Disjoint systems over set ideals
(On a generalization of the usual conception of almost disjoint set systems)

by

Géza **Fodor** and Attila **Máté** (Szeged)

Introduction. Let E be an infinite set. We shall denote by $\boldsymbol{P}(E)$ the set of all subsets of E. Let m be an infinite cardinal number; then a non-empty system $\boldsymbol{I}$ of the elements of $\boldsymbol{P}(E)$ is called an *m-additive ideal in E* if the sum of any system of elements of power smaller than m of $\boldsymbol{I}$ is again a set belonging to $\boldsymbol{I}$, and if $X \in \boldsymbol{I}$ and $Y \subseteq X$ imply $Y \in \boldsymbol{I}$. *Throughout this paper we shall assume that m is a regular cardinal number, the power of the set E is $\geqslant m$ and if $\boldsymbol{I}$ is an m-additive ideal, then the empty set belongs to $\boldsymbol{I}$.* We shall denote by $H^{<m}$ the m-additive ideal of the subsets of power $< m$ of the set H. Let X and Y be two elements of $\boldsymbol{P}(E)$. We shall say that X and Y are *disjoint over $\boldsymbol{I}$* if $X \cap Y \in \boldsymbol{I}$. Let $\boldsymbol{H}$ be a subset of $\boldsymbol{P}(E)$. If any two distinct elements of $\boldsymbol{H}$ are disjoint over $\boldsymbol{I}$, then we say that $\boldsymbol{H}$ is a *disjoint system over $\boldsymbol{I}$*; if, moreover, for every $X \in \boldsymbol{H}$ the relation $X \notin \boldsymbol{I}$ holds, then we say that $\boldsymbol{H}$ is a *strongly disjoint system over $\boldsymbol{I}$*. If $\boldsymbol{I} = \{0\}$, then we obtain the well-known definition of disjoint sets and if $\boldsymbol{I} = E^{<m}$, then we obtain the well-known definition of the almost disjoint sets. Let $\boldsymbol{G}$ be a subset of $\boldsymbol{P}(E)$ and let $\boldsymbol{I} \subseteq \boldsymbol{P}(E)$ be the smallest m-additive ideal which contains $\boldsymbol{G}$. We shall call $\boldsymbol{I}$ the *m-additive ideal induced by $\boldsymbol{G}$*. If $\boldsymbol{G}$ consists of (strongly) disjoint sets over $\boldsymbol{J}$, then we say that $\boldsymbol{I}$ is *induced by (strongly) disjoint sets over $\boldsymbol{J}$*. Let $\boldsymbol{I}$ be a given m-additive ideal; then the m-additive ideal $\boldsymbol{J}$ is said to be *$T_0(\boldsymbol{I})$ ideal* if for every subset $F \notin \boldsymbol{I}$ of E the relation $\boldsymbol{J} \cap \boldsymbol{P}(F) \nsubseteq \boldsymbol{I}$ holds, *$T_0'(\boldsymbol{I})$ ideal* if for every subset $F \notin \boldsymbol{J}$ of E the relation $\boldsymbol{J} \cap \boldsymbol{P}(F) \nsubseteq \boldsymbol{I}$ holds, and *$T_1(\boldsymbol{I})$ ideal* if for every subset $F \notin \boldsymbol{J}$ of E, $\boldsymbol{J} \cap \boldsymbol{P}(F)$ cannot be induced by disjoint sets over $\boldsymbol{I}$, respectively. It is obvious that a $T_1(\boldsymbol{I})$ ideal is always $T_0'(\boldsymbol{I})$ ideal, further a $T_0'(\boldsymbol{I})$ ideal is always $T_0(\boldsymbol{I})$ ideal and the $T_0(\boldsymbol{I})$ ideal $\boldsymbol{J}$ is $T_0'(\boldsymbol{I})$ ideal if and only if $\boldsymbol{I} \subseteq \boldsymbol{J}$. A proper m-additive ideal $\boldsymbol{J}$ (i.e. for which $\boldsymbol{J} \neq \boldsymbol{P}(E)$) is called *$T_2(\boldsymbol{I})$ ideal* if every proper m-additive ideal $\boldsymbol{K}$ with $\boldsymbol{K} \supseteq \boldsymbol{J}$ is a $T_1(\boldsymbol{I})$ ideal. Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be two m-additive ideals and $\boldsymbol{S}$ a (strongly) disjoint system over $\boldsymbol{I}$. We say that $\boldsymbol{S}$ *is a complete (strongly) disjoint system in $\boldsymbol{J}$ over $\boldsymbol{I}$* if there is no element X of $\boldsymbol{J}$ such that $X \notin \boldsymbol{I}$ and

$S \cup \{X\}$ is (strongly) disjoint over $\boldsymbol{I}$. It is easy to see that each *complete strongly disjoint* system $\boldsymbol{S}$ in $\boldsymbol{J}$ over $\boldsymbol{I}$ is also a *complete disjoint* system in $\boldsymbol{J}$ over $\boldsymbol{I}$. If $\boldsymbol{J} = \boldsymbol{P}(E)$, then we say that $\boldsymbol{S}$ *is a complete (strongly) disjoint system over I.*

A *set mapping* $S(x)$ on E is defined as a function from E into $P(E)$. We assume that for every $x \in E$ the relation $x \notin S(x)$ holds. Two distinct elements of E, x and y, are called *independent* if $x \notin S(y)$ and $y \notin S(x)$. A subset F of E is called *free* if any two of the elements of it are independent. For any element x of E, let $S^{-1}(x) = \{y \in E\colon x \in S(y)\}$ and $S[x] = S(x) \cup S^{-1}(x)$.

If $\boldsymbol{I}$ is an m-additive ideal, $\boldsymbol{S} \subseteq \boldsymbol{P}(E)$ and $H \subseteq E$, then we shall denote by the symbol $[\boldsymbol{S}, \boldsymbol{I}, H]$ the set $\{X \in \boldsymbol{S}\colon H \cap X \notin \boldsymbol{I}\}$. If $\boldsymbol{G}$ is a set of sets, then we denote by $\langle \boldsymbol{G} \rangle$ the set $\bigcup_{G \in \boldsymbol{G}} G$. The symbol $\overline{\overline{S}}$ denotes the power of the set S, $\omega(m)$ the initial number of the cardinality m and m^+ the cardinality following m immediately.

In this paper we are going to prove some results concerning the powers of disjoint systems over a proper m-additive ideal and some results concerning $T_0(\boldsymbol{I})$, $T_0'(\boldsymbol{I})$, $T_1(I)$ and $T_2(\boldsymbol{I})$ ideals.

1. Basic results. THEOREM 1. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a disjoint system over $\boldsymbol{I}$, and $\boldsymbol{J}$ the m-additive ideal induced by $\boldsymbol{S}$. If for a subset H of E the power of $[\boldsymbol{S}, \boldsymbol{I}, H]$ is $\geqslant m$, then $H \notin \boldsymbol{J}$.*

Proof. Suppose, on the contrary, that $H \in \boldsymbol{J}$. Then there exists a subset $\boldsymbol{S}'$ of $\boldsymbol{S}$ such that the power of $\boldsymbol{S}$ is $< m$ and $\boldsymbol{H} \subseteq \langle \boldsymbol{S}' \rangle$. Since the power of $[\boldsymbol{S}, \boldsymbol{I}, H]$ is $\geqslant m$, we obtain that

$$[\boldsymbol{S}, \boldsymbol{I}, H] - \boldsymbol{S}' \neq 0 .$$

Let F be an arbitrary element of this set. Since $\boldsymbol{S}' \subseteq \boldsymbol{S}$, further the power of $\boldsymbol{S}'$ is $< m$ and the elements of $\boldsymbol{S}$ are disjoint over $\boldsymbol{I}$, we obtain that $F \cap \langle \boldsymbol{S}' \rangle \in \boldsymbol{I}$. On the other hand, the definition of $[\boldsymbol{S}, \boldsymbol{I}, H]$ implies that $F \cap H \notin \boldsymbol{I}$. Consequently $F \cap H - \langle \boldsymbol{S}' \rangle \notin \boldsymbol{I}$. Therefore $H - \langle \boldsymbol{S}' \rangle \notin \boldsymbol{I}$. But this contradicts the fact $H \subseteq \langle \boldsymbol{S}' \rangle$. The theorem is proved.

COROLLARY 2. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a strongly disjoint system over $\boldsymbol{I}$, of power $\geqslant m$. Then the m-additive ideal induced by $\boldsymbol{S}$ is proper.*

The proof is obvious.

THEOREM 3. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a complete disjoint system over $\boldsymbol{I}$, and $\boldsymbol{J}$ the m-additive ideal induced by $\boldsymbol{S}$. If $\boldsymbol{I} \subseteq \boldsymbol{J}$ and $H \notin \boldsymbol{J}$, then the power of $[\boldsymbol{S}, \boldsymbol{I}, H]$ is $\geqslant m$.*

Proof. Suppose, on the contrary, that $[\boldsymbol{S}, \boldsymbol{I}, H]$ has power $< m$. Then $\langle [\boldsymbol{S}, \boldsymbol{I}, H] \rangle \in \boldsymbol{J}$, because $[\boldsymbol{S}, \boldsymbol{I}, H] \subseteq \boldsymbol{S}$. Consequently the set $G = H - \langle [\boldsymbol{S}, \boldsymbol{I}, H] \rangle$ does not belong to $\boldsymbol{J}$. Since $\boldsymbol{I} \subseteq \boldsymbol{J}$, we have that $G \notin \boldsymbol{I}$. On the other hand, for every $X \in \boldsymbol{S}$ the relation $G \cap X \in \boldsymbol{I}$ holds. Thus $\boldsymbol{S} \cup \{G\}$ is a disjoint (or strongly disjoint) system over $\boldsymbol{I}$. This contradicts the assumption that $\boldsymbol{S}$ is complete disjoint system over $\boldsymbol{I}$.

2. The cardinality of disjoint systems. THEOREM 4. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a disjoint system over $\boldsymbol{I}$ such that the power of $[\boldsymbol{S}, \boldsymbol{I}, E]$ is $\geqslant m$. Then there exists a sequence $\{H_\xi\}_{\xi<\omega(m)}$ of type $\omega(m)$ of mutually disjoint subsets of E such that for every $\xi < \omega(m)$ the set $[\boldsymbol{S}, \boldsymbol{I}, H_\xi]$ has power $\geqslant m$.*

Proof. Let A be the set of all the ordered pairs (α, β), where $\alpha < \beta < \omega(m)$, i.e. $A = \{(\alpha, \beta)\colon \alpha < \beta < \omega(m)\}$. By the familiar way we define a well ordering in A as follows. If (α_1, β_1) and (α_2, β_2) are two elements of A, then $(\alpha_1, \beta_1) \precsim (\alpha_2, \beta_2)$ means either $\beta_1 < \beta_2$ or else $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$. The relation $\precsim$ well orders A. Since for every $(\alpha, \beta) \in A$ the inequality $\alpha < \beta$ holds, the set A has the ordinal type $\omega(m)$.

Let $\boldsymbol{S}'$ be a subset of power m of $[\boldsymbol{S}, \boldsymbol{I}, E]$ and let $(\alpha, \beta) \to S^\alpha_\beta$ be a one-to-one mapping of A onto $\boldsymbol{S}'$. Then $\boldsymbol{S}' = \{S^\alpha_\beta\}_{\alpha<\beta<\omega(m)}$. Put

$$G^\xi_\eta = S^\xi_\eta - \bigcup_{(\alpha,\beta)\precsim(\xi,\eta)} S^\alpha_\beta .$$

Since the set $\{(\alpha, \beta) \in A\colon (\alpha, \beta) \precsim (\xi, \eta)\}$ has power $< m$ and the sets S^α_β $\big((\alpha, \beta) \in A\big)$ are strongly disjoint over $\boldsymbol{I}$, we obtain that $G^\xi_\eta \notin \boldsymbol{I}$. Put

$$H_\xi = \bigcup_{\xi<\eta<\omega(m)} G^\xi_\eta .$$

It is easy to see that the sets H_ξ $\big(\xi < \omega(m)\big)$ are mutually disjoint and the power of $[\boldsymbol{S}, \boldsymbol{I}, H_\xi]$ is $\geqslant m$ for every $\xi < \omega(m)$.

THEOREM 5. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a disjoint system over $\boldsymbol{I}$, $\boldsymbol{J}$ the m-additive ideal induced by $\boldsymbol{S}$, and $\boldsymbol{G}$ a disjoint system of power m over $\boldsymbol{J}$. Suppose that for every $Z \in \boldsymbol{G}$ the set $[\boldsymbol{S}, \boldsymbol{I}, Z]$ has power $\geqslant m$. Then there exists a subset $F \notin \boldsymbol{G}$ of E such that the power of $[\boldsymbol{S}, \boldsymbol{I}, F]$ is $\geqslant m$ and $\boldsymbol{G} \cup \{F\}$ is a disjoint system over $\boldsymbol{J}$.*

Proof. Let $\{G_\xi\}_{\xi<\omega(m)}$ be a well ordering of type $\omega(m)$ of $\boldsymbol{G}$ and $F_\xi = G_\xi - \bigcup_{\alpha<\xi} G_\alpha$. Obviously the sets F_ξ $\big(\xi < \omega(m)\big)$ are mutually disjoint. On the other hand, for every $\xi < \omega(m)$

$$[\boldsymbol{S}, \boldsymbol{I}, F_\xi] = [\boldsymbol{S}, \boldsymbol{I}, G_\xi] .$$

Thus the power of $[\boldsymbol{S}, \boldsymbol{I}, F_\xi]$ is $\geqslant m$, where $\xi < \omega(m)$. Let X_0 be an arbitrary element of $[\boldsymbol{S}, \boldsymbol{I}, F_0]$. Let $\xi > 0$ and suppose that the sets X_α, where $0 \leqslant \alpha < \xi$, have been already defined. Then let X_ξ be an arbitrary element of $[\boldsymbol{S}, \boldsymbol{I}, F_\xi]$ for which $X_\xi \notin \{X_\alpha\}_{\alpha<\xi}$. Such an X_ξ clearly exists, because the power of $[\boldsymbol{S}, \boldsymbol{I}, F_\xi]$ is $\geqslant m$. Thus we can define the sets X_ξ for every $\xi < \omega(m)$. Put

$$F = \bigcup_{\xi<\omega(m)} (X_\xi \cap F_\xi) .$$

It is easy to see that

$$\{X_\xi\}_{\xi<\omega(m)} \subseteq [\boldsymbol{S}, \boldsymbol{I}, F];$$

consequently the power of $[\boldsymbol{S}, \boldsymbol{I}, F]$ is $\geqslant m$. Therefore, by Theorem 1, the relation $F \notin \boldsymbol{J}$ holds. Since $G_\xi \subseteq \bigcup_{\alpha\leqslant\xi} F_\alpha$ and the sets F_ξ $(\xi < \omega(m))$ are mutually disjoint, we obtain:

$$F \cap G_\xi \subseteq \Big(\bigcup_{\lambda<\omega(m)} (X_\lambda \cap F_\lambda)\Big) \cap \Big(\bigcup_{\alpha\leqslant\xi} F_\alpha\Big) = \bigcup_{\lambda<\xi} (X_\lambda \cap F_\lambda) \in \boldsymbol{J},$$

where $\xi < \omega(m)$; so F is disjoint to each elements of $\boldsymbol{G}$ over $\boldsymbol{J}$, and so cannot belong to $\boldsymbol{G}$, either. The theorem is proved.

COROLLARY 6. *If the conditions of Theorem 5 hold except for the power of $\boldsymbol{G}$ but $\boldsymbol{G}$ is complete strongly disjoint over $\boldsymbol{J}$, then the power of $\boldsymbol{G} \neq m$.*

The proof is obvious (see Theorem 5).

COROLLARY 7. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a complete disjoint system over $\boldsymbol{I}$, $\boldsymbol{J}$ the proper m-additive ideal induced by $\boldsymbol{S}$ and suppose that $\boldsymbol{I} \subseteq \boldsymbol{J}$. If $\boldsymbol{G}$ is a complete strongly disjoint system over $\boldsymbol{J}$, then $\overline{\overline{\boldsymbol{G}}} \neq m$.*

Proof. This follows from Theorem 3 and Corollary 6.

THEOREM 8. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a disjoint system over $\boldsymbol{I}$, $\overline{\overline{[\boldsymbol{S}, \boldsymbol{I}, E]}} \geqslant m$ and $\boldsymbol{J}$ the m-additive ideal induced by $\boldsymbol{S}$. Then there exists a strongly disjoint system over $\boldsymbol{J}$ with power $> m$.*

Proof. Let $\mathfrak{G}$ be the set of all strongly disjoint systems $\boldsymbol{G}$ over $\boldsymbol{J}$ for which

(1) $$\overline{\overline{\boldsymbol{G}}} \geqslant m,$$

(2) $$\overline{\overline{[\boldsymbol{S}, \boldsymbol{I}, Z]}} \geqslant m \quad \text{for every } Z \in \boldsymbol{G}.$$

By Theorem 4 the set $\mathfrak{G}$ is not empty. By Zorn's lemma there is a maximal element $\boldsymbol{G}_1$ of $\mathfrak{G}$ with respect to the relation of inclusion. It follows from Theorem 5 that $\overline{\overline{\boldsymbol{G}}}_1 > m$.

Now we prove Theorem 8 in another way using the following for $p = m$:

THEOREM A. *Every infinite set of power p is the sum of more than p almost disjoint sets each of which is of power p* (see [1]).

Another proof of Theorem 8. Let $\boldsymbol{S}'$ be a subset of power m of $[\boldsymbol{S}, \boldsymbol{I}, E]$ and $\{S_\xi\}_{\xi<\omega(m)}$ a well ordering of type $\omega(m)$ of $\boldsymbol{S}'$. Put

$$Q_\xi = S_\xi - \bigcup_{\alpha<\xi} S_\alpha \quad (\xi < \omega(m)).$$

It is clear that the sets Q_ξ are mutually disjoint and $Q_\xi \notin \boldsymbol{I}$ for every $\xi < \omega(m)$. Let

$$\boldsymbol{Q} = \{Q_\xi\}_{\xi<\omega(m)}.$$

By Theorem A, $\boldsymbol{Q}$ is the sum $\bigcup_{\lambda<\tau} \boldsymbol{Q}_\lambda$ of more than m almost disjoint sets $\boldsymbol{Q}_\lambda$ $(\lambda < \tau)$ each of which is of power m. It is easy to see by Theorem 1 that the sets $P_\lambda = \langle \boldsymbol{Q}_\lambda \rangle$, where $\lambda < \tau$, form a strongly disjoint system of power $> m$ over $\boldsymbol{J}$.

3. Existence of T_1 and T_2 ideals. THEOREM 9. *Let $\boldsymbol{I}$ be a proper m-additive ideal, $\boldsymbol{S}$ a disjoint system over $\boldsymbol{I}$, $\overline{\overline{[\boldsymbol{S}, \boldsymbol{I}, E]}} \geqslant m$ and $\boldsymbol{J}$ the m-additive ideal induced by $\boldsymbol{S}$. Then there exists a $T_1(\boldsymbol{J})$ ideal.*

Proof. Let $\boldsymbol{S}'$ be a subset of power m of $[\boldsymbol{S}, \boldsymbol{I}, E]$ and consider $\boldsymbol{S}'$ in the form:

$$\boldsymbol{S}' = \{S_\eta^\xi\}_{\eta<\omega(m)}^{\xi<\omega(m)}.$$

Put $E' = \langle \boldsymbol{S}' \rangle$ and $\boldsymbol{S}_\eta = \{S_\eta^\xi\}_{\xi<\omega(m)}$ for every $\eta < \omega(m)$. Let $\mathfrak{R}$ be the set of the sets $\boldsymbol{R} = \{R_\eta\}_{\eta<\omega(m)} \subseteq \boldsymbol{S}'$ which have only the element R_η common with the set $\boldsymbol{S}_\eta$ for every $\eta < \omega(m)$. Let $\boldsymbol{Q}$ be the set of all the sets Q for which $Q \cap S_\eta^\xi \in \boldsymbol{I}$, where $\xi, \eta < \omega(m)$. Let

$$\mathfrak{P} = \mathfrak{R} \cup \{\boldsymbol{S}_\eta\}_{\eta<\omega(m)}.$$

Further let

$$\boldsymbol{V} = \{\langle \boldsymbol{M} \rangle\colon \boldsymbol{M} \in \mathfrak{P}\} \cup \boldsymbol{Q}.$$

Let $\boldsymbol{K}$ be the m-additive ideal induced by $\boldsymbol{V}$. We shall prove that $\boldsymbol{K}$ is a $T_1(\boldsymbol{J})$ ideal in E' (¹). (Then the m-additive ideal $\boldsymbol{K}_1$ induced by $\boldsymbol{V} \cup \{E - E'\}$, is a $T_1(\boldsymbol{J})$ ideal in E.)

We prove that $\boldsymbol{K}$ is a $T_1(\boldsymbol{J})$ ideal in E'. Let $F \subseteq E'$ and $F \notin \boldsymbol{K}$ and suppose, on the contrary, that $\boldsymbol{K} \cap \boldsymbol{P}(F)$ can be induced by a disjoint system $\boldsymbol{G}$ over $\boldsymbol{J}$.

We need, for the proof, the system $\{F_\xi\}_{\xi<\omega(m)}$ of the sets F_ξ which we shall define later on such that

(1) $$F = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_\xi \supseteq \ldots \quad (\xi < \omega(m))$$

and

(2) $$F_0 - F_\xi \in \boldsymbol{K} \quad \text{for every } \xi < \omega(m).$$

If ξ and $\eta < \omega(m)$, then we put

$$\boldsymbol{A}_\eta^\xi = \{S_\eta^\gamma \cap F_\xi\colon S_\eta^\gamma \cap F_\xi \notin \boldsymbol{I},\ \gamma < \omega(m)\}.$$

It is obvious (particularly in the case $\xi = 0$) that for every cardinal number $n < m$ the set

$$W^0|n = \{\eta < \omega(m)|\colon \overline{\overline{\boldsymbol{A}_\eta^0}} > n\}$$

(¹) More precisely said, if we consider the ideals only in the set E', then $\boldsymbol{K} \cap \boldsymbol{P}(E')$ is a $T_1(\boldsymbol{J} \cap \boldsymbol{P}(E'))$ ideal.

has power m, because $F_0 = F \notin \boldsymbol{K}$. It is also obvious that the set $W = \{\eta\colon \eta < \omega(m)\}$ has a partition $W = \bigcup_{\lambda<\omega(m)} W_\lambda$ such that for every cardinal number $n < m$ the set

$$W_\lambda^0 \,|n = \{\eta \in W_\lambda\colon\ \overline{\overline{A_\eta^0}} > n\}$$

has power m. Let now λ and $\xi < \omega(m)$ and let for $n < m$

$$W_\lambda^\xi|n = \{\eta \in W_\lambda\colon\ \overline{\overline{A_\eta^\xi}} > n\}\,.$$

According to (2) the fact $\overline{\overline{W_\lambda^0|n}} = m$ implies that

(3) $$\overline{\overline{W_\lambda^\xi|n}} = m \quad \text{for every } \lambda,\ \xi < \omega(m) \text{ and } n < m\,.$$

It is obvious that if $\xi_1 < \xi_2$, then

(4) $$W_\lambda^{\xi_1}|n \supseteq W_\lambda^{\xi_2}|n\,.$$

Let by definition

$$W_\lambda^\xi|0 = W_\lambda^\xi$$

for $\lambda,\ \xi < \omega(m)$.

Now we define by transfinite induction the sets F_ξ $\big(\xi < \omega(m)\big)$ as follows. Let $\boldsymbol{G}$ be a system of disjoint sets over $\boldsymbol{J}$ which induces the ideal $\boldsymbol{K} \cap \boldsymbol{P}(F)$. Put $F_0 = F$ and let $\boldsymbol{B}_0$ be an arbitrary set for which

(A_0) $$\boldsymbol{B}_0 \subseteq \bigcup_{\eta \in W_0^0} \boldsymbol{A}_\eta^0$$

and

(B_0) $$\overline{\overline{\boldsymbol{B}_0 \cap \boldsymbol{A}_\eta^0}} = 1 \quad \text{for every } \vartheta \in W_0^0\,.$$

(Such a set $\boldsymbol{B}_0$ clearly exists.)

It is easy to see that $\langle \boldsymbol{B}_0\rangle \in \boldsymbol{K}$, but by Theorem 1 $\langle \boldsymbol{B}_0\rangle \notin \boldsymbol{J}$. Since $\boldsymbol{G}$ induces the ideal $\boldsymbol{K} \cap \boldsymbol{P}(F)$, there is a subset $\boldsymbol{G}_0$ of power $< m$ of $\boldsymbol{G}$ such that

$$\langle \boldsymbol{B}_0\rangle \subseteq \langle \boldsymbol{G}_0\rangle \qquad (\notin \boldsymbol{J})\,.$$

Since $\overline{\overline{\boldsymbol{G}_0}} < m$, there exists a set $G_0 \in \boldsymbol{G}$ such that

(C_0) $$\langle \boldsymbol{B}_0\rangle \subseteq G_0 \in \boldsymbol{J},$$

Put $F_1 = F_0 - G_0$. Let $0 < \xi < \omega(m)$ and suppose that we have defined the sets F_α, $\boldsymbol{B}_\alpha$ and G_α for every $\alpha < \xi$ such that $G_\alpha \in \boldsymbol{G}_\alpha$; moreover,

(A_α) $$\boldsymbol{B}_\alpha \subseteq \bigcup_{\eta \in W_\alpha^\alpha} \boldsymbol{A}_\eta^\alpha,$$

(B_α) $$\overline{\overline{\boldsymbol{B}_\alpha \cap \boldsymbol{A}_\eta^\alpha}} = 1 \quad \text{for every } \eta \in W_\alpha^\alpha\,.$$

(C_α) $$\langle \boldsymbol{B}_\alpha\rangle \cap G_\alpha \notin \boldsymbol{J}$$

and

(D_α) $$G_\alpha \neq G_\beta \quad \text{for every } \beta < \alpha\,.$$

Let

$$F_\xi = F_0 - \bigcup_{\alpha<\xi} G_\alpha\,.$$

Since $G_\alpha \in \boldsymbol{G}$, relations (1) and (2) hold obviously.

Now we define the sets $\boldsymbol{B}_\xi$ and G_ξ as follows. Let $\boldsymbol{B}_\xi$ be an arbitrary set such that

(A_ξ) $$\boldsymbol{B}_\xi \subseteq \bigcup_{\eta \in W_\xi^\xi} \boldsymbol{A}_\eta^\xi,$$

and

(B_ξ) $$\overline{\overline{\boldsymbol{B}_\xi \cap \boldsymbol{A}_\eta^\xi}} = 1 \quad \text{for every } \eta \in W_\xi^\xi.$$

(Such a set $\boldsymbol{B}_\xi$ clearly exists.)

It is easy to see that $\langle \boldsymbol{B}_\xi\rangle \in \boldsymbol{K}$, but by Theorem 1 $\langle \boldsymbol{B}_\xi\rangle \notin \boldsymbol{J}$. Since $\boldsymbol{G}$ induces the ideal $\boldsymbol{K}$, there exists a subset $\boldsymbol{G}_\xi$ of power $< m$ of $\boldsymbol{G}$ such that

$$\langle \boldsymbol{B}_\xi\rangle \subseteq \langle \boldsymbol{G}_\xi\rangle \qquad (\notin \boldsymbol{J})\,.$$

Since $\overline{\overline{\boldsymbol{G}_\xi}} < m$, there exists a set $G_\xi \in \boldsymbol{G}_\xi$ such that

(C_ξ) $$\langle \boldsymbol{B}_\xi\rangle \cap G_\xi \notin \boldsymbol{J}.$$

It is obvious that

(D_ξ) $$G_\xi \neq G_\alpha$$

if $\alpha < \xi$, because $\langle \boldsymbol{B}_\xi\rangle \subseteq F_\xi$ and $G_\alpha \cap F_\xi = 0$ for every $\alpha < \xi$, so otherwise the relation (C_ξ) could not hold for G_α if $\alpha < \xi$.

Thus we can define the sets F_ξ, $\boldsymbol{B}_\xi$ and G_ξ such that $G_\xi \in \boldsymbol{G}$, (A_ξ), (B_ξ), (C_ξ) and (D_ξ) hold.

It follows from (A_ξ) and (B_ξ) that

$$B = \bigcup_{\xi<\omega(m)} \langle \boldsymbol{B}_\xi\rangle \in \boldsymbol{K}\,.$$

It follows from (C_ξ) and (D_ξ) that

$$\overline{\overline{[\boldsymbol{G}, \boldsymbol{J}, B]}} \geqslant m\,.$$

This means by Theorem 1 that

$$B \notin \boldsymbol{K}\,.$$

But this is a contradiction. Consequently the ideal $\boldsymbol{K}$ is a $T_1(\boldsymbol{J})$ ideal in E'. Thus the theorem is proved.

THEOREM 10. *There exists a* $T_2(E^{<m})$ *ideal.*

We need, for the proof of this theorem, some lemmas [2].

[2] The undermentioned lemmas concerning set mappings and free sets are contained partly in [2] and [3].

LEMMA A. *If $\overline{\overline{E}} > m$, then the ideal $\boldsymbol{J} = E^{<m^+}$ is a $T_2(E^{<m})$ ideal.*

Proof. Suppose, on the contrary, that there exists an m-additive ideal $\boldsymbol{K}' \supseteq E^{<m^+}$ such that for a subset F of E the proper m-additive ideal $\boldsymbol{K} = \boldsymbol{K}' \cap \boldsymbol{P}(F)$ defined in F can be induced by the disjoint system $\boldsymbol{G} \subseteq \boldsymbol{P}(F)$ over $F^{<m}$ and let

$$\boldsymbol{G}' = \{X \in \boldsymbol{G}\colon \overline{\overline{X}} \geqslant m\}.$$

Then, by $\boldsymbol{K}' \supseteq E^{<m^+}$, we obtain $\boldsymbol{K} \supset F^{<m^+}$, and so $\overline{\overline{\boldsymbol{G}'}} \geqslant m$. Let $\boldsymbol{G}''$ be a subset of power m of $\boldsymbol{G}'$ and let us correspond to every element X of $\boldsymbol{G}''$ a subset $H(X)$ of power m of X. Let

$$H = \bigcup_{X \in \boldsymbol{G}''} H(X).$$

Since $\overline{\overline{H}} = m$, $H \in F^{<m^+} \subseteq \boldsymbol{K}$. On the other hand, according to Theorem 1, we have $H \notin \boldsymbol{K}$, because $[\boldsymbol{G}, F^{<m}, H] \supseteq \boldsymbol{G}''$ and $\boldsymbol{G}'' = m$. This contradiction proves Lemma A.

By Lemma A we can assume that $\overline{\overline{E}} = m$.

LEMMA B. *Let $\overline{\overline{E}} = m$ and let $\boldsymbol{J}$ be a proper m-additive ideal which can be induced by a system $\boldsymbol{G}$ of power $\leqslant m$ of sets. If $S(x)$ is a set mapping such that $S[x] \in \boldsymbol{J}$ for every $x \in E$, then there exists a free subset M of E for which $M \in \boldsymbol{J}$.*

Proof. Let $\{G_\xi\}_{\xi<\tau}$ be a well ordering of type $\tau \leqslant \omega(m)$ of $\boldsymbol{G}$. Put

$$H_\xi = G_\xi - \bigcup_{\alpha<\xi} G_\alpha.$$

It is clear that the set $\{H_\xi\}_{\xi<\tau}$ of the disjoint sets H_ξ induces $\boldsymbol{J}$.

We consider two cases:

(a) The power of the set of the sets H_ξ for which $H_\xi \neq 0$ is smaller than m.

(b) The power of the set of the sets H_ξ for which $H_\xi \neq 0$ is m.

Ad (A). It is obvious that the non-empty set $M = E - \bigcup_{\xi<\tau} H_\xi$ satisfies the conditions of the lemma, since $\langle \boldsymbol{I} \rangle \subseteq \bigcup_{\xi<\tau} H_\xi \in \boldsymbol{I}$.

Ad (b). We define a free set $\{x_\lambda\}_{\lambda<\omega(m)}$ by transfinite induction as follows. Let ξ_0 be the smallest ordinal number for which $H_{\xi_0} \neq 0$, and let x_0 be an arbitrary element of H_{ξ_0}; further let λ be a given ordinal number, $1 \leqslant \lambda < \omega(m)$. Suppose that we have defined the elements x_α for each $\alpha < \lambda$ such that the set $\{x_\alpha\}_{\alpha<\lambda}$ is free and $\overline{\overline{\{x_\alpha\}_{\alpha<\lambda} \cap H_\xi}} \leqslant 1$ for every $\xi < \tau$. Let

$$\boldsymbol{V}_\lambda = \{H_\xi\colon \{x_\alpha\}_{\alpha<\lambda} \cap H_\xi \neq 0 \text{ for } \xi < \tau\}$$

and

$$Q_{\lambda,\xi} = H_\xi - \bigcup_{\alpha<\lambda} (S[x_\alpha] - \langle \boldsymbol{V}_\alpha \rangle).$$

Obviously there exists a ξ for which $Q_{\lambda,\xi} \neq 0$, because

$$\bigcup_{\xi<\tau} H_\xi \notin \boldsymbol{J}$$

and, on the other hand,

$$\bigcup_{\alpha<\xi} (S[x_\alpha] \cup \langle \boldsymbol{V}_\alpha \rangle) \in \boldsymbol{J}.$$

Let ξ_λ be the smallest ordinal number ξ for which $Q_{\lambda,\xi} \neq 0$, and x_λ an arbitrary element of Q_{λ,ξ_λ}. Thus we can define the elements x_λ for every $\lambda < \omega(m)$. It is easy to see that the set $M = \{x_\lambda\}_{\lambda<\omega(m)}$ is free. On the other hand, we have the relation $M \notin \boldsymbol{J}$. Indeed, the set $\{H_\xi\}_{\xi<\tau}$ is a disjoint system (over $\{0\}$) which induces $\boldsymbol{J}$, the inclusion

$$[\{H_\xi\}_{\xi<\tau}, \{0\}, M] \supseteq \{H_{\xi_\lambda}\}_{\lambda<\omega(m)}$$

holds and, since the ordinal numbers ξ_λ $(\lambda < \omega(m))$ are distinct, we have the inequality

$$\overline{\overline{[\{H_\xi\}_{\xi<\tau}, \{0\}, M]}} \supseteq \overline{\overline{\{H_{\xi_\lambda}\}_{\lambda<\omega(m)}}} = m.$$

So we may apply Theorem 1. The proof is complete.

LEMMA C. *Let $\overline{\overline{E}} = m$ and let $\boldsymbol{J}$ be a proper m-additive ideal which can be induced by the almost disjoint system $\boldsymbol{G}$. If $S(x)$ is a set mapping such that $S[x] \in \boldsymbol{J}$ for every $x \in E$, then there exists a free subset M of E for which $M \notin \boldsymbol{J}$.*

Proof. We consider two cases:

(a) $[\boldsymbol{G}, E^{<m}, E] = \{X \in \boldsymbol{G}\colon \overline{\overline{X}} \geqslant m\}$ has power $\leqslant m$,

(b) $[\boldsymbol{G}, E^{<m}, E] = \{X \in \boldsymbol{G}\colon \overline{\overline{X}} \geqslant m\}$ has power $> m$.

Ad (a). In this case the ideal $\boldsymbol{J}$ can be induced by the system

$$\boldsymbol{G}' = \{X \in \boldsymbol{G}\colon \overline{\overline{X}} \geqslant m\} \cup \big\{\{x\}\colon \{x\} \in \boldsymbol{J}\big\},$$

which has power $\leqslant m$. Therefore we can use Lemma B.

Ad (b). Let

$$\boldsymbol{G}' = [\boldsymbol{G}, E^{<m}, E] \quad (= \{X \in \boldsymbol{G}\colon \overline{\overline{X}} \geqslant m\})$$

and

$$\boldsymbol{G}(x) = \big[\boldsymbol{G}, E^{<m}, S[x]\big],$$

where x is an arbitrary element of E. Since $S[x] \in \boldsymbol{J}$, by Theorem 1 we have $\overline{\overline{\boldsymbol{G}(x)}} < m$. Let

$$\boldsymbol{G}'' = \boldsymbol{G}' - \bigcup_{x \in E} \boldsymbol{G}(x).$$

It is clear that $\overline{\overline{\boldsymbol{G}''}} > m$, because $\overline{\overline{\boldsymbol{G}'}} > m$ and $\overline{\overline{\boldsymbol{G}(x)}} < m$. Let $\boldsymbol{H}$ be a subset of power m of $\boldsymbol{G}''$, further let $\{H_\xi\}_{\xi<\omega(m)}$ be a well ordering of type $\omega(m)$ of $\boldsymbol{H}$. Let $\boldsymbol{A}$ be the set of all ordered pairs (α, β), where $\alpha < \beta < \omega(m)$, well ordered in the same way as in the proof of Theorem 4. We define the set M by transfinite induction as follows. Let x_0^0 be an arbitrary element of H_0 and suppose that the elements x_β^α have been already defined for every $(\alpha, \beta) \prec (\xi, \eta)$, wherein (ξ, η) is an arbitrary element of A with $(0, 0) \prec (\xi, \eta)$. Let

$$H_\eta^\xi = H_\xi - \bigcup_{(\alpha,\beta)\prec(\xi,\eta)} (\{x_\beta^\alpha\} \cup S[x_\beta^\alpha]) .$$

Since $\overline{\overline{H_\xi}} \geqslant m$ and $\overline{\overline{S[x] \cap H_\xi}} < m$, we obtain $H_\eta^\xi \neq 0$. Let x_η^ξ be an arbitrary element of H_η^ξ. Thus we can define the elements x_η^ξ for each $(\xi, \eta) \in A$. Let

$$M = \{x_\eta^\xi\}_{(\xi,\eta)\in A} .$$

By definition, M is a free set. On the other hand, $\overline{\overline{M \cap H_\xi}} = m$ for every $\xi < \omega(m)$. Since

$$[G, E^{<m}, M] \supseteq \{H_\xi\}_{\xi<\omega(m)} ,$$

the power of $[G, E^{<m}, M]$ is $\geqslant m$. Thus, by Theorem 1, $M \notin J$.

LEMMA D. *Let $\overline{\overline{E}} = m$ and let J be a proper m-additive ideal such that $\{x\} \in J$ for every $x \in E$. Let $S(x)$ be a set mapping such that $S[x] \in J$ for every $x \in E$. If J is not $T_2(E^{<m})$ ideal, then there exists a free subset M of E such that $M \notin J$.*

Proof. If J is not $T_2(E^{<m})$ ideal, then there exists a proper m-additive ideal $K \supseteq J$ which is not $T_1(E^{<m})$ ideal. Thus there exists a subset F of E for which $F \notin K$ and the ideal $K \cap P(F)$ can be induced by almost disjoint sets. Since $\{x\} \in J$ for every $x \in E$, we obtain $\overline{\overline{F}} = m$. Using Lemma C for F instead of E, we obtain that there exists a free subset M of F such that $M \notin K$. It follows that $M \notin J$, because $K \supseteq J$. The lemma is proved.

LEMMA E. *If $\overline{\overline{E}} = m$, then there exist a proper m-additive ideal J and a set mapping $S(x)$ such that $S[x]$, $\{x\}$ $(x \in E)$ and every free subset of E belong to J.*

Proof. Let $E = \bigcup_{\xi<\omega(m)} E_\xi$ be a decomposition of E into m mutually disjoint sets of power m. Let

$$R = \{R \subseteq E:\ \overline{\overline{R \cap E_\xi}} = 1 \text{ for every } \xi < \omega(m)\}$$

and let J be the m-additive ideal induced by the set

$$G = R \cup \{E_\xi\}_{\xi<\omega(m)} .$$

If $x \in E_\xi$, then we put $S(x) = E_\xi - \{x\}$. It is easy to see that the ideal J and the set mapping $S(x)$ defined in this way satisfy the conditions of Lemma E.

It follows from Lemma D that the ideal J defined in Lemma E is $T_2(E^{<m})$ ideal. Thus Theorem 10 is proved.

4. Structural remarks [3]. THEOREM 11. *Let I be an arbitrary m-additive ideal, and J a $T_0(I)$ ideal; moreover, let S be a disjoint system over I. If S is complete in J over I, then S is complete in $P(E)$ too.*

Proof. Let F be a subset of E not belonging to I. If there exists a subset $H \notin I$ of F such that $H \in J$, then, since S is complete in J, there exists a set K in S such that $H \cap K \notin I$, consequently $F \cap K \notin I$. If for every subset $H \notin I$ of F the relation $H \notin J$ holds, then $J \cap P(F) \subseteq I$, which contradicts the fact that J is a $T_0(I)$ ideal. Thus we may conclude that S is complete in $P(E)$.

THEOREM 12. *Let I be a proper m-additive ideal, S a complete disjoint system over I, and let J be the m-additive ideal induced by S. Then J is a $T_0(I)$ ideal.*

Proof. Suppose, on the contrary, that J is not a $T_0(I)$ ideal, i.e. there exists a subset $F \notin I$ of E such that $J \cap P(F) \subseteq I$. Then because of $S \subseteq J$, the relation $S \cap P(F) \subseteq I$ holds, and thus the set $S \cup \{F\}$ is also (strongly) disjoint over I, which contradicts the completeness of S.

COROLLARY 13. *Let I be a proper m-additive ideal, S a complete disjoint system over I, and let J be the m-additive ideal induced by S. If $I \subseteq J$, then J is a $T_0'(I)$ ideal.*

This follows from Theorem 12 applying the very simple fact mentioned in the introduction that the $T_0(I)$ ideal J is a $T_0'(I)$ ideal if and only if $I \subseteq J$.

THEOREM 14. *Let I be a proper m-additive ideal including all subsets of one element of E (i.e. let I be a proper $T_0(\{0\})$ ideal) and let J be a proper $T_0'(I)$ ideal. Then J cannot be induced by disjoint sets (in the usual sense, i.e. over $\{0\}$).*

Proof. Suppose the contrary and let G be a disjoint inducing system of J. Choose an element out of each element of G and let F be the set of the chosen elements. Then it is obvious that $F \notin J$; on the other hand $J \cap P(F) = F^{<m}$ which is a subset of I. This contradicts the fact that J is a $T_0'(I)$ ideal.

COROLLARY 15. *Let I be a proper m-additive ideal including all subsets of one element of E and let J be a proper $T_0'(I)$ ideal. If G is an inducing system of J, then $\overline{\overline{G}} > m$.*

Proof. Suppose, on the contrary, that $\overline{\overline{G}} \leqslant m$ and let $\{G_\xi\}_{\xi<\tau}$ be a well ordering of type $\tau \leqslant \omega(m)$ of G. Now let

$$H_\xi = G_\xi - \bigcup_{\alpha<\xi} G_\alpha \quad (\xi < \tau).$$

Then it is easy to see that $\{H_\xi\}_{\xi<\tau}$ is a disjoint inducing system of G contradicting Theorem 14. The proof is complete.

---

[3] Each of the following results are given in [2] (pp. 363-365) in the special case $I = E^{<m}$.

References

[1] A. Tarski, *Sur la décomposition des ensembles en sous-ensembles presque disjoints*, Fund. Math. 12 (1928), pp. 186-304 and 14 (1929), 205-215.

[2] G. Fodor and A. Máté, *Free sets and the structure of additive ideals*, Annali di Mat. pura ed applicata 65 (1964), pp. 361-276.

[3] A. Máté, *Additive Ideale und unabhängige Mengen*, Acta Sci. Math. 25 (1964), pp. 149-159.

Reçu par la Rédaction le 11. 11. 1966

icm©

Concerning homotopy properties of compacta

by

Karol Borsuk (Warszawa)

The aim of this note is to introduce some notions which allow us to compare the global homotopy properties of two compacta X and Y lying in the Hilbert space H. The basic notion in this study is the notion of the fundamental class from X to Y, which is in fact a generalization of the classical notion of the homotopy class of a map of X into Y. If Y is a polyhedron or, more generally, an ANR-space, then this notion differs only formally from the classical notion of the homotopy class. However, in the case of arbitrary compacta, the situation is different and the notion of fundamental class has a more intimate connection with the global topological structure of spaces than the classical notion of homotopy class. The category consisting of fundamental classes as mappings and of compacta (in H) as objects allows us to study the global homotopy properties of compacta from a new point of view.

The author acknowledges his gratitude to Dr. H. Patkowska, who read the manuscript and made several valuable suggestions.

§ 1. Homotopy classes. Let X_0 be a subset of a space X and Y_0 a subset of a space Y. By a *map* of the pair (X, X_0) into the pair (Y, Y_0) we understand a continuous function $f\colon (X, X_0)\to(Y, Y_0)$. If $(A, A_0) \subset (X, X_0)$, i.e. $A\subset X$ and $A_0\subset A\cap X_0$, then the map $f'\colon (A, A_0)\to(Y, Y_0)$ defined by the formula $f'(x)=f(x)$ for every $x\in A$ is called the *restriction* of f and denoted by $f/(A, A_0)$.

If X is a subset of a space M, then a pair (Z, Z_0) is said to be a *neighborhood* of the pair (X, X_0) in M if Z is a neighborhood of X, and Z_0 a neighborhood of X_0 (in M). If Z is an open (closed) neighborhood of X, and Z_0 is an open (closed) neighborhood of X_0, then the pair (Z, Z_0) is said to be an *open* (*closed*) *neighborhood of* (X, X_0) *in* M. In the case of $X_0 = 0$, the pair $(X, 0)$ is considered as identical with X.

Two maps $f, g\colon (X, X_0)\to(M, M_0)$ are said to be *homotopic in the pair* $(Z, Z_0)\subset(M, M_0)$ if there exists a map

$$\varphi\colon (X\times\langle 0,1\rangle;\ X_0\times\langle 0,1\rangle)\to(M, M_0)$$

such that $\varphi(X\times\langle 0,1\rangle)\subset Z$, $\varphi(X_0\times\langle 0,1\rangle)\subset Z_0$ and $\varphi(x,0)=f(x)$, $\varphi(x,1)=g(x)$ for every point $x\in X$. Then we write $f\simeq g$ in (Z, Z_0) and