, \ldots) \,\&\, X_n = X_n\big) \supset B'_j .$$

It is easily shown by mathematical induction on j that $\vdash_5 B''_j$ for each j from 1 to p. For suppose that B_j counts as an axiom of $\mathrm{QC}\overset{1}{=}$ and hence as an axiom of $\mathrm{QC}\overset{5}{=}$. Then B'_j counts as an axiom of $\mathrm{QC}\overset{5}{=}$, and hence in view of L3(h) $\vdash_5 B''_j$. Or suppose that B_j follows from two previous entries, say, B_g and $B_g \supset B_j$ $(= B_h)$, by means of rule R1, and suppose that $\vdash_5 B''_g$ and $\vdash_5 B''_h$. Then in view of L3(i) $\vdash_5 B''_g \supset B''_j$, and hence in view of L3(g) $\vdash_5 B''_j$. Or suppose that B_j follows from a previous entry, say, B_h, by means of rule R7, and suppose that $\vdash_5 B''_h$. Then in view of L3(o) $\vdash_5 B''_j$, which completes the induction, Hence $\vdash_5 B''_p$. But in view L3(b) and L3(l)

$$\vdash_5 (\ldots(X_1 = X_1 \,\&\, X_2 = X_2) \,\&\, \ldots) \,\&\, X_n = X_n .$$

Hence in view of L3(g) $\vdash_5 B'_p$ $(=A)$.

For proof of L9(a). Let $S \vdash_1 A$, where S and A are as in L9. Then in view of L3(a) there is a finite subset of S, say, $\{B_1, B_2, \ldots, B_n\}$ $(n \geqslant 0)$, such that $\{B_1, B_2, \ldots, B_n\} \vdash_1 A$, hence in view of L3(f) $\vdash_1 B_1 \supset \big(B_2 \supset (\ldots(B_n \supset A)\ldots)\big)$, hence in view of L8(a) $\vdash_5 B_1 \supset \big(B_2 \supset (\ldots(B_n \supset A)\ldots)\big)$, and hence in view of L3(c) $\{B_1, B_2, \ldots, B_n\} \vdash_5 B_1 \supset \big(B_2 \supset (\ldots(B_n \supset A)\ldots)\big)$. But in view of L3(b) $\{B_1, B_2, \ldots, B_n\} \vdash_5 B_j$ for each j from 1 to n. Hence in view of L3(g) $\{B_1, B_2, \ldots, B_n\} \vdash_5 A$, and hence in view of L3(c) $S \vdash_5 A$.

For proof of L12(a). Let $S \vdash_1 A$, where S and A are as in L12. Then there is a finite subset S' of S such that $S' \vdash_1 A$.

Case 1: Every member of $S' \cup \{A\}$ is constant-free. Then in view of L7 there is a constant-free derivation of A from S' in $\mathrm{QC}\overset{1}{=}$, and hence a derivation of A from S' in $\mathrm{QC}\overset{9}{=}$. Hence in view of L11(a) $S \vdash_5 A$.

Case 2: At least one member of $S' \cup \{A\}$ (and hence of S) is not constant-free. Since $S' \vdash_1 A$, there is a finite column of formulas of $\mathrm{QC}\overset{1}{=}$, say, the formulas $B_1, B_2, \ldots$, and B_p, that counts as a derivation of A from S' in $\mathrm{QC}\overset{1}{=}$. Let $X_1, X_2, \ldots, X_n$ $(n > 0)$ be all the individual constants of $\mathrm{QC}\overset{1}{=}$ that occur in one or more of $B_1, B_2, \ldots$, and B_p, but not in any member of S (nor, as a result, in A); let Y be the alphabetically earliest individual constant of $\mathrm{QC}\overset{1}{=}$ to occur in a member of S; and for each j

from 1 to p let B'_j be the result of replacing by an occurrence of Y every occurrence in B_j of every one of $X_1, X_2, \ldots$, and X_n. Then the column made up of $B'_1, B'_2, \ldots$, and B'_p counts as a derivation of A from $S' \cup \{C\}$ in $\mathrm{QC}\overset{9}{=}$, where C is the aphabetically earliest member of S that contains an occurrence of Y. Hence in view of L11(a) $S \vdash_9 A$.

3. Semantics. We next attend to the semantics of $\mathrm{QC}\overset{i}{=}$ $(0 \leqslant i \leqslant 10)$ and $\mathrm{QC}\overset{\infty}{=}$. Of the two domains D and D' mentioned in D13 and later definitions, D is the inner domain of Section 1, D' the outer one. Note in connection with item (ii) of D13 that when an individual constant X of $\mathrm{QC}\overset{i}{=}$ $[\mathrm{QC}\overset{\infty}{=}]$ is assigned a member of $D \cup D'$ that belongs to D, then X designates a value of 'x', 'y', 'z', etc.; when X is a assigned a member of $D \cup D'$ that belongs to D', then X designates something not a value of 'x', 'y', 'z', etc.; and when X is not assigned any member of $D \cup D'$ (which is bound to be the case if $D \cup D'$ is empty, but may also happen when $D \cup D'$ has members), then X fails to designate at all.

D13. Let D and D' be disjoint domains, and $\mathrm{Int}_{\langle D,D'\rangle}$ be any result of assigning:

(i) exactly one of the two truth-values T and F to each sentence variable P of $\mathrm{QC}\overset{i}{=}$, the truth-value in question to be known as the value of P under $\mathrm{Int}_{\langle D,D'\rangle}$;

(ii) if D is not empty, exactly one member of D to each individual variable X of $\mathrm{QC}\overset{i}{=}$, the member in question to be known as the value of X under $\mathrm{Int}_{\langle D,D'\rangle}$;

(iii) if $D \cup D'$ is not empty, *at most one* member of $D \cup D'$ to each individual constant X of $\mathrm{QC}\overset{i}{=}$, the member in question to be known as the value of X under $\mathrm{Int}_{\langle D,D'\rangle}$; and

(iv) exactly one subset of $(D \cup D')^m$ to each m-adic $(m = 1, 2, \ldots)$ predicate variable F of $\mathrm{QC}\overset{i}{=}$, the subset in question to be known as the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$.

Then $\mathrm{Int}_{\langle D,D'\rangle}$ counts as a $\langle D, D'\rangle$-interpretation of the variables and constants of $\mathrm{QC}\overset{i}{=}$ or, for short, as a $\langle D, D'\rangle$-interpretation of VC^i.

D13$^\infty$. Like D13, but with '$\mathrm{QC}\overset{\infty}{=}$' in place of '$\mathrm{QC}\overset{i}{=}$', and '$\mathrm{VC}^\infty$' in place of '$\mathrm{VC}^i$'.

Remark. Except D19-D21, every further definition in this section will be understood to carry along its analogue for $\mathrm{QC}\overset{\infty}{=}$.

D14. Let D, D', and $\mathrm{Int}_{\langle D,D'\rangle}$ be as in D13. If every individual constant of $\mathrm{QC}\overset{i}{=}$ has a value under $\mathrm{Int}_{\langle D,D'\rangle}$, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to be C-exhaustive.

D15. Let D be a non-empty domain; let D' be a domain disjoint from D; let $\mathrm{Int}_{\langle D,D'\rangle}$ and $\mathrm{Int}'_{\langle D,D'\rangle}$ be $\langle D, D'\rangle$-interpretations of VC^i; and let X be an individual variable of $\mathrm{QC}\overset{i}{=}$. If $\mathrm{Int}'_{\langle D,D'\rangle}$ is like $\mathrm{Int}_{\langle D,D'\rangle}$

except possibly for assigning to X a different member of D than $\mathrm{Int}_{\langle D, D'\rangle}$ does, then $\mathrm{Int}'_{\langle D, D'\rangle}$ counts as an X-*variant* of $\mathrm{Int}_{\langle D, D'\rangle}$.

We next define three notions of satisfaction: satisfaction$_T$, satisfaction$_F$, and (plain) satisfaction. The first is intended for $\mathrm{QC}\overset{3}{=}$-$\mathrm{QC}\overset{8}{=}$, which—the reader will recall—automatically pronounce true any formula that contains a non-designating constant; the second is intended for $\mathrm{QC}\overset{7}{=}$-$\mathrm{QC}\overset{10}{=}$, which automatically pronounce false any such formula; and the third is intended for $\mathrm{QC}\overset{1}{=}$, $\mathrm{QC}\overset{2}{=}$, and $\mathrm{QC}\overset{\infty}{=}$, which under our official account of things require every individual constant to designate something, either a value of a variable or something not a value of a variable. D16-D18 run along anticipated lines when D is not empty, and hence the variables 'x', 'y', 'z', etc., have values under $\mathrm{Int}_{\langle D, D'\rangle}$. When, on the other hand, D is $\varnothing$, then an open formula or a closed one of the kind $(\forall X)A$ is automatically held, e.g., satisfied$_T$ by $\mathrm{Int}_{\langle D, D'\rangle}$, this in the second case because $(\forall X)A$ is held tantamount to $\sim(\exists X)\sim A$, in the first because the formula is held tantamount to its universal closure.

D16. Let A be a formula of $\mathrm{QC}\overset{i}{=}$; D and D' be disjoint domains; and $\mathrm{Int}_{\langle D, D'\rangle}$ be a $\langle D, D'\rangle$-interpretation of VC^i.

Case 1: At least one individual constant of $\mathrm{QC}\overset{i}{=}$ occurring in A has no value under $\mathrm{Int}_{\langle D, D'\rangle}$. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A.

Case 2: Every individual constant of $\mathrm{QC}\overset{i}{=}$ occurring in A has a value under $\mathrm{Int}_{\langle D, D'\rangle}$.

Case 2.1: D is not empty.

(a) If A is a sentence variable of $\mathrm{QC}\overset{i}{=}$ and the value of A under $\mathrm{Int}_{\langle D, D'\rangle}$ is T, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(b) If A is of the kind $F(X_1, X_2, \ldots, X_m)$ and the m-tuple made up to the values of $X_1, X_2, \ldots, X_m$ (in that order) under $\mathrm{Int}_{\langle D, D'\rangle}$ belongs to the value of F under $\mathrm{Int}_{\langle D, D'\rangle}$, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(c) If A is of the kind $X = Y$ and the value of X under $\mathrm{Int}_{\langle D, D'\rangle}$ is the same as that of Y, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(d) If A is of the kind $\sim B$ and $\mathrm{Int}_{\langle D, D'\rangle}$ does not satisfy$_T$ B, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(e) If A is of the kind $B \supset C$ and $\mathrm{Int}_{\langle D, D'\rangle}$ does not satisfy$_T$ B or satisfies$_T$ C, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(f) If A is of the kind $(\forall X)B$ and every X-variant of $\mathrm{Int}_{\langle D, D'\rangle}$ satisfies$_T$ B, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(g) $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A pursuant only to one or another of (a)-(f).

Case 2.2: D is empty.

Case 2.2.1: A is open. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A.

Case 2.2.2: A is closed.

(a)-(e) Like (a)-(e) under Case 2.1;

(f) If A is of the kind $(\forall X)B$, then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A;

(g) Like (g) under Case 2.1.

D17. Let A, D, D', and $\mathrm{Int}_{\langle D, D'\rangle}$ be as in D16.

Case 1: At least one individual constant of $\mathrm{QC}\overset{i}{=}$ occurring in A has no value under $\mathrm{Int}_{\langle D, D'\rangle}$. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said not to satisfy$_F$ A.

Case 2: Every individual constant of $\mathrm{QC}\overset{i}{=}$ occurring in A has a value under $\mathrm{Int}_{\langle D, D'\rangle}$. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy$_T$ A if $\mathrm{Int}_{\langle D, D'\rangle}$ satisfies$_T$ A.

D18. Let A, D, and D' be as in D16; and let $\mathrm{Int}_{\langle D, D'\rangle}$ be a C-exhaustive $\langle D, D'\rangle$-interpretation of VC^i. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to satisfy A if $\mathrm{Int}_{\langle D, D'\rangle}$ satisfies$_T$ A.

Remarks. It follows from D16-D18 that, where $\mathrm{Int}_{\langle D, D'\rangle}$ is a C-exhaustive $\langle D, D'\rangle$-interpretation of VC^i [VC^∞], then a formula of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] is satisfied by $\mathrm{Int}_{\langle D, D'\rangle}$ if and only if satisfied$_T$, and hence satisfied$_F$, by $\mathrm{Int}_{\langle D, D'\rangle}$. It likewise follows from D16-D18 that a closed formula of $\mathrm{QC}\overset{i}{=}$ [$\mathrm{QC}\overset{\infty}{=}$] of the kind $\mathrm{E}!X$ is both satisfied and satisfied$_F$ by $\mathrm{Int}_{\langle D, D'\rangle}$ if and only if the value of X under $\mathrm{Int}_{\langle D, D'\rangle}$ belongs to D, and is satisfied$_T$ by $\mathrm{Int}_{\langle D, D'\rangle}$ unless the value of X under $\mathrm{Int}_{\langle D, D'\rangle}$ belongs to D'.

D19. Let S be a set of formulas of $\mathrm{QC}\overset{i}{=}$, and D and D' be as in D16.

(a) Let $\mathrm{Int}_{\langle D, D'\rangle}$ be a $\langle D, D'\rangle$-interpretation of VC^i. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to simultaneously satisfy$_T$ [simultaneously satisfy$_F$] S if $\mathrm{Int}_{\langle D, D'\rangle}$ satisfies [satisfies$_F$] each and every member of S.

(b) Let $\mathrm{Int}_{\langle D, D'\rangle}$ be a C-exhaustive $\langle D, D'\rangle$-interpretation of VC^i. Then $\mathrm{Int}_{\langle D, D'\rangle}$ is said to simultaneously satisfy S if $\mathrm{Int}_{\langle D, D'\rangle}$ simultaneously satisfies$_T$ S.

We then turn to the notion of implication and the attendant one of validity. Eleven different cases are in order.

D20. Let S be a set of formulas of $\mathrm{QC}\overset{i}{=}$, and A be a formula of $\mathrm{QC}\overset{i}{=}$.

(a) S is said to imply A in $\mathrm{QC}\overset{0}{=}$ if, for every non-empty domain D, every C-exhaustive $\langle D, \varnothing\rangle$-interpretation of VC^0 that simultaneously satisfies S also satisfies A.

(b) S is said to imply A in $\mathrm{QC}\overset{1}{=}$ if, for every non-empty domain D and every domain D' disjoint from D, every C-exhaustive $\langle D, D'\rangle$-interpretation of VC^1 that simultaneously satisfies S also satisfies A.

(c) S is said to imply A in $\mathrm{QC}\overset{2}{=}$ if, for every domain D and every domain D' disjoint from D, every C-exhaustive $\langle D, D'\rangle$-interpretation of VC^2 that simultaneously satisfies S also satisfies A.

(d) S is said to imply A in $\mathrm{QC}\overset{3}{=}$ if, for every non-empty domain D, every $\langle D, \varnothing\rangle$-interpretation of VC^3 that simultaneously satisfies$_T$ S also satisfies$_T$ A.

(e) S is said to imply A in $\mathrm{QC}\overset{4}{=}$ if, for every domain D, every $\langle D, \emptyset\rangle$-interpretation of VC^4 that simultaneously satisfies$_T$ S also satisfies$_T$ A.

(f) S is said to imply A in $\mathrm{QC}\overset{5}{=}$ if, for every non-empty domain D and every domain D' disjoint from D, every $\langle D, D'\rangle$-interpretation of VC^5 that simultaneously satisfies$_T$ S also satisfies$_T$ A.

(g) S is said to imply A in $\mathrm{QC}\overset{6}{=}$ if, for every domain D and every domain D' disjoint from D_0 every $\langle D, D'\rangle$-interpretation of VC^6 that simultaneously satisfies$_T$ S also satisfies$_T$ A.

(h) S is said to imply A in $\mathrm{QC}\overset{7}{=}$ if, for every non-empty domain D, every $\langle D, \emptyset\rangle$-interpretation of VC^7 that simultaneously satisfies$_F$ S also satisfies$_F$ A.

(i) S is said to imply A in $\mathrm{QC}\overset{8}{=}$ if, for every domain D, every $\langle D, \emptyset\rangle$-interpretation of VC^8 that simultaneously satisfies$_F$ S also satisfies$_F$ A.

(j) S is said to imply A in $\mathrm{QC}\overset{9}{=}$ if, for every non-empty domain D and every domain D' disjoint from D, every $\langle D, D'\rangle$-interpretation of VC^9 that simultaneously satisfies$_F$ S also satisfies$_F$ A.

(k) S is said to imply A in $\mathrm{QC}\overset{10}{=}$ if, for every domain D and every domain D' disjoint from D, every $\langle D, D'\rangle$-interpretation of VC^{10} that simulatenously satisfies$_F$ S also satisfies$_F$ A.

Remark. It is evident that no $\langle \emptyset, \emptyset\rangle$-interpretation of VC^2 can be C-exhaustive; hence S implies A in $\mathrm{QC}\overset{2}{=}$ if and only if, for every domain D and every domain D' that is disjoint from D and non-empty if D is empty, every C-exhaustive $\langle D, D'\rangle$-interpretation of VC^2 that simultaneously satisfies S also satisfies A.

The various restrictions placed in D20(a)-(k) upon D, D', and $\mathrm{Int}_{\langle D,D'\rangle}$, and the kind of satisfaction that is in order in each case, may be tabulated as follows:

TABLE II

	D	D'	Int	kind of satisfaction
0	non-empty	empty	C-exhaustive	satisfaction
1	non-empty	disjoint from D	C-exhaustive	satisfaction
2	arbitrary	disjoint from D	C-exhaustive	satisfaction
3	non-empty	empty	arbitrary	satisfaction$_T$
4	arbitrary	empty	arbitrary	satisfaction$_T$
5	non-empty	disjoint from D	arbitrary	satisfaction$_T$
6	arbitrary	disjoint from D	arbitrary	satisfaction$_T$
7	non-empty	empty	arbitrary	satisfaction$_F$
8	arbitrary	empty	arbitrary	satisfaction$_F$
9	non-empty	disjoint from D	arbitrary	satisfaction$_F$
10	arbitrary	disjoint from D	arbitrary	satisfaction$_F$

D21. A formula A of $\mathrm{QC}\overset{i}{=}$ is said to be *valid in* $\mathrm{QC}\overset{i}{=}$ if $\emptyset$ implies A in $\mathrm{QC}\overset{i}{=}$.

Remarks. It follows from D20-D21 that a formula A of $\mathrm{QC}\overset{i}{=}$ is not valid in $\mathrm{QC}\overset{i}{=}$ for any i from 7 to 10 unless A is constant-free. To abridge matters, we shall write, e.g., 'A is implied$_1$ by S' for 'A is implied by S in $\mathrm{QC}\overset{1}{=}$', and '$A$ is valid$_1$' for 'A is valid in $\mathrm{QC}\overset{1}{=}$'.

The following consequences of D13-D21 merit separate recording as lemmas.

L14. *Let* $(\forall X)A$ *be a formula of* $\mathrm{QC}\overset{\infty}{=}$, D *be a non-empty domain,* D' *be a domain disjoint from* D, *and* $\mathrm{Int}_{\langle D,D'\rangle}$ *be a* C*-exhaustive* $\langle D, D'\rangle$*-interpretation of* VC^{∞}.

(a) *Let each member of* D *be the value under* $\mathrm{Int}_{\langle D,D'\rangle}$ *of some individual constant or other of* $\mathrm{QC}\overset{\infty}{=}$. *If* $\mathrm{Int}_{\langle D,D'\rangle}$ *satisfies* $A(Y/X)$ *for every individual constant* Y *of* $\mathrm{QC}\overset{\infty}{=}$ *whose value under* $\mathrm{Int}_{\langle D,D'\rangle}$ *belongs to* D, *then* $\mathrm{Int}_{\langle D,D'\rangle}$ *satisfies* $(\forall X)A$.

(b) *If* $\mathrm{Int}_{\langle D,D'\rangle}$ *does not satisfy* $A(Y/X)$ *for at least one individual constant* Y *of* $\mathrm{QC}\overset{\infty}{=}$ *whose value under* $\mathrm{Int}_{\langle D,D'\rangle}$ *belongs to* D, *then* $\mathrm{Int}_{\langle D,D'\rangle}$ *does not satisfy* $(\forall X)A$.

L15. *Let* A *be a closed formula of* $\mathrm{QC}\overset{2}{=}$, A' *be the* $\emptyset$*-associate of* A, D' *be a non-empty domain, and* $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ *be a* $\langle \emptyset, D'\rangle$*-interpretation of* VC^2. *Then* $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ *satisfies* A *if and only if* $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ *satisfies* A'.

L16. *Let* A *be a closed formula of* $\mathrm{QC}\overset{i}{=}$, S' *consist of the closed members of* S, *and* $i = 2$ *or* 6. *If* S *implies*$_i$ A, *then it is not the case that there is a non-empty domain* D' *and a* C*-exhaustive* $\langle \emptyset, D'\rangle$*-interpretation* $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ *of* VC^i *such that* $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ *simultaneously satisfies* $S' \cup \{\sim A\}$.

L17. (a) *If* S *implies*$_2$ A, *then* S *implies*$_1$ A.

(b) *If* S *implies*$_5$ A, *then* S *implies*$_1$ A.

(c) *If* S *implies*$_6$ A, *then* S *implies*$_5$ A.

(d) *If* S *implies*$_9$ A, *then* S *implies*$_1$ A.

(e) *If* S *implies*$_{10}$ A, *then* S *implies*$_2$ A.

L18. *Let* S' *consist of every member of* S *in which there occurs an individual constant of* $\mathrm{QC}\overset{i}{=}$ *not occurring in* A, S'' *be* $S - S'$, *and* $i = 5$ *or* 6. *If* S *implies*$_i$ A, *then* S'' *implies*$_i$ A.

L19. *Let* S' *consist of every closed formula of* $\mathrm{QC}\overset{1}{=}$ *(hence,* $\mathrm{QC}\overset{5}{=}$ *and* $\mathrm{QC}\overset{6}{=}$*) of the kind* $\mathrm{E!}X$.

(a) *If* S *implies*$_0$ A, *then* $S \cup S'$ *implies*$_1$ A.

(b) *If* S *implies*$_3$ A, *then* $S \cup S'$ *implies*$_5$ A.

(c) *If* S *implies*$_4$ A, *then* $S \cup S'$ *implies*$_6$ A.

L20. *Let at least one individual constant of* $\mathrm{QC}\overset{i}{=}$ *that occurs in* A *fail to occur in any member of* S, *and* $7 \leqslant i \leqslant 10$. *If* S *implies*$_i$ A, *then* S *implies*$_i$ 'p & $\sim p$'.

L21. *Let S' consist of every closed formula of* $\mathrm{QC}\overset{9}{=}$ (*hence,* $\mathrm{QC}\overset{10}{=}$) *of the kind* E!$X$, *where X occurs in one or more members of S.*

(a) *If S implies$_7$ A, then $S \cup S'$ implies$_9$ A.*

(b) *If S implies$_8$ A, then $S \cup S'$ implies$_{10}$ A.*

Proof of L14 can be retrieved from [8], [20] proof of L15, L16, and L17 is obvious; proof of L19(c) is like that of L19(b); and proof of L21(b) is like that of L21(a). We shall therefore restrict ourselves here to L18, L19(a), L19(b), L20, and L21(a).

For proof of L18. Let S', S'', and i be as in L18, and let S imply$_i$ A.

Case 1: $i = 5$. Let be a non-empty domain, D' be a domain disjoint from D, $\mathrm{Int}_{\langle D,D'\rangle}$ be a $\langle D, D'\rangle$-interpretation of VC^i that simultaneously satisfies$_T$ S'', and $\mathrm{Int}_{\langle D,D'\rangle}$ be like $\mathrm{Int}_{\langle D,D'\rangle}$ except for not assigning any member of $D \cup D'$ to the individual constants of $\mathrm{QC}\overset{i}{=}$ that do not occur in A. Since $\mathrm{Int}'_{\langle D,D'\rangle}$ simultaneously satisfies$_T$ S', then clearly $\mathrm{Int}'_{\langle D,D'\rangle}$ simultaneously satisfies$_T$ $S' \cup S''$ ($= S$) as well. Hence, since S implies$_i$ A, then $\mathrm{Int}_{\langle D,D'\rangle}$ satisfies$_T$ A. Hence, clearly, so does $\mathrm{Int}_{\langle D,D'\rangle}$. Hence S'' implies$_i$ A.

Case 2: $i = 6$. Proof like that of Case 1, but with D allowed to be empty.

For proof of L19(a). Let S' be as in L19, and suppose that $S \cup S'$ does not imply$_1$ A. Then there is a non-empty domain D, a domain D' disjoint from D, and a C-exhaustive $\langle D, D'\rangle$-interpretation $\mathrm{Int}_{\langle D,D'\rangle}$ of VC^1 that simultaneously satisfies $S \cup S'$ (and, hence, assigns a member of D to every individual constant of $\mathrm{QC}\overset{1}{=}$), but does not satisfy A. Now let $\mathrm{Int}_{\langle D,\emptyset\rangle}$ be like $\mathrm{Int}_{\langle D,D'\rangle}$ except for assigning to each m-adic predicate variable F of $\mathrm{QC}\overset{1}{=}$ the subset of D^m consisting of every member of D^m that belongs to the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$. Clearly, $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies $S \cup S'$ if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does, and fails to satisfy A if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does. Hence S does not imply$_0$ A. Hence, if S implies$_0$ A, then $S \cup S'$ implies$_1$ A.

For proof of L19(b). Let S' be as in L19, and suppose that $S \cup S'$ does not imply$_5$ A. Then there is a non-empty domain D, a domain D' disjoint from D, and a $\langle D, D'\rangle$-interpretation $\mathrm{Int}_{\langle D,D'\rangle}$ of VC^5 that simultaneously satisfies$_T$ $S \cup S'$ (and hence, does not assign a member of D' to any individual constant of $\mathrm{QC}\overset{5}{=}$), but does not satisfy$_T$ A. Now let $\mathrm{Int}_{\langle D,\emptyset\rangle}$ be like $\mathrm{Int}_{\langle D,D'\rangle}$ except for assigning to each m-adic predicate variable F of $\mathrm{QC}\overset{5}{=}$ the subset of D^m consisting of every member of D^m that belongs to the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$. Clearly, $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies$_T$ $S \cup S'$ if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does, and fails to satisfy$_T$ A if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does. Hence S does not imply$_3$ A. Hence, if S implies$_3$ A, then $S \cup S'$ implies$_5$ A.

[20] See the proof of MT2.5.35 on p. 158.

For proof of L20. Let S and A be as in L20.

Case 1: $i = 7$. Suppose that there is a non-empty domain D and a $\langle D, \emptyset\rangle$-interpretation $\mathrm{Int}_{\langle D,\emptyset\rangle}$ of VC^i such that $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies$_F$ S; and let $\mathrm{Int}'_{\langle D,\emptyset\rangle}$ be like $\mathrm{Int}_{\langle D,\emptyset\rangle}$ except for not assigning any member of D to the individual constants of $\mathrm{QC}\overset{i}{=}$ that occur in A but do not occur in any member of S. Since $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies$_F$ S, then clearly so does $\mathrm{Int}'_{\langle D,\emptyset\rangle}$. But $\mathrm{Int}'_{\langle D,\emptyset\rangle}$ does not satisfy$_F$ A. Hence S does not imply$_i$ A. Hence, if S implies$_i$ A, then it is not the case that there is a non-empty domain D and a $\langle D, \emptyset\rangle$-interpretation of VC^i such that $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies$_F$ S. Hence, if S implies$_i$ A, then S implies$_i$ 'p & $\sim p$'.

Case 2: $8 \leqslant i \leqslant 10$. Proof like that of Case 1.

For proof of L21(a). Let S' be as in L21, and suppose $S \cup S'$ does not imply$_9$ A. Then there is a non-empty domain D, a domain D' disjoint from D, and a $\langle D, D'\rangle$-interpretation $\mathrm{Int}_{\langle D,D'\rangle}$ of VC^9 that simultaneously satisfies$_F$ $S \cup S'$ (and hence assigns a member of D to every individual constant of $\mathrm{QC}\overset{9}{=}$ that occurs in one or more members of S), but does not satisfy$_F$ A. Now let $\mathrm{Int}_{\langle D,\emptyset\rangle}$ be the result of assigning to each sentence variable of $\mathrm{QC}\overset{9}{=}$ the same truth-value as in $\mathrm{Int}_{\langle D,D'\rangle}$, to each individual variable of $\mathrm{QC}\overset{9}{=}$ the same member of D as in $\mathrm{Int}_{\langle D,D'\rangle}$, to each individual constant of $\mathrm{QC}\overset{9}{=}$ that occurs in one or more members of S the same member of D as in $\mathrm{Int}_{\langle D,D'\rangle}$, and to each m-adic predicate variable F of $\mathrm{QC}\overset{9}{=}$ the subset of D^m consisting of every member of D^m that belongs to the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$. Clearly, $\mathrm{Int}_{\langle D,\emptyset\rangle}$ simultaneously satisfies$_F$ S if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does, and fails to satisfy$_F$ A if $\mathrm{Int}_{\langle D,D'\rangle}$ does. Note, in the latter case, that if any individual constant of $\mathrm{QC}\overset{9}{=}$ that occurs in A has no value under $\mathrm{Int}_{\langle D,\emptyset\rangle}$, then $\mathrm{Int}_{\langle D,\emptyset\rangle}$ fails to satisfy$_F$ A, whereas if every individual constant of $\mathrm{QC}\overset{9}{=}$ that occurs in A has a value under $\mathrm{Int}_{\langle D,\emptyset\rangle}$, then $\mathrm{Int}_{\langle D,\emptyset\rangle}$ fails to satisfy$_F$ A if and only if $\mathrm{Int}_{\langle D,D'\rangle}$ does. Hence S does not imply$_7$ A. Hence, if S implies$_7$ A, then $S \cup S'$ implies$_9$ A.

4. Completeness theorems. In this section we first establish that $\mathrm{QC}\overset{1'}{=}$ is complete both in Henkin's sense and in Gödel's, and then proceed to obtain similar results for the rest of our calculi. The proofs of the auxiliary theorems T1, T2, and T4 owe much, as the reader will gather, to [3].

T1. *Let S_0 be a set of formulas of $\mathrm{QC}\overset{1}{=}$ that is consistent in $\mathrm{QC}\overset{1}{=}$; for each j from 1 on let S_j be $S_{j-1} \cup \{(\exists X_j)A_j \supset A_j(Y/X_j),\ \mathrm{E!}Y\}$, where*

(i) $(\exists X_j)A_j$ *is in alphabetical order the* j*-th formula of* $\mathrm{QC}^{\infty}_{=}$ *of the kind* $(\exists X)A$, *and* (ii) Y *is the alphabetically earliest individual constant of* $\mathrm{QC}^{\infty}_{=}$ *not to occur in any member of* S_{j-1} *nor in* $(\exists X_j)A_j$; *and let* S_∞ *be the union of* $S_0, S_1, S_2, \ldots$ *Then:*

(a) *For each* j *from* 0 *on* S_j *is consistent in* $\mathrm{QC}^{\infty}_{=}$, *and*

(b) S_∞ *is consistent in* $\mathrm{QC}^{\infty}_{=}$.

Proof.

(a) Suppose that S_j $(j \geqslant 1)$ is inconsistent in $\mathrm{QC}^{\infty}_{=}$. Then in view of L1(o) and L1(r) so is S_{j-1}. But in view of L1(d) S_0 is consistent in $\mathrm{QC}^{\infty}_{=}$. Hence (a) by mathematical induction on j.

(b) Suppose that S_∞ is inconsistent in $\mathrm{QC}^{\infty}_{=}$. Then in view of L1(e) some finite subset of S_∞ is inconsistent in $\mathrm{QC}^{\infty}_{=}$. But every finite subset of S_∞ is a subset of S_j for some j from 0 on. Hence in view of L1(e) not all of $S_0, S_1, S_2, \ldots$, are consistent in $\mathrm{QC}^{\infty}_{=}$, as against (a). Hence (b).

T2. *Let* S^0_∞ *be the set* S_∞ *of* T1; A_k *being in alphabetical order the* k*-th formula of* $\mathrm{QC}^{\infty}_{=}$, *let* S^k_∞ *be for each* k *from* 1 *on* $S^{k-1}_\infty \cup \{A_k\}$ *or* S^{k-1}_∞ *according as* $S^{k-1}_\infty \cup \{A_k\}$ *is consistent*$_1$ *in* $\mathrm{QC}^{\infty}_{=}$ *or not; and let* S^∞_∞ *be the union of* $S^0_\infty, S^1_\infty, S^2_\infty, \ldots$ *Then:*

(a) *For each* k *from* 0 *on* S^k_∞ *is consistent in* $\mathrm{QC}^{\infty}_{=}$;

(b) S^∞_∞ *is consistent in* $\mathrm{QC}^{\infty}_{=}$;

(c) *If* $S^\infty_\infty \cup \{A\}$ *is consistent in* $\mathrm{QC}^{\infty}_{=}$, *then* A *belongs to* S^∞_∞;

(d) $S^\infty_\infty \vdash^\infty_1 A$ *if and only if it is not the case that* $S^\infty_\infty \vdash^\infty_1 \sim A$; *and*

(e) $(\exists X)A_j$ *being in alphabetical order the* j*-th formula of* $\mathrm{QC}^{\infty}_{=}$ *of the kind* $(\exists X)A$, *then for each* j *from* 1 *on there is an individual constant* Y *of* $\mathrm{QC}^{\infty}_{=}$ *such that* $S^\infty_\infty \vdash^\infty_1 (\exists X_j)A_j \supset A_j(Y/X_j)$ *and* $S^\infty_\infty \vdash^\infty_1 \mathrm{E}!Y$.

Proof.

(a) Suppose that S^k_∞ $(k \geqslant 1)$ is $S^{k-1}_\infty \cup \{A_k\}$; then S^k_∞ is consistent in $\mathrm{QC}^{\infty}_{=}$, and hence is consistent in $\mathrm{QC}^{\infty}_{=}$ if S^{k-1}_∞ is. Suppose, on the other hand, that S^k_∞ is S^{k-1}_∞; then S^k_∞ is consistent in $\mathrm{QC}^{\infty}_{=}$ if S^{k-1}_∞ is. But in view of T1(b) S^0_∞ is consistent in $\mathrm{QC}^{\infty}_{=}$. Hence (a) by mathematical induction on k.

(b) Proof like that of T1(b).

(c) Suppose that $S^\infty_\infty \cup \{A\}$ is consistent in $\mathrm{QC}^{\infty}_{=}$ and that A is in alphabetical order the kth formula of $\mathrm{QC}^{\infty}_{=}$. If S^{k-1}_∞ were inconsistent in $\mathrm{QC}^{\infty}_{=}$, then in view of L1(c) $S^\infty_\infty \cup \{A\}$ would be inconsistent in $\mathrm{QC}^{\infty}_{=}$, contrary to the assumption. Hence $S^{k-1}_\infty \cup \{A\}$ is consistent in $\mathrm{QC}^{\infty}_{=}$, hence A belongs to S^k_∞, and hence A belongs to S^∞_∞.

(d) Suppose $S^\infty_\infty \vdash^\infty_1 A$ and $S^\infty_\infty \vdash^\infty_1 \sim A$. Then in view of L1(f) S^∞_∞ is inconsistent in $\mathrm{QC}^{\infty}_{=}$, as against (b). Suppose, on the other hand, that it is not the case that $S^\infty_\infty \vdash^\infty_1 A$. Then in view of L1(b) A does not belong to S^∞_∞. Hence in view of (c) $S^\infty_\infty \cup \{A\}$ is inconsistent in $\mathrm{QC}^{\infty}_{=}$. Hence in view of L1(g) $S^\infty_\infty \vdash^\infty_1 \sim A$.

(e) Proof by T1 and L1(b).

T3. *Let* S *be a set of formulas of* $\mathrm{QC}^{1}_{=}$ *that is consistent in* $\mathrm{QC}^{1}_{=}$. *Then there is a set* S' *of formulas of* $\mathrm{QC}^{\infty}_{=}$ *such that:*

(a) S *is a subset of* S';

(b) $S' \vdash^\infty_1 A$ *if and only if it is not the case that* $S' \vdash^\infty_1 \sim A$; *and*

(c) $(\exists X_j)A$ *being as in* T2(e), *for each* j *from* 1 *on there is an individual constant* Y *of* $\mathrm{QC}^{\infty}_{=}$ *such that* $S' \vdash^\infty_1 (\exists X_j)A_j \supset A_j(Y/X_j)$ *and* $S' \vdash^\infty_1 \mathrm{E}!Y$.

Proof by T1-T2.

T4. *Let* S *be a set of formulas of* $\mathrm{QC}^{1}_{=}$. *If* S *is consistent in* $\mathrm{QC}^{1}_{=}$, *then there is a non-empty domain* D, *a domain* D' *disjoint from* D, *and a* C*-exhaustive* $\langle D, D'\rangle$*-interpretation* $\mathrm{Int}_{\langle D,D'\rangle}$ *of* VC^1 *such that* $\mathrm{Int}_{\langle D,D'\rangle}$ *simultaneously satisfies* S.

Proof. Let S be consistent in $\mathrm{QC}^{1}_{=}$. Then in view of T3 there is a set S' of formulas of $\mathrm{QC}^{\infty}_{=}$ of which (a)-(c) in T3 hold true.

Part One. Let T^∞ consist of all the individual terms of $\mathrm{QC}^{\infty}_{=}$ and R be a dyadic relation on T^∞ such that, for any members X and Y of T^∞, $R(X, Y)$ if and only if $S' \vdash^\infty_1 X = Y$. In view of L1(s)-(u) R is an equivalence relation on T^∞, and hence partitions T^∞ into one or more sets, say, $T^\infty_1, T^\infty_2, \ldots$, which by definition are pairwise disjoint and exhaustive of T^∞. Now, for each k from 1 on, let U_k be the alphabetically earliest individual variable of $\mathrm{QC}^{\infty}_{=}$ to belong to T^∞_k if any individual variable of $\mathrm{QC}^{\infty}_{=}$ belongs to T^∞_k, otherwise the alphabetically earliest individual constant of $\mathrm{QC}^{\infty}_{=}$ to belong to T^∞_k; and for each individual term X of $\mathrm{QC}^{\infty}_{=}$ let $\gamma(X)$ be U_k, where T^∞_k is the one subset of T^∞ to which X belongs. ([21]) It is easily verified that:

(1.1) *For each individual term* X *of* $\mathrm{QC}^{\infty}_{=}$, $S' \vdash^\infty_1 \mathrm{E}!X$ *if and only if* $S' \vdash^\infty_1 \mathrm{E}!\gamma(X)$;

(1.2) *For each individual variable* X *of* $\mathrm{QC}^{\infty}_{=}$, $S' \vdash^\infty_1 \mathrm{E}!\gamma(X)$;

(1.3) *There is at least one* k *such that* $S' \vdash^\infty_1 \mathrm{E}!U_k$;]

(1.4) $S' \vdash^\infty_1 F(X_1, X_2, \ldots, X_m)$ *if and only if* $S' \vdash^\infty_1 F(\gamma(X_1), \gamma(X_2), \ldots, \gamma(X_m))$; *and*

(1.5) $S' \vdash^\infty_1 X = Y$ *if and only if* $S' \vdash^\infty_1 \gamma(X) = \gamma(Y)$.

([21]) For each k from 1 on U_k can be thought of as the representative of the various individual terms of $\mathrm{QC}^{\infty}_{=}$ that belong to T^∞_k; and hence, for each individual term X of $\mathrm{QC}^{\infty}_{=}$, $\gamma(X)$ can be thought of as the representative of the various individual terms of $\mathrm{QC}^{\infty}_{=}$ that belong to the same subset of T^∞ as X does.

Note for proof of (1.1), (1.4), and (1.5) that $S' \vdash_1^\infty X = \gamma(X)$ by the very definition of $\gamma(X)$. Hence (1.1), (1.4), and (1.5) by repeated uses of L1(v). Note for proof of (1.2) that in view of L1(n) $S' \vdash_1^\infty \mathrm{E}!X$ for each individual variable X of $\mathrm{QC}\overset{\infty}{=}$. Hence (1.2) by (1.1). And hence (1.3).

Part Two. Let $D \cup D'$ be $\{U_1, U_2, \ldots\}$; for each k from 1 on let U_k belong to D if $S' \vdash_1^\infty \mathrm{E}!U_k$, otherwise to D'; and let $\mathrm{Int}'_{\langle D,D'\rangle}$ be the result of assigning:

(i) to each sentence variable P of $\mathrm{QC}\overset{\infty}{=}$ the truth-value T if $S' \vdash_1^\infty P$, otherwise the truth-value F;

(ii) to each individual term X of $\mathrm{QC}\overset{\infty}{=}$ the uniquely determined member $\gamma(X)$ of $D \cup D'$; and

(iii) to each m-adic ($m \geqslant 1$) predicate variable F of $\mathrm{QC}\overset{\infty}{=}$ the one subset of $(D \cup D')^m$ to which $\langle U_{i_1}, U_{i_2}, \ldots, U_{i_m}\rangle$ belongs if and only if $S' \vdash_1^\infty F(U_{i_1}, U_{i_2}, \ldots, U_{i_m})$.

It is easily verified that:

(2.1) *D is not empty and D' is disjoint from D;*

(2.2) $\mathrm{Int}'_{\langle D,D'\rangle}$ *assigns a member of D to each individual variable of* $\mathrm{QC}\overset{\infty}{=}$, *and a member of $D \cup D'$ to each individual constant of* $\mathrm{QC}\overset{\infty}{=}$;

(2.3) $\mathrm{Int}'_{\langle D,D'\rangle}$ *counts as a C-exhaustive $\langle D, D'\rangle$-interpretation of* VC^∞;

(2.4) $S' \vdash_1^\infty \mathrm{E}!X$, *where X is an individual constant of* $\mathrm{QC}\overset{\infty}{=}$, *if and only if the value of X under* $\mathrm{Int}'_{\langle D,D'\rangle}$ *belongs to D; and*

(2.5) *Each member of D is the value under* $\mathrm{Int}'_{\langle D,D'\rangle}$ *of some individual constant or other of* $\mathrm{QC}\overset{\infty}{=}$.

Of these, (2.1) follows from (1.3); (2.2) from (1.2) and (ii); and (2.3) from (2.1)-(2.2) and (i)-(iii). As for (2.4), note that in view of (1.1) $S' \vdash_1^\infty \mathrm{E}!X$ if and only if $S' \vdash_1^\infty \mathrm{E}!\gamma(X)$; hence (2.4) by (ii) and the definition of D. As for (2.5), if U_k belongs to D, then $S' \vdash_1^\infty \mathrm{E}!U_k$, and hence in view of L1(p) $S' \vdash_1^\infty (\exists X)(X = U_k)$, where X is the alphabetically earliest individual variable of $\mathrm{QC}\overset{\infty}{=}$ to differ from U_k. Hence in view of T3(c) and L1(i) $S' \vdash_1^\infty Y = U_k$ for some individual constant Y of $\mathrm{QC}\overset{\infty}{=}$. Hence Y belongs to Σ_k. Hence U_k is the value of Y under $\mathrm{Int}'_{\langle D,D'\rangle}$.

Part Three. Let A be a formula of $\mathrm{QC}\overset{\infty}{=}$. It is easily shown by mathematical induction on the number of occurrences of '$\sim$', '$\supset$', and '$\forall$' in A that $\mathrm{Int}_{\langle D,D'\rangle}$ satisfies A if and only if $S' \vdash_1^\infty A$.

Base Step. By (1.4)-(1.5) and (i)-(iii).

Inductive Step. Suppose that A is of the kind $\sim B$. Then in view of T3(b) $S' \vdash_1^\infty A$ if and only if it is not the case that $S' \vdash_1^\infty B$, hence in view of the hypothesis of the induction if and only if $\mathrm{Int}'_{\langle D,D'\rangle}$ does not satisfy B, and hence if and only if $\mathrm{Int}'_{\langle D,D'\rangle}$ satisfies A. Or suppose that A is of the kind $B \supset C$. Then view of T3(b) and L1(j) $S' \vdash_1^\infty A$ if and only if $S' \vdash_1^\infty C$ or it is not the case that $S' \vdash_1^\infty B$, hence in view of the hypothesis of the induction if and only if $\mathrm{Int}'_{\langle D,D'\rangle}$ does not satisfy B or $\mathrm{Int}'_{\langle D,D'\rangle}$ satisfies C, and hence if and only if $\mathrm{Int}'_{\langle D,D'\rangle}$ satisfies A. Or suppose that A is of the kind $(\forall X)B$ and $S' \vdash_1^\infty A$. Then in view of L1(q) $S' \vdash_1^\infty \mathrm{E}!Y \supset B(Y/X)$ for every individual constant Y of $\mathrm{QC}\overset{\infty}{=}$; hence in view of (2.4) and L1(i) $S' \vdash_1^\infty B(Y/X)$ for every individual constant Y of $\mathrm{QC}\overset{\infty}{=}$ whose value under $\mathrm{Int}'_{\langle D,D'\rangle}$ belongs to D; and hence in view of (2.5) and L14(a) $\mathrm{Int}'_{\langle D,D'\rangle}$ satisfies A. Or suppose that A is of the kind $(\forall X)B$ and it is not the case that $S' \vdash_1^\infty A$. Then in view of T3(b) $S' \vdash_1^\infty \sim(\forall X)B$; hence in view of L1(m) $S' \vdash_1^\infty (\exists X)\sim B$; hence in view of T3(c) and L1(i) $S' \vdash_1^\infty \sim B(Y/X)$ and $S' \vdash_1^\infty \mathrm{E}!Y$ for at least one individual constant Y of $\mathrm{QC}\overset{\infty}{=}$; hence in view of T3(b) and (2.4) it is not the case that $S' \vdash_1^\infty B(Y/X)$ for at least one individual constant Y of $\mathrm{QC}\overset{\infty}{=}$ whose value under $\mathrm{Int}'_{\langle D,D'\rangle}$ belongs to D; hence in view of the hypothesis of the induction $\mathrm{Int}'_{\langle D,D'\rangle}$ does not satisfy $B(Y/X)$ for at least one individual constant Y of $\mathrm{QC}\overset{\infty}{=}$ whose value under $\mathrm{Int}'_{\langle D,D'\rangle}$ belongs to D; and hence in view of L14(b) $\mathrm{Int}'_{\langle D,D'\rangle}$ does not satisfy A.

Part Four. Let A be a member of S, and $\mathrm{Int}_{\langle D,D'\rangle}$ be the result of assigning to each sentence variable of $\mathrm{QC}=$ the same truth-value as in $\mathrm{Int}'_{\langle D,D'\rangle}$, to each individual term of $\mathrm{QC}=$ the same member of $D \cup D'$ as in $\mathrm{Int}'_{\langle D,D'\rangle}$, and to each m-adic predicate variable of $\mathrm{QC}=$ the same subset of $(D \cup D')^m$ as in $\mathrm{Int}'_{\langle D,D'\rangle}$. Since in view of T3(a) A belongs to S', then in view of L1(b) $S' \vdash_1^\infty A$, hence in view of Part Three $\mathrm{Int}'_{\langle D,D'\rangle}$ satisfies A, and hence so does $\mathrm{Int}_{\langle D,D'\rangle}$. Hence T4 in view of (2.1) and (2.3).

T5. *Let S be a set of formulas of* $\mathrm{QC}\overset{1}{=}$, *and A be a formula of* $\mathrm{QC}\overset{1}{=}$.

(a) *If S implies$_1$ A, then* $S \vdash_1 A$.

(b) *If A is valid$_1$, then* $\vdash_1 A$.

Proof. (a). Let S imply$_1$ A. Then it is not the case that there is a non-empty domain D, a domain D' disjoint from D, and a C-exhaustive $\langle D, D'\rangle$-interpretation $\mathrm{Int}_{\langle D,D'\rangle}$ of VC^1 such that $\mathrm{Int}_{\langle D,D'\rangle}$ simultaneously satisfies $S \cup \{\sim A\}$. Hence in view of T4 $S \cup \{\sim A\}$ is inconsistent in $\mathrm{QC}\overset{1}{=}$. Hence in view of L3(d) $S \vdash_1 A$. (b) Let A be valid$_1$. Then $\varnothing$ implies$_1$ A. Hence $\vdash_1 A$ in view of (a).

In view of T5(a) $\mathrm{QC}\overset{1}{=}$ may be said to be *complete in Henkin's sense*, and in view of T5(b) to be *complete in Gödel's*.

We next establish that $\mathrm{QC}\overset{2}{=}$ is likewise complete in both senses.

T6. *Let S be a set of closed formulas of* $\mathrm{QC}\overset{2}{=}$, *and let* $S \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\}$ *be consistent in* $\mathrm{QC}\overset{2}{=}$. *Then there is a non-empty domain D' and a C-exhaustive $\langle \varnothing, D'\rangle$-interpretation* $\mathrm{Int}_{\langle \varnothing,D'\rangle}$ *of* VC^2 *such that* $\mathrm{Int}_{\langle \varnothing,D'\rangle}$ *simultaneously satisfies S.*

Proof. Part One. Let S'_0 consist of the $\emptyset$-associates of the members of S; A_j being in alphabetical order the jth closed and quantifier-free formula of $\mathrm{QC}\overset{2}{=}$, let S'_j be for each j from 1 on $S'_{j-1} \cup \{A_j\}$ or S'_{j-1} according as $S'_{j-1} \cup \{A_j\}$ is consistent in $\mathrm{QC}\overset{2}{=}$ or not; and let S'_∞ be the union of $S'_0, S'_1, \ldots$. In view of L4(a) S'_0 is consistent in $\mathrm{QC}\overset{2}{=}$. Hence by the same reasoning as in the proof of T2(a)-(d), but with L3(b), L2(a), L3(c), L2(b), and L2(c) respectively doing duty for L1(b), L1(e), L1(c), L1(f), and L1(g):

(1) *$S'_\infty \vdash_2 A$ if and only if it is not the case that $S'_\infty \vdash_2 \sim A$.*

Part Two: Let C^2 consist of all the individual constants of $\mathrm{QC}\overset{2}{=}$, and let R be a dyadic relation on C^2 such that, for any members X and Y of C^2, $R(X, Y)$ if and only if $S'_\infty \vdash_2 X = Y$. In view of L2(e)-(g) R is an equivalence relation on C^2, and hence partitions C^2 into one or more sets, say, $C_1^2, C_2^2, \ldots$, which by definition are pairwise disjoint and exhaustive of C^2. Now, for each k from 1 on, let U_k be the alphabetically earliest individual constant of $\mathrm{QC}\overset{2}{=}$ to belong to C_k^2; and for each individual constant X of $\mathrm{QC}\overset{2}{=}$, let $\gamma(X)$ be U_k, where C_k^2 is the one subset of C^2 to which X belongs. It is easily verified with the aid of L2(h) that:

(2) *$S'_\infty \vdash_2 F(X_1, X_2, \ldots, X_m)$, where $X_1, X_2, \ldots$, and X_m are individual constants of $\mathrm{QC}\overset{2}{=}$, if and only if $S'_\infty \vdash_2 F(\gamma(X_1), \gamma(X_2), \ldots, \gamma(X_m))$, and*

(3) *$S'_\infty \vdash_2 X = Y$, where X and Y are individual constants of $\mathrm{QC}\overset{2}{=}$, if and only if $S'_\infty \vdash_2 \gamma(X) = \gamma(Y)$.*

Part Three. Let D' be $\{U_1, U_2, \ldots\}$, and let $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ be the result of assigning:

(i) to each sentence variable P of $\mathrm{QC}\overset{2}{=}$ the truth-value T if $S'_\infty \vdash_2 P$, otherwise the truth-value F;

(ii) to each individual constant X of $\mathrm{QC}\overset{2}{=}$ the uniquely determined member $\gamma(X)$ of D'; and

(iii) to each m-adic predicate variable F of $\mathrm{QC}\overset{2}{=}$ the one subset of D'^m to which $\langle U_{i_1}, U_{i_2}, \ldots, U_{i_m}\rangle$ belongs if and only if $S'_\infty \vdash_2 F(U_{i_1}, U_{i_2}, \ldots, U_{i_m})$.

It is easily verified that:

(4) *D' is not empty and $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ counts as a C-exhaustive $\langle\emptyset, D'\rangle$-interpretation of VC^2.*

Part Four. By the same reasoning as in Part Three of the proof of T4, but with the induction carried on the number of occurrences of '$\sim$' and '$\supset$' (rather than '$\sim$', '$\supset$', and '$\forall$') in A, and with (1)-(3) above and L2(d) doing duty for T3(b), (1.4), (1.5), and L1(j), it is easily shown that, where A is a closed and quantifier-free formula of $\mathrm{QC}\overset{2}{=}$, $\mathrm{Int}_{\langle\emptyset,D\rangle}$ satisfies A if and only if $S'_\infty \vdash_2 A$. Hence $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ simultaneously satisfies S'_∞, hence S'_0 (S'_0 being a subset of S'_∞), and hence in view of L15 S itself. Hence T6 in view of (4).

T7. *Let S be a set of formulas of $\mathrm{QC}\overset{2}{=}$, and A be a formula of $\mathrm{QC}\overset{2}{=}$.*

(a) *If S implies$_2$ A, then $S \vdash_2 A$.*

(b) *If A is valid$_2$, then $\vdash_2 A$.*

Proof. (a) Let S imply$_2$ A. Then in view of L17(a) S implies$_1$ A, and hence in view of T5(a) $S \vdash_1 A$.

Case 1: A is open. Then in view of L5(b) $S \cup \{(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$. But in view of L4(b) and L3(c) $S \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$. Hence $S \vdash_2 A$ in view of L3(e).

Case 2: A is closed. Let S' consist of all the closed members of S. Since S implies$_2$ A, then in view of L16 it is not the case that there is a non-empty domain D' and a C-exhaustive $\langle\emptyset, D'\rangle$-interpretation $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ of VC^2 such that $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ simultaneously satisfies $S' \cup \{\sim A\}$. Hence in view of T6 $S' \cup \{\sim(\exists x)(f(x) \vee \sim f(x)), \sim A\}$ is inconsistent in $\mathrm{QC}\overset{2}{=}$, hence in view of L3(d) $S' \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$, and hence in view of L3(c), $S \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$. On the other hand, since $S \vdash_1 A$, then in view of L5(b) $S \cup \{(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$. Hence $S \vdash_2 A$ in view of L3(e).

(b) Proof like that of T5(b).

Banking on T5 and T7, we next establish that $\mathrm{QC}\overset{5}{=}$ and $\mathrm{QC}\overset{6}{=}$ are complete in Henkin's sense and in Gödel's.

T8. *Let S be a set of formulas of $\mathrm{QC}\overset{5}{=}$, and A be a formula of $\mathrm{QC}\overset{5}{=}$.*

(a) *If S implies$_5$ A, then $S \vdash_5 A$.*

(b) *If A is valid$_5$, then $\vdash_5 A$.*

Proof. (a) Let S imply$_5$ A, and S'' be as in L18. Then in view of L18 S'' implies$_5$ A, hence in view of L17(b) S'' implies$_1$ A, hence in view of T5(a) $S'' \vdash_1 A$, hence in view of L9(a) $S'' \vdash_5 A$, and hence in view of L3(c) $S \vdash_5 A$.

(b) Proof like that of T5(b).

T9. *Let S be a set of formulas of $\mathrm{QC}\overset{6}{=}$, and A be a formula of $\mathrm{QC}\overset{6}{=}$.*

(a) *If S implies$_6$ A, then $S \vdash_6 A$.*

(b) *If A is valid$_6$, then $\vdash_6 A$.*

Proof. (a) Let S imply$_6$ A. Then in view of L17(c) S implies$_5$ A, and hence in view of T8(a) $S \vdash_5 A$.

Case 1: A is open. Then $S \vdash_6 A$ by the same reasoning as in the proof of T7(a), Case 1, but with L5(c) doing duty for L5(b).

Case 2: A is closed. Let S'' be as in L18, and let S''' consist of all the closed members of S''. Since S implies$_6$ A, then in view of L18 S'' implies$_6$ A, and hence in view of L16 it is not the case that there is a non-empty domain D' and a C-exhaustive $\langle\emptyset, D'\rangle$-interpretation $\mathrm{Int}_{\langle\emptyset,D'\rangle}$

of VC^6 such that $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ simultaneously satisfies $S''' \cup \{\sim A\}$. Hence in view of T6 $S''' \cup \{\sim(\exists x)(f(x) \vee \sim f(x)), \sim A\}$ is inconsistent in $\mathrm{QC}\overset{2}{=}$, hence in view of L3(d) $S''' \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_2 A$, hence in view of L9(b) $S''' \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_6 A$, and hence in view of L3(c) $S \cup \{\sim(\exists x)(f(x) \vee \sim f(x))\} \vdash_6 A$. On the other hand, since $S \vdash_5 A$, then in view of L5(c) $S \cup \{(\exists x)(f(x) \vee \sim(fx))\} \vdash_6 A$. Hence $S \vdash_6 A$ in view of L3(e).

(b) Proof like that of T5(b).

Banking on T5 and T8-T9, we next establish that $\mathrm{Q3}\overset{0}{=}$, $\mathrm{QC}\overset{3}{=}$, and $\mathrm{QC}\overset{4}{=}$ are complete in Henkin's sense and in Gödel's.

T10. *Let S be a set of formulas of $\mathrm{QC}\overset{i}{=}$, A be a formula of $\mathrm{QC}\overset{i}{=}$, and $i = 0$, 3, or 4.*

(a) *If S implies$_i$ A, then $S \vdash_i A$.*

(b) *If A is valid$_i$, then $\vdash_i A$.*

Proof. (a) Let S' consist of every closed formula of $\mathrm{QC}\overset{i}{=}$ of the kind $\mathrm{E}!X$.

Case 1: $i = 0$. If S implies$_i$ A, then in view of L9(a) $S \cup S'$ implies$_1$ A, hence in view of T5(a) $S \cup S' \vdash_1 A$, and hence in view of L10(a) $S \vdash_i A$.

Case 2: $i = 3$ or 4. Proof like that of Case 1, but with L19(b)-(c) doing duty for L19(a), T8(a)-T9(a) for T5(a), and L10(b)-(c) for L10(a).

(b) Proof like that of T5(b).

Finally, banking on T5 and T7, we establish that $\mathrm{QC}\overset{7}{=}$, $\mathrm{QC}\overset{8}{=}$, $\mathrm{QC}\overset{9}{=}$, and $\mathrm{QC}\overset{10}{=}$ are complete in Henkin's sense and in Gödel's.

T11. *Let S and A be as in* T10, *and* $7 \leqslant i \leqslant 10$.

(a) *If S implies$_i$ A, then $S \vdash_i A$.*

(b) *If A is valid$_i$, then $\vdash_i A$.*

Proof. (a) Case 1: $i = 9$. Subcase 1.1: Every individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A occurs in one or more members of S. If S implies$_i$ A, then in view of L17(c) S implies$_1$ A, hence in view of T5(a) $S \vdash_1 A$, and hence in view of L12(a) $S \vdash_i A$.

Subcase 1.2: At least one individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A does not occur in any member of S. If S implies$_i$ A, then in view of L20 S implies$_i$ 'p & $\sim p$', hence in view of Subcase 1.1 $S \vdash_i p \mathbin{\&} \sim p$, and hence in view of L11(b) $S \vdash_i A$.

Case 2: $i = 10$. Proof like that of Case 1, but with L17(d) doing duty for L17(c), T7(b) for T5(b), and L12(b) for L12(a).

Case 3: $i = 7$. Subcase 3.1: Every individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A occurs in one or more members of S. Let S' be as in L21. If S implies$_i$ A, then in view of L21(a) $S \cup S'$ implies$_9$ A, hence in view of Case 1 $S \cup S' \vdash_9 A$, and hence in view of L13(a) $S \vdash_7 A$.

Subcase 3.2: At least one individual constant of $\mathrm{QC}\overset{i}{=}$ that occurs in A does not occur in any member of S. If S implies$_i$ A, then in view of L20 S implies$_i$ 'p & $\sim p$', hence in view of Subcase 3.1 $S \vdash_i p \mathbin{\&} \sim p$, and hence in view of L11(b) $S \vdash_i A$.

Case 4: $i = 8$. Proof like that of Case 3, but with L21(b) doing duty for L21(a), and L13(b) for L13(a).

(b) Proof like that of T5(b).

The converse of each one of our completeness theorems also holds true, as the reader may verify:

T12. *Let S and A be as in* T10, *and* $0 \leqslant i \leqslant 10$.

(a) *If $S \vdash_i A$, then S implies$_i$ A.*

(b) *If $\vdash_i A$, then A is valid$_i$.*

In view of T12(a) $\mathrm{QC}\overset{0}{=}$-$\mathrm{QC}\overset{10}{=}$ may be said to be sound in Henkin's sense; and in view of T12(b) to be sound in Gödel's sense. (22)

5. Closing remarks. With T5 and T7-T12 at hand, the claims made in the introduction for $\mathrm{QC}\overset{1}{=}$-$\mathrm{QC}\overset{10}{=}$ are easily defended. (i) The inner domains that figure in clauses (c), (e), (g), (i), and (k) of D20 need not have members. (ii) The $\langle D, D'\rangle$-interpretations that figure in the last eight clauses of the definition need not be C-exhaustive. (iii) And those that figure in clauses (b), (c), (f), (g), (j), and (k), when C-exhaustive, may assign a member of D' rather than D to any individual constant of $\mathrm{QC}{=}$. Hence, in view of T7-T12 and (i), $\mathrm{QC}\overset{2}{=}$, $\mathrm{QC}\overset{4}{=}$, $\mathrm{QC}\overset{6}{=}$, $\mathrm{QC}\overset{8}{=}$, and $\mathrm{QC}\overset{10}{=}$ *do* lift the restriction usually placed on the individual variables of $\mathrm{QC}{=}$, namely, that they have values. Hence, in view of T5, T7-T12, and (ii)-(iii), $\mathrm{QC}\overset{1}{=}$, $\mathrm{QC}\overset{2}{=}$, $\mathrm{QC}\overset{5}{=}$, $\mathrm{QC}\overset{6}{=}$, $\mathrm{QC}\overset{9}{=}$, and $\mathrm{QC}\overset{10}{=}$ allow the individual constants of $\mathrm{QC}{=}$ to designate something not a value of a variable; $\mathrm{QC}\overset{2}{=}$-$\mathrm{QC}\overset{10}{=}$ allow them not to designate at all; and, hence, all ten of $\mathrm{QC}\overset{1}{=}$-$\mathrm{QC}\overset{10}{=}$ *do* lift the restriction always placed on the individual constants of $\mathrm{QC}{=}$, namely, that they each designate a value of a variable. (23)

(22) In [18] van Fraassen mentions the possibility of denying a truth-value to any formula of $\mathrm{QC}{=}$ that contains a non-designating constant, but, given a pair of formulas A and B of $\mathrm{QC}{=}$ at least one of which contains a non-designating constant, letting A imply B if—*in case every individual constant that occurs in A or B designated something*—A would imply B. In view of T5, T7, and T12, two modifications $\mathrm{QC}\overset{11}{=}$ and $\mathrm{QC}\overset{12}{=}$ of $\mathrm{QC}{=}$ are readily designed that comply with van Fraassen's suggestion: (1) a formula A of $\mathrm{QC}\overset{11}{=}$ being understood to be derivable in $\mathrm{QC}\overset{11}{=}$ from a set S of formulas of $\mathrm{QC}\overset{11}{=}$ if $S \vdash_1 A$ and, in case where S is $\emptyset$, A is constant-free, and (2) a formula A of $\mathrm{QC}\overset{12}{=}$ being understood to be derivable in $\mathrm{QC}\overset{12}{=}$ from a set S of formulas of $\mathrm{QC}\overset{12}{=}$ if $S \vdash_2 A$ and, in case S is $\emptyset$, A is constant-free.

(23) Our thanks go to Professors Hintikka and Hiż, who commented at the 1966 meeting of the American Philosophical Association on an early draft of this paper (see [11]), and to Professor van Fraassen whose results in [16] and [17] considerably influenced our thinking on non-designating constants.

Appendix I. We abide in the main text by Hailperin's suggestion in [2] that a formula A of $\mathrm{QC}^{i}_{=}$ of the kind $(\forall X)B$ be held satisfied by any $\langle\emptyset, D'\rangle$-interpretation $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ of the variables and constants of $\mathrm{QC}^{i}_{=}$. Another course is open, and was adopted by Mostowski in [13]: letting A be satisfied by $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ so long as X occurs free in B, otherwise requiring that B be satisfied by $\mathrm{Int}_{\langle\emptyset,D'\rangle}$ if A is to be satisfied by $\mathrm{Int}_{\langle\emptyset,D'\rangle}$. The changes that must be brought to Sections 2-3 when Mostowski's policy towards $(\forall X)B$ is enforced, are as follows:

(1) Amend (d) in D6, Case 2, to read: If A is of the kind $(\forall X)B$, where X occurs free in B, then '$p \supset p$' counts as the $\emptyset$-associate of A; if A is of the kind $(\forall X)B$, where X does not occur free in B, and B' is the $\emptyset$-associate of B, then B' counts as the $\emptyset$-associate of A.

(2) Amend (d) in D7 to read: A formula of $\mathrm{QC}^{i}_{=}$ counts as an axiom of $\mathrm{QC}^{2}_{=}$, $\mathrm{QC}^{6}_{=}$, and $\mathrm{QC}^{10}_{=}$ if it is of one of the eight kinds listed under (a), but with Y understood to be an individual variable of $\mathrm{QC}^{i}_{=}$ when the formula is of the kind $(\forall X)A \supset A(Y/X)$, and X understood to occur free in B when the formula is of the kind $(\forall X)(A \supset B) \supset \big((\forall X)A \supset (\forall X)B\big)$.

(3) Amend (e) in D7 to read: A formula of $\mathrm{QC}^{i}_{=}$ counts as an axiom of $\mathrm{QC}_{=}$ and $\mathrm{QC}^{8}_{=}$ if it is of one of the eight kinds listed under (a), but with X understood to occur free in B when the formula is of the kind $(\forall X)(A \supset B) \supset \big((\forall X)A \supset (\forall X)\big)B$.

(4) In L3(p) require X to occur free in B when $i = 2$, 4, or 6.

(5) Retrieve proof L4(a) from [13] (rather than [15], which follows Hailperin).

(6) Amend (a) in L5 to read: Let X be an individual variable of $\mathrm{QC}^{i}_{=}$ that does not occur in B, and $i = 2$ or 6. Then $\vdash_i (\exists x)\big(f(x) \vee \sim f(x)\big) \supset \big((\forall X)(A \supset B)\big) \supset \big((\forall X)A \supset (\forall X)B\big)$. Proof of the lemma is as follows.

Part One. The following column of formulas of $\mathrm{QC}^{i}_{=}$,

(1) $(\forall X)(A \supset B)$

(2) $1 \supset (A \supset B)$ (Axiom)

(3) $A \supset B$ (R2 or R4, 1, 2)

(4) $3 \supset \Big(\big(f(X) \vee \sim f(X)\big) \supset (A \supset B)\Big)$ (Axiom)

(5) $\big(f(X) \vee \sim f(X)\big) \supset (A \supset B)$ (R2 or R4, 3, 4)

(6) $5 \supset \bigg(\Big(\big(f(X) \vee \sim f(X)\big) \supset A\Big) \supset \Big(\big(f(X) \vee \sim f(X)\big) \supset B\Big)\bigg)$ (Axiom)

(7) $\Big(\big(f(X) \vee \sim f(X)\big) \supset A\Big) \supset \Big(\big(f(X) \vee \sim f(X)\big) \supset B\Big)$ (R2 or R4, 5, 6)

(8) A

(9) $8 \supset \Big(\big(f(X) \vee \sim f(X)\big) \supset A\Big)$ (Axiom)

(10) $\big(f(X) \vee \sim f(X)\big) \supset A$ (R2 or R4, 8, 9)

(11) $\big(f(X) \vee \sim f(X)\big) \supset B$ (R2 or R4, 7, 10)

(12) $11 \supset \Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)$ (Axiom)

(13) $\sim B \supset \sim \big(f(X) \vee f(X)\big)$ (R2 or R4, 11, 12)

counts as a derivation of $\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)$ from $(\forall X)(A \supset B)$, A in $\mathrm{QC}^{i}_{=}$. Hence in view of L3(f)

$$\{(\forall X)(A \supset B)\} \vdash_i A \supset \Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big),$$

hence in view of L3(m)

$$\{(\forall X)(A \supset B)\} \vdash_i (\forall X)\bigg(A \supset \Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\bigg),$$

hence in view of L3(p)

$$\{(\forall X)(A \supset B)\} \vdash_i (\forall X)A \supset (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big),$$

and hence in view of L3(c)

$$\big\{(\exists x)\big(f(x) \vee \sim f(x)\big), (\forall X)(A \supset B)\big\} \vdash_i$$
$$\vdash_i (\forall X)A \supset (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big).$$

Part Two. In view of L3(b) and L3(p)

$$\Big\{(\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i (\forall X) \sim B \supset (\forall X) \sim \big(f(X) \vee \sim f(X)\big),$$

hence in view of L3(k)

$$\Big\{(\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i (\exists X)\big(f(X) \vee \sim f(X)\big) \supset (\exists X)B,$$

and hence in view of L3(c)

$$\Big\{(\exists x)\big(f(x) \vee \sim f(x)\big), (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i$$
$$\vdash_i (\exists X)\big(f(X) \vee \sim f(X)\big) \supset (\exists X)B.$$

But in view of L3(b) and L3(q)

$$\Big\{(\exists x)\big(f(x) \vee \sim f(x)\big), (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i$$
$$\vdash_i (\exists X)\big(f(X) \vee \sim f(X)\big).$$

Hence in view of L3(g)

$$\Big\{(\exists x)\big(f(x) \vee \sim f(x)\big), (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i (\exists X)B,$$

hence in view of L3(n)

$$\Big\{(\exists x)\big(f(x) \vee \sim f(x)\big), (\forall X)\Big(\sim B \supset \sim \big(f(X) \vee \sim f(X)\big)\Big)\Big\} \vdash_i B,$$

hence in view of L3(m)

$$\{(\exists x)(f(x) \vee \sim f(x)),\ (\forall X)\big(\sim B \supset \sim (f(X) \vee \sim f(X))\big)\} \vdash_i (\forall X)B\,,$$

hence in view of L3(f)

$$\{(\exists x)(f(x) \vee \sim f(x))\} \vdash_i (\forall X)\big(\sim B \supset \sim (f(X) \vee \sim f(X))\big) \supset (\forall X)B\,,$$

hence in view of L3(h)

$$\{(\exists x)(f(x) \vee \sim f(x))\} \vdash_i (\forall X)A \supset (\forall X)\big(\sim B \supset \sim (f(X) \vee \sim f(X))\big) \supset (\forall X)B\,,$$

hence in view of L3(i)

$$\{(\exists x)(f(x) \vee \sim f(x))\} \vdash_i$$
$$\vdash_i \Big((\forall X)A \supset (\forall X)\big(\sim B \supset \sim (f(X) \vee \sim f(X))\big)\Big) \supset \big((\forall X)A \supset (\forall X)B\big)\,,$$

and hence in view of L3(c)

$$\{(\exists x)(f(x) \vee \sim f(x)),\ (\forall X)(A \supset B)\} \vdash_i$$
$$\vdash_i \Big((\forall X)A \supset (\forall X)\big(\sim B \supset \sim (f(X) \vee \sim f(X))\big)\Big) \supset \big((\forall X)A \supset (\forall X)B\big)\,.$$

Part Three. In view of Parts One-Two and L3(g)

$$\{(\exists x)(f(x) \vee \sim f(x)),\ (\forall X)(A \supset B)\} \vdash_i (\forall X)A \supset (\forall X)B\,.$$

Hence in view of L3(f)

$$\vdash_i (\exists x)(f(x) \vee \sim f(x)) \supset \big((\forall X)(A \supset B) \supset \big((\forall X)A \supset (\forall X)B\big)\big)\,.$$

(7) In L6(e) require X to occur free in both B and C or in neither (as well as not to occur free in A). The restriction does not affect the proof of L6(f) on p. 140.

(8) Amend (f) in D16, Case 2.2.2, to read: If A is of the kind $(\forall X)B$, where X occurs free in B, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy$_T$ A; if A is of the kind $(\forall X)B$, where X does not occur free in B, and $\mathrm{Int}_{\langle D,D'\rangle}$ satisfies$_T$ B, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy$_T$ A.

Since all of theorems T1-T12 hold true as before, and by the same proofs as before, we end up with five extra variants of $\mathrm{QC}{=}$ that lift one or both of the restrictions normally placed on the individual terms of $\mathrm{QC}{=}$.

Appendix II. The alternative accounts of $\mathrm{QC}\overset{1}{=}$ and $\mathrm{QC}\overset{2}{=}$ that we mentioned in Section 1 call for the following definitions:

D21. Let D and D' be disjoint domains, $i = 1$ or 2, and $\mathrm{Int}_{\langle D,D'\rangle}$ be any result of assigning:

(i) exactly one of the two truth-values T and F to each sentence variable P of $\mathrm{QC}\overset{i}{=}$, the truth-value in question to be known as the value of P under $\mathrm{Int}_{\langle D,D'\rangle}$;

(ii) if D is not empty, exactly one member of D to each individual variable X of $\mathrm{QC}\overset{i}{=}$, the member in question to be known as the value of X under $\mathrm{Int}_{\langle D,D'\rangle}$;

(iii) if $D \cup D'$ is not empty, *at most one* member of $D \cup D'$ to each individual constant X of $\mathrm{QC}\overset{i}{=}$, the member in question to be known as the value of X under $\mathrm{Int}_{\langle D,D'\rangle}$;

(iv) exactly one subset of $(D \cap D')^m$ to each m-adic $(m = 1, 2, \ldots)$ predicate variable F of $\mathrm{QC}\overset{i}{=}$, the subset in question to be known as the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$; and

(v) if one or more individual constants of $\mathrm{QC}\overset{i}{=}$ have no value under $\mathrm{Int}_{\langle D,D'\rangle}$, exactly one of the two truth-values T and F to each atomic formula A of $\mathrm{QC}\overset{i}{=}$ that contains such a constant, the truth-value in question to be known as the value of A under $\mathrm{Int}_{\langle D,D'\rangle}$.

If (a) for every individual constant X of $\mathrm{QC}\overset{i}{=}$ that has no value under $\mathrm{Int}_{\langle D,D'\rangle}$ the atomic formula $X = X$ of $\mathrm{QC}\overset{i}{=}$ has the value T under $\mathrm{Int}_{\langle D,D'\rangle}$ and (b) for every two individual constants X and Y of $\mathrm{QC}\overset{i}{=}$ at least one of which has no value under $\mathrm{Int}_{\langle D,D'\rangle}$, every atomic formula A of $\mathrm{QC}\overset{i}{=}$ that contains X, and every (atomic) formula A' of $\mathrm{QC}\overset{i}{=}$ of the kind $A(Y//X)$, A and A' have the same value under $\mathrm{Int}_{\langle D,D'\rangle}$ if the atomic formula $X = Y$ of $\mathrm{QC}\overset{i}{=}$ has the value T under $\mathrm{Int}_{\langle D,D'\rangle}$, then $\mathrm{Int}_{\langle D,D'\rangle}$ counts as a $\langle D, D'\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{i}{=}$.

D22-D23. Like D14-D15, but with $\mathrm{Int}_{\langle D,D'\rangle}$ in the first case, each one of $\mathrm{Int}_{\langle D,D'\rangle}$ and $\mathrm{Int}'_{\langle D,D'\rangle}$ in the other, understood to be a $\langle D, D'\rangle$-interpretation of the variables, constants and atomic formulas of $\mathrm{QC}\overset{i}{=}$.

D24. Let A be a formula of $\mathrm{QC}\overset{i}{=}$; D and D' be disjoint domains; $\mathrm{Int}_{\langle D,D'\rangle}$ be a $\langle D, D'\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{i}{=}$; and $i = 1$ or 2.

Case 1: D is not empty.

(a) If A is a sentence variable of $\mathrm{QC}\overset{i}{=}$ and has the value T under $\mathrm{Int}_{\langle D,D'\rangle}$, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(b1) If A is of the kind $F(X_1, X_2, \ldots, X_m)$, each one of $X_1, X_2, \ldots,$ and X_m has a value under $\mathrm{Int}_{\langle D,D'\rangle}$, and the m-tuple made up of the values of $X_1, X_2, \ldots,$ and X_m (in that order) under $\mathrm{Int}_{\langle D,D'\rangle}$ belongs to the value of F under $\mathrm{Int}_{\langle D,D'\rangle}$, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(b2) If A is of the kind $F(X_1, X_2, \ldots, X_m)$, at least one of $X_1, X_2, \ldots,$ and X_m has no value under $\mathrm{Int}_{\langle D,D'\rangle}$, and the value of A under $\mathrm{Int}_{\langle D,D'\rangle}$ is T, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(c1) If A is of the kind $X = Y$, each one of X and Y has a value under $\mathrm{Int}_{\langle D,D'\rangle}$, and the value of X under $\mathrm{Int}_{\langle D,D'\rangle}$ is the same as that of Y, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(c2) If A is of the kind $X = Y$, at least one of X and Y has no value under $\mathrm{Int}_{\langle D,D'\rangle}$, and the value of A under $\mathrm{Int}_{\langle D,D'\rangle}$ is T, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(d) If A is of the kind $\sim B$ and $\mathrm{Int}_{\langle D,D'\rangle}$ does not satisfy B, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(e) If A is of the kind $B \supset C$ and $\mathrm{Int}_{\langle D,D'\rangle}$ does not satisfy B or satisfies C, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(f) If A is of the kind $(\forall X)B$ and every X-variant of $\mathrm{Int}_{\langle D,D'\rangle}$ satisfies B, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(g) $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A pursuant only to one or another of (a)-(f).

Case 2: D is empty.

Case 2.1: A is open. Then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A.

Case 2.2: A is closed.

(a)-(e) Like (a)-(e) under Case 1;

(f) If A is of the kind $(\forall X)B$, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to satisfy A;

(g) Like (g) under Case 1.

D25. Let S be a set of ~~formulas of~~ $\mathrm{QC}\overset{i}{=}$, D, D', and $\mathrm{Int}_{\langle D,D'\rangle}$ be as in D24 and $i = 1$ or 2. If $\mathrm{Int}_{\langle D,D'\rangle}$ satisfies each and every member of S, then $\mathrm{Int}_{\langle D,D'\rangle}$ is said to simultaneously satisfy S.

D26. Let S be a set of formulas of $\mathrm{QC}\overset{i}{=}$, A be a formula of $\mathrm{QC}\overset{i}{=}$, where $i = 1$ or 2. (a) S is said to imply$'_1$ A if, for every non-empty domain D and every domain D' disjoint from D, every $\langle D, D'\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{1}{=}$ that simultaneously satisfies S also satisfies A.

(b) S is said to imply$'_2$ A if, for all disjoint domains D and D', every $\langle D, D'\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{2}{=}$ that simultaneously satisfies S also satisfies A.

D27. Let S, A, and i be as in D26.

(a) S is said to imply$''_1$ A if, for every non-empty domain D, every $\langle D, \emptyset\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{1}{=}$ that simultaneously satisfies S also satisfies A.

(b) S is said to imply$''_2$ A if, for every domain D, every $\langle D, \emptyset\rangle$-interpretation of the variables, constants, and atomic formulas of $\mathrm{QC}\overset{2}{=}$ that simultaneously satisfies S also satisfies A.

It is readily verified (details are left to the reader) that if $S \vdash_1 A$, then S implies$'_1$ A. But, if S implies$'_1$ A, then S implies$_1$ A (in the sense of D20(b)), [24] and hence in view of T5(a) $S \vdash_1 A$. Hence S implies$'_1$ A if and only if $S \vdash_1 A$, and by the same reasoning (but with T7(a) doing

[24] Concerning this point, see the first half of footnote (6).

duty for T5(a)) S implies$'_2$ A if and only if $S \vdash_2 A$. Hence $\mathrm{QC}\overset{1}{=}$ and $\mathrm{QC}\overset{2}{=}$, which are sound and complete under Account Two, are also sound and complete under Account One, which allows their individual constants to designate a value of a variable, or designate something not a value of a variable, or not designate at all.

It is readily verified also that if $S \vdash_1 A$, then S implies$''_1$ A. But, if S implies$''_1$ A, then S implies$_1$ A, [25] and hence $S \vdash_1 A$. Hence S implies$''_1$ A if and only if $S \vdash_1 A$, and by the same reasoning S implies$''_2$ A if and only if $S \vdash_2 A$. Hence $\mathrm{QC}\overset{1}{=}$ and $\mathrm{QC}\overset{2}{=}$, which are sound and complete under either one of Accounts One and Two, are also sound and complete under Account Three, which requires their individual constants to designate a value of a variable or not designate at all.

Three accounts of $\mathrm{QC}\overset{1}{=}$ and $\mathrm{QC}\overset{2}{=}$ are thus available. We personally prefer Account One, but for reasons already stated employ Account Two as our official one in the paper.

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Über die Mächtigkeiten und Unabhängigkeitsgrade der Basen freier Algebren, I*

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C. J. Everett hat 1942 in [4] ein Kriterium angegeben, wann ein endlich erzeugter Vektorraum über einem Ring Basen verschiedener Mächtigkeiten besitzt, und auch gleich ein Beispiel eines solchen Vektorraumes. B. Jónsson and A. Tarski haben 1956 in [6] ein solches Beispiel für eine andere Klasse von Algebren angegeben und eine Bedingung dafür aufgestellt, wann es in einer Klasse $\mathfrak{K}$ von Algebren keine $\mathfrak{K}$-frei erzeugten Algebren mit $\mathfrak{K}$-Basen verschiedener Mächtigkeiten geben kann. E. Marczewski hat dann gezeigt (vgl. [8]), daß für Algebren mit endlichstelligen Operationen, die Basen verschiedener Mächtigkeiten besitzen, diese Basismächtigkeiten eine arithmetische Folge bilden. Gleichzeitig warf er die Frage auf, welche arithmetischen Folgen dabei auftreten können. Diese wurde von A. Goetz und C. Ryll-Nardzewski in [5] teilweise und von S. Świerczkowski in [21] vollständig beantwortet für den Fall, daß man es mit Algebren mit endlichstelligen Operationen zu tun hat; dann ist nämlich jede arithmetische Folge „realisierbar", die nicht die Null enthält.

Zu einer primitiven Klasse $\mathfrak{A}$ von (partiellen) Algebren, die eine mindestens zweielementige Algebra enthält (d.h. die „nichttrivial" ist), bezeichne $\boldsymbol{F}(M, \mathfrak{A})$ die—durch die Mächtigkeit $|M|$ der Menge M bis auf Isomorphie eindeutig bestimmte—von einer Menge M $\mathfrak{A}$-frei erzeugte $\mathfrak{A}$-Algebra. In der Klasse $\boldsymbol{K}$ aller Kardinalzahlen definieren wir dann zu $\mathfrak{A}$ eine Äquivalenzrelation $R_{\mathfrak{A}}$ vermöge

$$(1)\qquad (\mathfrak{m}, \mathfrak{n}) \in R_{\mathfrak{A}} \text{ genau dann, wenn } \boldsymbol{F}(\mathfrak{m}, \mathfrak{A}) \cong \boldsymbol{F}(\mathfrak{n}, \mathfrak{A}) \qquad (\text{für alle } \mathfrak{m}, \mathfrak{n} \in \boldsymbol{K}).$$

* Die Resultate dieses ersten Teiles sind im wesentlichen schon in der Diplomarbeit des Verfassers enthalten, die im April 1965 an der Freien Universität Berlin eingereicht wurde. Unabhängig hiervon hat inzwischen G. Grätzer fast die gleichen Ergebnisse (d.h. eine etwas schwächere Form von Korollar 1 zum Hauptsatz der vorliegenden Arbeit bzw. zum Satz 4.14 der Diplomarbeit) ohne Beweis im Juni 1966 bei den Notices of the American Mathematical Society zur Ankündigung eingereicht (vgl. Notices Amer. Math. Soc. 13 (1966), Seite 632f). Vgl. Dissertation Bonn (D5) 1966, Teil I.