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Concerning homotopy properties of compacta

by

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The aim of this note is to introduce some notions which allow us to compare the global homotopy properties of two compacta X and Y lying in the Hilbert space H. The basic notion in this study is the notion of the fundamental class from X to Y, which is in fact a generalization of the classical notion of the homotopy class of a map of X into Y. If Y is a polyhedron or, more generally, an ANR-space, then this notion differs only formally from the classical notion of the homotopy class. However, in the case of arbitrary compacta, the situation is different and the notion of fundamental class has a more intimate connection with the global topological structure of spaces than the classical notion of homotopy class. The category consisting of fundamental classes as mappings and of compacta (in H) as objects allows us to study the global homotopy properties of compacta from a new point of view.

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§ 1. Homotopy classes. Let X_0 be a subset of a space X and Y_0 a subset of a space Y. By a *map* of the pair (X, X_0) into the pair (Y, Y_0) we understand a continuous function $f\colon (X, X_0)\to(Y, Y_0)$. If $(A, A_0) \subset (X, X_0)$, i.e. $A\subset X$ and $A_0\subset A\cap X_0$, then the map $f'\colon (A, A_0)\to(Y, Y_0)$ defined by the formula $f'(x) = f(x)$ for every $x\in A$ is called the *restriction* of f and denoted by $f/(A, A_0)$.

If X is a subset of a space M, then a pair (Z, Z_0) is said to be a *neighborhood* of the pair (X, X_0) in M if Z is a neighborhood of X, and Z_0 a neighborhood of X_0 (in M). If Z is an open (closed) neighborhood of X, and Z_0 is an open (closed) neighborhood of X_0, then the pair (Z, Z_0) is said to be an *open* (*closed*) *neighborhood of* (X, X_0) *in* M. In the case of $X_0 = 0$, the pair $(X, 0)$ is considered as identical with X.

Two maps $f, g\colon (X, X_0)\to(M, M_0)$ are said to be *homotopic in the pair* $(Z, Z_0)\subset(M, M_0)$ if there exists a map

$$\varphi\colon (X\times\langle 0, 1\rangle;\ X_0\times\langle 0, 1\rangle)\to(M, M_0)$$

such that $\varphi(X\times\langle 0,1\rangle)\subset Z$, $\varphi(X_0\times\langle 0,1\rangle)\subset Z_0$ and $\varphi(x, 0) = f(x)$, $\varphi(x, 1) = g(x)$ for every point $x\in X$. Then we write $f\simeq g$ in (Z, Z_0) and

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we call the map φ a *homotopy joining the map f with the map g in* (Z, Z_0). If $f, g\colon (X, X_0)\to(M, M_0)$ are homotopic in (M, M_0), then we say shortly that f and g are *homotopic* and we write $f \simeq g$. It is evident that both the relation of homotopy and the relation of homotopy in (Z, Z_0) are reflexive, symmetric and transitive. Let us observe that $f \simeq g$ in (Z, Z_0) implies $f(X)\cup g(X)\subset Z$ and $f(X_0)\cup g(X_0)\subset Z_0$.

§ 2. Weak homotopy classes. If $f\colon (X, X_0)\to(M, M_0)$ is a map, then we denote by $[f]$ the homotopy class with the representative f, i.e. the collection of all maps $g\colon (X, X_0)\to(M, M_0)$ satisfying the condition $f\simeq g$.

In the sequel X, X_0, Y, Y_0, Z, Z_0 will always denote compact subsets of the Hilbert space H such that $X_0\subset X$, $Y_0\subset Y$, $Z_0\subset Z$. By a neighborhood of a subset of H we always mean a neighborhood in the space H.

It two maps $f, g\colon (X, X_0)\to(Y, Y_0)$ are homotopic in each neighborhood (V, V_0) of the pair (Y, Y_0), then they will be said to be *weakly homotopic*. It is clear that the relation of weak homotopy is reflexive, symmetric and transitive. Thus the collection of all maps

$$f\colon (X, X_0)\to(Y, Y_0)$$

decomposes uniquely into disjoint classes of maps called *weak homotopy classes*. The weak homotopy class with the representative f will be denoted by $[f]_w$.

(2.1) THEOREM. *If $Y, Y_0\in$ ANR, then for every map $f\colon (X, X_0)\to(Y, Y_0)$ the weak homotopy class $[f]_w$ coincides with the homotopy class $[f]$.*

First let us prove the following

(2.2) LEMMA. *If $Y, Y_0\in$ ANR, then there exist a closed neighborhood (W, W_0) of the pair (Y, Y_0) and a map $\beta\colon (W, W_0)\to(Y, Y_0)$ such that*

(1) *Y is a retract of W;*

(2) *the restriction $\beta/(Y, Y_1)$ is homotopic to identity.*

Proof. Since $Y\in$ ANR, there exist an open neighborhood V of Y and a retraction $r\colon \overline{V}\to Y$. Since $Y_0\in$ ANR, there exist an open neighborhood $V_0\subset V$ of Y_0 and a retraction $r_0\colon \overline{V}_0\to Y_0$ such that for every point $y\in\overline{V}_0$ the segment $\overline{yr_0(y)}$ lies in V. Let us set $W=\overline{V}$ and let us consider a closed neighborhood $W_0\subset V_0$ of Y_0 and a map $\alpha\colon H\to\langle 0,1\rangle$ such that

$$\alpha(y)=\begin{cases}0 & \text{for every point } y\in W_0,\\ 1 & \text{for every point } y\in H-V_0.\end{cases}$$

Moreover, there exist a map $s\colon H\to H$ such that $s(y)=r_0(y)$ for every point $y\in\overline{V}_0$.

Since the point $\alpha(y)\cdot y+(1-\alpha(y))\cdot r_0(y)$ belongs for $y\in\overline{V}_0$ to the segment $\overline{yr_0(y)}\subset V\subset W$, we infer that

$$\alpha(y)\cdot y+(1-\alpha(y))\cdot s(y)\in V \quad \text{for every point } y\in\overline{V}_0.$$

Moreover,

$$\alpha(y)\cdot y+(1-\alpha(y))\cdot s(y)=r_0(y)\in Y_0 \quad \text{for every point } y\in W_0,$$

and we infer that the formula

$$\beta(y)=r[\alpha(y)\cdot y+(1-\alpha(y))\cdot s(y)] \quad \text{for every point } y\in W$$

defines a map $\beta\colon (W, W_0)\to(Y, Y_0)$.

Now let us observe that for every $y\in\overline{V}_0$ and $0\leqslant u\leqslant 1$, the point $(1-u+u\cdot\alpha(y))\cdot y+(u-u\cdot\alpha(y))\cdot s(y)$ lies on the segment $\overline{yr_0(y)}\subset V$, and for $y\in V-V_0$ we have $(1-u+u\cdot\alpha(y))\cdot y+(u-u\cdot\alpha(y))\cdot s(y)=y\in V$. It follows that the formula

$$\gamma(y, u)=r\Big[(1-u+u\cdot\alpha(y))\cdot y+(u-u\cdot\alpha(y))\cdot s(y)\Big] \quad \text{for } y\in Y \text{ and } u\in\langle 0,1\rangle$$

defines a homotopy $\gamma\colon (Y\times\langle 0,1\rangle,\ Y_0\times\langle 0,1\rangle)\to(Y, Y_0)$ such that

$$\gamma(y, 0)=y \quad\text{and}\quad \gamma(y, 1)=\beta(y) \quad \text{for every point } y\in Y.$$

Thus we have shown that the restriction $\beta/(Y, Y_0)\colon (Y, Y_0)\to(Y, Y_0)$ is homotopic to identity. The proof of Lemma (2.2) is finished.

Proof of Theorem (2.1). Consider two weakly homotopic maps $f, g\colon (X, X_0)\to(Y, Y_0)$. Let (W, W_0) and $\beta\colon (W, W_0)\to(Y, Y_0)$ be as in Lemma (2.2). Then there exists a homotopy

$$\varphi\colon (X\times\langle 0,1\rangle,\ X_0\times\langle 0,1\rangle)\to(W, W_0)$$

such that $\varphi(x, 0)=f(x)$ and $\varphi(x, 1)=g(x)$ for every point $x\in X$. Setting $\psi=\beta\varphi$, we get a homotopy $\psi\colon (X\times\langle 0,1\rangle, X_0\times\langle 0,1\rangle)\to(Y, Y_0)$ such that $\psi(x, 0)=\beta f(x)$ and $\psi(x, 1)=\beta g(x)$. Since the values of f and g belong to Y and since $\beta/(Y, Y_0)\colon (Y, Y_0)\to(Y, Y_0)$ is homotopic to the identity map, we infer that $f\simeq g$. Thus the proof of Theorem (2.1) is finished.

§ 3. Fundamental sequences and fundamental classes. By a *fundamental sequence from (X, X_0) to (Y, Y_0)* we understand an ordered triple consisting of the pairs (X, X_0), (Y, Y_0) and of a sequence of maps $f_k\colon H\to H$, $k=1,2,\ldots$, such that for every neighborhood (V, V_0) of (Y, Y_0) there exists a neighborhood (U, U_0) of (X, X_0) such that

(3.1) $\quad f_k/(U, U_0)\simeq f_{k+1}/(U, U_0)$ in (V, V_0) for almost all k.

We shall denote this fundamental sequence by $\{f_k, (X, X_0), (Y, Y_0)\}$ or, shortly, by $\underline{f}$. Manifestly condition (3.1) is equivalent to the following one:

(3.2) *There exists an index k_0 such that $f_k/(U, U_0)\simeq f_l/(U, U_0)$ in (V, V_0) for every $k, l\geqslant k_0$.*

Let us observe that if $f = \{f_k, (X, X_0), (Y, Y_0)\}$ is a fundamental sequence and the maps $g_k\colon H \to H$ satisfy, for $k = 1, 2, \ldots$, the condition

(3.3) *For every neighborhood (V, V_0) of (Y, Y_0) there exists a neighborhood (U, U_0) of (X, X_0) such that $f_k/(U, U_0) \simeq g_k/(U, U_0)$ in (V, V_0) for almost all k,*

then $\underline{g} = \{g_k, (X, X_0), (Y, Y_0)\}$ is also a fundamental sequence. We shall say that the fundamental sequence $\underline{g}$ satisfying condition (3.3) is *homotopic* to the fundamental sequence $\underline{f}$ (notation $\underline{f} \simeq \underline{g}$). Clearly this relation is reflexive, symmetric and transitive and so it decomposes the collection of all fundamental sequences into mutually exclusive classes called *fundamental classes* (from (X, X_0) to (Y, Y_0)). The fundamental class with a representative $\underline{f}$ will be denoted by $[\underline{f}]$.

One can easily see that

(3.4) *If $f_k, g_k\colon H \to H$ are maps and if there exists a sequence of indices $i_k \to \infty$ such that $g_k = f_{i_k}$ for almost all indices k and, moreover, if $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$ is a fundamental sequence, then $\underline{g} = \{g_k, (X, X_0), (Y, Y_0)\}$ is a fundamental sequence homotopic to $\underline{f}$.*

We say that a sequence of maps $\{f'_k\}$ is obtained from the sequence $\{f_k\}$ by an *infinitesimal translation*, if there exists a sequence $\{\varepsilon_k\}$ of positive numbers converging to zero and such that

$$\varrho\big(f_k(x), f'_k(x)\big) < \varepsilon_k \text{ for every point } x \in H \text{ and } k = 1, 2, \ldots$$

One can easily see that

(3.5) *If $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$ is a fundamental sequence, then every sequence of maps $f'_k\colon H \to H$ obtained from $\{f_k\}$ by an infinitesimal translation is a fundamental sequence homotopic to the fundamental sequence $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$.*

§ 4. Fundamental classes generated by maps. If $f\colon (X, X_0) \to (Y, Y_0)$ is a map, then there exists a map $\hat{f}\colon H \to H$ such that $\hat{f}(x) = f(x)$ for every point $x \in X$. It is evident that setting $f_k = \hat{f}$ for every $k = 1, 2, \ldots$, we get a sequence of maps $f_k\colon H \to H$ such that $\{f_k, (X, X_0), (Y, Y_0)\}$ is a fundamental sequence. Let us show that its fundamental class does not depend on the choice of the map $\hat{f}$ satisfying the condition $\hat{f}(x) = f(x)$ for every point $x \in X$. Let us prove more:

(4.1) *If $f, g\colon (X, X_0) \to (Y, Y_0)$ are weakly homotopic maps and if the maps $\hat{f}, \hat{g}\colon H \to H$ satisfy the condition $\hat{f}(x) = f(x)$, $\hat{g}(x) = g(x)$ for every point $x \in X$, then setting $f_k = \hat{f}$, $g_k = \hat{g}$ for $k = 1, 2, \ldots$ we get two homotopic fundamental sequences:*

$$\{f_k, (X, X_0), (Y, Y_0)\}, \quad \{g_k, (X, X_0), (Y, Y_0)\}.$$

First let us prove the following

(4.2) Lemma. *Let $f, g\colon (X, X_0) \to (Y, Y_0)$ be two maps homotopic in an open neighborhood (V, V_0) of (Y, Y_0) and let $\hat{f}, \hat{g}\colon H \to H$ be two maps such that $\hat{f}(x) = f(x)$, $\hat{g}(x) = g(x)$ for every $x \in X$. Then there exists a neighborhood (U, U_0) of (X, X_0) such that $\hat{f}/(U, U_0) \simeq \hat{g}/(U, U_0)$ in (V, V_0).*

Proof. Let $\varphi\colon (X \times \langle 0, 1\rangle, X_0 \times \langle 0, 1\rangle) \to (H, H)$ be a homotopy joining f and g in (V, V_0). Then there exists a closed neighborhood (W, W_0) of the pair (X, X_0) such that $\hat{f}(W) \cup \hat{g}(W) \subset V$ and $\hat{f}(W_0) \cup \hat{g}(W_0) \subset V_0$. Setting

$$\psi(x, 0) = \hat{f}(x), \quad \psi(x, 1) = \hat{g}(x) \quad \text{for every point } x \in W,$$

$$\psi(x, t) = \varphi(x, t) \quad \text{for every } (x, t) \in X \times \langle 0, 1\rangle,$$

we get a map ψ of the closed subset

$$T = \big(W \times (0)\big) \cup (X \times \langle 0, 1\rangle) \cup \big(W \times (1)\big)$$

of the space $H \times \langle 0, 1\rangle$, having values in V and satisfying the condition $\psi(x, t) \in V_0$ for every point $(x, t) \in T \cap (W_0 \times \langle 0, 1\rangle)$. But the sets V and V_0 are both open in H, whence the map ψ has a continuous extension ψ' onto a set $G \supset T$ open in the space $H \times \langle 0, 1\rangle$ such that $\psi'(G) \subset V$. If we recall that X and X_0 are compacta and V_0 is open in H, we infer that there exists a neighborhood $(U, U_0) \subset (W, W_0)$ of the pair (X, X_0) such that $U \times \langle 0, 1\rangle \subset G$ and that $\psi'(U_0 \times \langle 0, 1\rangle) \subset V_0$. It follows that the restriction $\psi'/(U \times \langle 0, 1\rangle, U_0 \times \langle 0, 1\rangle)$ is a homotopy which joins in (V, V_0) the map $\hat{f}/(U, U_0)$ with the map $\hat{g}/(U, U_0)$. Thus the proof of Lemma (4.2) is finished.

In order to prove (4.1), it suffices to show that for every open neighborhood (V, V_0) of the pair (Y, Y_0) there exists a neighborhood (U, U_0) of the pair (X, X_0) such that $f_k/(U, U_0) \simeq g_k/(U, U_0)$ in (V, V_0) for almost all k. This follows by Lemma (4.2), because $f \simeq g$ in (V, V_0) implies that there is a neighborhood (U, U_0) of (X, X_0) such that $\hat{f}/(U, U_0) \simeq \hat{g}/(U, U_0)$ in (V, V_0), i.e. $f_k/(U, U_0) \simeq g_k/(U, U_0)$ in (V, V_0) for every $k = 1, 2, \ldots$ Thus (4.1) is proved.

We infer by (4.1) that for every map $f\colon (X, X_0) \to (Y, Y_0)$ and for every map $\hat{f}\colon H \to H$ satisfying the condition $\hat{f}(x) = f(x)$ for $x \in X$, the fundamental class $\underline{f}$ with the representative $\{f_k, (X, X_0), (Y, Y_0)\}$, where $f_k = \hat{f}$ for every $k = 1, 2, \ldots$, does not depend on the choice of the map $\hat{f}$. Thus we can say that the *fundamental class* $[\underline{f}]$ *is generated by the map* f. Moreover, (4.1) implies that $[\underline{f}]$ remains fixed if one replaces f by another representative of the weak homotopy class $[f]_w$. Let us observe that Theorem (2.1) implies that for $Y, Y_0 \in$ ANR the adjective "weak" is superfluous in the last proposition.

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Now let us prove that maps belonging to different weak homotopy classes generate different fundamental classes, i.e., that

(4.3) *If two maps $f, g\colon (X, X_0)\to(Y, Y_0)$ generate the same fundamental class, then $[f]_W = [g]_W$.*

Proof. Let $\hat{f}, \hat{g}\colon H\to H$ be two maps such that $\hat{f}(x) = f(x)$, $\hat{g}(x) = g(x)$ for every point $x \in X$. Setting $f_k = \hat{f}$, $g_k = \hat{g}$ for $k = 1, 2, \ldots$, we get two homotopic fundamental sequences $\{f_k, (X, X_0), (Y, Y_0)\}$ and $\{g_k, (X, X_0), (Y, Y_0)\}$. Thus for every neighborhood (V, V_0) of (Y, Y_0) there is a neighborhood (U, U_0) of (X, X_0) such that

$$f_k/(U, U_0) \simeq g_k/(U, U_0) \text{ in } (V, V_0) \text{ for almost all } k.$$

Since $(X, X_0) \subset (U, U_0)$ and since $f(x) = f_k(x)$, $g(x) = g_k(x)$ for every point $x \in X$, it follows that $f \simeq g$ in (V, V_0). Thus we have shown that the maps f and g are weakly homotopic and the proof is finished.

In particular, the fundamental class generated by the identity map $i\colon (X, X_0)\to(X, X_0)$ is said to be the *fundamental identity class for the pair* (X, X_0).

If U is a subset of X, open (in X) and such that $\overline{U} \subset X_0 - \overline{X - X_0}$, then the fundamental class generated by the inclusion $j\colon (X - U, X_0 - U)\to(X, X_0)$ is said to be a *fundamental excision class.*

§ 5. A special case.

We shall see in the sequel (in § 9) that in general there exist fundamental classes which are not generated by any map. We have a different situation if the sets Y and Y_0 are both ANR's. Then we have the following

(5.1) Theorem. *If $Y, Y_0 \in$ ANR, then every fundamental class $\underline{f}$ from an arbitrary pair (X, X_0) to (Y, Y_0) is generated by a map $f\colon (X, X_0)\to(Y, Y_0)$.*

First let us prove two lemmas.

(5.2) Lemma. *If there exists a retraction r of a neighborhood $\hat{V}$ of Y to Y, then for every neighborhood (V, V_0) of (Y, Y_0) there are a neighborhood (V', V'_0) of (Y, Y_0) and a homotopy*

$$\varphi\colon (V'\times\langle 0,1\rangle, V'_0\times\langle 0,1\rangle)\to(V, V_0)$$

such that $\varphi(y, 0) = y$ and $\varphi(y, 1) = r(y)$ for every point $y \in V'$.

Proof. We can assume that $V \subset \hat{V}$. Consider a neighborhood $(V', V'_0) \subset (V, V_0)$ such that for every point $y \in V'$ the segment $\overline{yr(y)}$ lies in V, and if $y \in V'_0$, then $\overline{yr(y)} \subset V_0$. It suffices to set

$$\varphi(y, t) = t\cdot r(y) + (1-t)\cdot y \quad \text{for every } (y, t) \in V'\times\langle 0,1\rangle,$$

in order to obtain a homotopy φ satisfying all the required conditions.

(5.3) Lemma. *If $Y, Y_0 \in$ ANR, then there exists a compactum $Y_1 \subset Y$ such that $\overline{Y - Y_1} \cap Y_0 = 0$ and there exists a homotopy*

$$\vartheta\colon (Y\times\langle 0,1\rangle, Y_0\times\langle 0,1\rangle)\to(Y, Y_0)$$

satisfying the following conditions:

(1) *$\vartheta(y, t) = y$ for every point $y \in Y_0$ and $0 \leqslant t \leqslant 1$;*

(2) *$\vartheta(y, 0) = y$ for every point $y \in Y$;*

(3) *$\vartheta(Y_1, 1) = Y_0$.*

Proof. Let r be a retraction of a neighborhood V of Y to Y and r_0 a retraction of a neighborhood V_0 of Y_0 to Y_0. Moreover, let ε be a positive number so small that

(5.4) $$\varrho(y, H - V) > \varepsilon \quad \text{for every point } y \in Y.$$

It is evident that there exists a compactum $Y_1 \subset Y \cap V_0$ such that $\overline{Y - Y_1} \cap Y_0 = 0$ and that

$$\varrho(y, r_0(y)) < \varepsilon \quad \text{for every point } y \in Y_1.$$

Setting

$$\alpha_1(y) = y - r_0(y) \quad \text{for every point } y \in Y_1,$$

we get a map $\alpha_1\colon Y_1\to H$ such that $|\alpha_1(y)| \leqslant \varepsilon$ for every $y \in Y_1$. Manifestly, there exists a map $\alpha\colon H\to H$ which is an extension of α_1 and satisfies the condition

(5.5) $$|\alpha(y)| \leqslant \varepsilon \quad \text{for every point } y \in H.$$

It follows by (5.4) and (5.5) that

(5.6) $$y - t\cdot\alpha(y) \in V \quad \text{for every point } y \in Y \text{ and } 0 \leqslant t \leqslant 1.$$

Condition (5.6) allows us to define on the set $Y\times\langle 0,1\rangle$ a map ϑ by the formula

$$\vartheta(y, t) = r[y - t\cdot\alpha(y)].$$

If $y \in Y_0$, then $\vartheta(y, t) = r(y) = y \in Y_0$, because $\alpha(y) = \alpha_1(y) = y - r_0(y) = 0$. Hence

$$\vartheta\colon (Y\times\langle 0,1\rangle, Y_0\times\langle 0,1\rangle)\to(Y, Y_0),$$

and condition (1) is satisfied. Moreover, $\vartheta(y, 0) = r(y)$, whence $\vartheta(y, 0) = y$ for every point $y \in Y$, i.e., condition (2) is satisfied. Finally, if $y \in Y_1$, then $\alpha(y) = \alpha_1(y) = y - r_0(y)$, whence $\vartheta(y, 1) = rr_0(y) = r_0(y) \in Y_0$ and we infer that $\vartheta(Y_1, 1) \subset Y_0$. If we observe that $Y_0 \subset Y_1$ and (by (1)) that $\vartheta(Y_0, 1) = Y_0$, we conclude that condition (3) is also satisfied.

Proof of Theorem (5.1). Let r be a retraction of a neighborhood $\hat{V}$ of Y to Y and let Y_1 and ϑ be as in Lemma (5.3). Let $\hat{V}_0$ be a neighborhood of Y_0 such that $\hat{V}_0 \subset \hat{V}$ and $Y \cap \hat{V}_0 \subset Y_1$. Let us consider a representative $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$ of the fundamental class $[\underline{f}]$.

Let $\hat{V}_0' \subset \hat{V}$ be a neighborhood of Y_0 such that $r(\hat{V}_0') \subset \hat{V}_0$, Since f is a fundamental sequence, there exist a closed neighborhood (F, F_0) of (X, X_0) and an index k_0 such that

(5.7) $f_k/(F, F_0) \simeq f_{k_0}/(F, F_0)$ in $(\hat{V}, \hat{V}_0')$ for every $k \geqslant k_0$.

If we recall that $r(\hat{V}) \subset Y$, $r(\hat{V}_0') \subset \hat{V}_0$ and $Y \cap \hat{V}_0 \subset Y_1$, we infer by (5.7) that

(5.8) $rf_k/(F, F_0) \simeq rf_{k_0}/(F, F_0)$ in (Y, Y_1) for $k \geqslant k_0$.

Then $\vartheta(rf_k(x), t)$ is defined for every $(x, t) \in F \times \langle 0, 1\rangle$ and $k \geqslant k_0$, and the condition $\vartheta(rf_k(x), t) \in Y$ is satisfied. Moreover, $rf_k(F_0) \subset Y_1$ implies that $\vartheta(rf_k(x), 1) \in Y_0$ for every point $x \in F_0$. It follows that

(5.9) *Setting $\chi(x, t) = \vartheta(rf_k(x), t)$ for every $(x, t) \in F \times \langle 0, 1\rangle$ and $k \geqslant k_0$, one gets a homotopy χ joining in (Y, Y) the map $\vartheta(rf_k, 0)/(F, F_0)$ with the map $\vartheta(rf_k, 1)/(F, F_0)$.*

If we recall that $\vartheta(Y_1, 1) \subset Y_0$, we infer by (5.8) that

(5.10) $\vartheta(rf_k, 1)/(F, F_0) \simeq \vartheta(rf_{k_0}, 1)/(F, F_0)$ in (Y, Y_0) for $k \geqslant k_0$.

Since F is closed in H, there exists a map f_k': $H \to H$ (for every $k \geqslant k_0$) such that

$$f_k'(x) = \vartheta(rf_k(x), 1) \text{ for every point } x \in F.$$

Let us prove that for every neighborhood (V, V_0) of (Y, Y_0) there exist a neighborhood $(U, U_0) \subset (F, F_0)$ of (X, X_0) and an index $k_1 \geqslant k_0$ such that

(5.11) $f_k/(U, U_0) \simeq f_k'/(U, U_0)$ in (V, V_0) for every $k \geqslant k_1$.

We may assume that $(V, V_0) \subset (\hat{V}, \hat{V}_0')$. The inclusion $(U, U_0) \subset (F, F_0)$ implies that $f_k'(x) = \vartheta(rf_k(x), 1)$ for every point $x \in U$ and $k \geqslant k_0$. Since $\vartheta(y, t) = y$ for every point $y \in Y_0$ and for $0 \leqslant t \leqslant 1$, there exists a neighborhood W_0 of Y_0 such that $\vartheta(W_0 \cap Y, t) \subset V_0$ for every $0 \leqslant t \leqslant 1$. We can assume that the neighborhood U_0 is so small, and the index $k_1 \geqslant k_0$ is so great that $rf_k(U_0) \subset W_0$ for every $k \geqslant k_1$. Then (5.9) implies that the restriction $f_k'/(U, U_0)$ is homotopic in (V, V_0) to the map f_k'' given by the formula

$$f_k''(x) = \vartheta(rf_k(x), 0) \text{ for every point } x \in U.$$

But $rf_k(x) \in Y$ and $\vartheta(y, 0) = y$ for every point $y \in Y$, whence $f_k''(x) = rf_k(x)$ for every point $x \in U$. Thus we have shown that

(5.12) $f_k'/(U, U_0) \simeq rf_k/(U, U_0)$ in (V, V_0) for $k \geqslant k_1$.

By Lemma (5.2) there exist a neighborhood $(V', V_0') \subset (V, V_0)$ of (Y, Y_0) and a homotopy

$$\varphi\colon (V' \times \langle 0, 1\rangle, V_0' \times \langle 0, 1\rangle) \to (V, V_0)$$

such that $\varphi(y, 0) = y$ and $\varphi(y, 1) = r(y)$ for every point $y \in V'$. Now we can assume that the neighborhood (U, U_0) of (X, X_0) is so small and the index $k_1 \geqslant k_0$ so great that

$$f_k(U) \subset V' \text{ and } f_k(U_0) \subset V_0' \text{ for every } k \geqslant k_1.$$

Then setting $\psi_k(x, t) = \varphi(f_k(x), t)$ for every $x \in U$ and $0 \leqslant t \leqslant 1$, we get a homotopy ψ_k: $(U \times \langle 0, 1\rangle, U_0 \times \langle 0, 1\rangle) \to (V, V_0)$ such that $\psi_k(x, 0) = \varphi(f_k(x), 0) = f_k(x)$ and $\psi_k(x, 1) = \varphi(f_k(x), 1) = rf_k(x)$ for every point $x \in U$ and $k \geqslant k_1$. Hence

(5.13) $rf_k/(U, U_0) \simeq f_k/(U, U_0)$ in (V, V_0) for $k \geqslant k_1$.

Thus relation (5.11) is a consequence of relations (5.12) and (5.13).

Now let us show that

(5.14) $f_k'/(U, U_0) \simeq f_{k_0}'/(U, U_0)$ in (V, V_0) for every $k \geqslant k_0$.

Since $U \subset F$, the maps $f_k'/(U, U_0)$ and $f_{k_0}'/(U, U_0)$ are the same as the maps $\vartheta(rf_k, 1)$ and $\vartheta(rf_{k_0}, 1)$. Thus relation (5.14) follows by relation (5.10), if we recall that $(U, U_0) \subset (F, F_0)$ and $(Y, Y_0) \subset (V, V_0)$.

Relations (5.11) and (5.14) both imply that

(5.15) $f_k/(U, U_0) \simeq f_{k_0}'/(U, U_0)$ in (V, V_0) for $k \geqslant k_1$.

We infer by (5.15) that the fundamental class $[\underline{f}]$ is generated by the map f: $(X, X_0) \to (Y, Y_0)$ given by the formula

$$f(x) = f_{k_0}'(x) = \vartheta(rf_{k_0}(x), 1) \text{ for every point } x \in X.$$

Thus the proof of Theorem (5.1) is completed.

§ 6. Composition of fundamental classes. The definition of the composition of fundamental classes is based on the following

(6.1) Lemma. *If $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$ and $\underline{g} = \{g_k, (Y, Y_0), (Z, Z_0)\}$ are two fundamental sequences, then $\{g_k f_k, (X, X_0), (Z, Z_0)\}$ is a fundamental sequence.*

Proof. Let (W, W_0) be a neighborhood of the pair (Z, Z_0). Then there exist a neighborhood (V, V_0) of (Y, Y_0) and an index k_1 such that

(6.2) $g_k/(V, V_0) \simeq g_{k+1}/(V, V_0)$ in (W, W_0) for every $k \geqslant k_1$.

Moreover, there exist a neighborhood (U, U_0) of (X, X_0) and an index k_2 such that

(6.3) $f_k/(U, U_0) \simeq f_{k+1}/(U, U_0)$ in (V, V_0) for every $k \geqslant k_2$.

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Setting $k_0 = \mathrm{Max}(k_1, k_2)$, we infer by (6.2) and (6.3) that

$$g_k f_k/(U, U_0) \simeq g_{k+1} f_{k+1}/(U, U_0) \text{ in } (W, W_0) \text{ for every } k \geqslant k_0 .$$

Hence $\{g_k f_k, (X, X_0), (Z, Z_0)\}$ is a fundamental sequence. It will be denoted by $\underline{g}\underline{f}$ and called the *composition of the fundamental sequences* $\underline{f}$ *and* $\underline{g}$.

Now let us prove that

(6.4) *If $\underline{f}$ and $\underline{f}'$ are two homotopic fundamental sequences from (X, X_0) to (Y, Y_0) and $\underline{g}$ and $\underline{g}'$ are two homotopic fundamental sequences from (Y, Y_0) to (Z, Z_0), then the compositions $\underline{g}\underline{f}$ and $\underline{g}'\underline{f}'$ are homotopic.*

Proof. Let

$$\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}, \quad \underline{f}' = \{f'_k, (X, X_0), (Y, Y_0)\},$$
$$\underline{g} = \{g_k, (Y, Y_0), (Z, Z_0)\}, \quad \underline{g}' = \{g'_k, (Y, Y_0), (Z, Z_0)\}.$$

If (W, W_0) is a neighborhood of (Z, Z_0), then there exist a neighborhood (V, V_0) of (Y, Y_0) and an index k_1 such that

$$g_k/(V, V_0) \simeq g'_k/(V, V_0) \text{ in } (W, W_0) \text{ for every } k \geqslant k_1 .$$

Moreover, there are a neighborhood (U, U_0) of (X, X_0) and an index k_2 such that

$$f_k/(U, U_0) \simeq f'_k/(U, U_0) \text{ in } (V, V_0) \text{ for every } k \geqslant k_2 .$$

Setting $k_0 = \mathrm{Max}(k_1, k_2)$, we see at once that

$$g_k f_k/(U, U_0) \simeq g'_k f'_k/(U, U_0) \text{ in } (W, W_0) \text{ for every } k \geqslant k_0,$$

and thus the proof of (6.4) is finished.

It follows by (6.1) and (6.4) that if $[\underline{f}]$ is a fundamental class from (X, X_0) to (Y, Y_0) and $[\underline{g}]$ is a fundamental class from (Y, Y_0) to (Z, Z_0), then all compositions of a representative of $[\underline{f}]$ and of a representative of $[\underline{g}]$ belong to the fundamental class $[\underline{g}\underline{f}]$ from (X, X_0) to (Z, Z_0). This fundamental class $[\underline{g}\underline{f}]$ will be denoted by $[\underline{g}][\underline{f}]$ and called the *composition* of the fundamental classes $[\underline{f}]$ and $[\underline{g}]$. It is clear that if $[\underline{f}]$ and $[\underline{g}]$ are fundamental classes generated by the maps f and g respectively, then their composition $[\underline{g}][\underline{f}]$ is generated by the map gf.

§ 7. The fundamental category.

It is evident that the definition of the composition of fundamental classes implies that

(7.1) *$[\underline{h}]([\underline{g}][\underline{f}])$ is defined if and only if $([\underline{h}][\underline{g}])[\underline{f}]$ is defined, and then $[\underline{h}]([\underline{g}][\underline{f}]) = ([\underline{h}][\underline{g}])[\underline{f}]$.*

Instead of $[\underline{h}]([\underline{g}][\underline{f}])$ (when it is defined) we write $[\underline{h}][\underline{g}][\underline{f}]$. Manifestly,

(7.2) *$[\underline{h}][\underline{g}][\underline{f}]$ is defined if and only if both $[\underline{h}][\underline{g}]$ and $[\underline{g}][\underline{f}]$ are defined.*

(7.3) *If $[\underline{f}]$ is a fundamental class from (X, X_0) to (Y, Y_0) and $[\underline{i}]$ is the fundamental identity class for (X, X_0) and $[\underline{j}]$ the fundamental identity class for (Y, Y_0), then $[\underline{f}] = [\underline{f}][\underline{i}] = [\underline{j}][\underline{f}]$.*

Propositions (7.1), (7.2) and (7.3) mean that one obtains a category if one considers the fundamental classes as morphisms and the pairs (X, X_0) of compacta in H as objects. The product is defined as the composition of fundamental classes and the identities as fundamental identity classes. Let us call this category the *fundamental category*, and denote it by $\mathfrak{F}$.

§ 8. Fundamental domination and fundamental equivalence.

A fundamental class $[\underline{f}]$ from (X, X_0) to (Y, Y_0) is *right-invertible* if there exists a fundamental class $[\underline{g}]$ from (Y, Y_0) to (X, X_0) such that $[\underline{f}][\underline{g}]$ is the fundamental identity class. One sees at once that the composition of two right-invertible fundamental classes (if defined) is also a right-invertible fundamental class. If there exists a right-invertible fundamental class $[\underline{f}]$ from (X, X_0) to (Y, Y_0), then we say that (X, X_0) *fundamentally dominates* (Y, Y_0) (notation: $(X, X_0) \underset{F}{\geqslant} (Y, Y_0)$). It is evident that

(8.1) $$(X, X_0) \underset{F}{\geqslant} (X, X_0)$$

and

(8.2) $(X, X_0) \underset{F}{\geqslant} (Y, Y_0)$ and $(Y, Y_0) \underset{F}{\geqslant} (Z, Z_0)$ imply $(X, X_0) \underset{F}{\geqslant} (Z, Z_0)$.

If the relations $(X, X_0) \underset{F}{\geqslant} (Y, Y_0)$ and $(Y, Y_0) \underset{F}{\geqslant} (X, X_0)$ both hold, then we write $(X, X_0) \underset{F}{=} (Y, Y_0)$ and say that (X, X_0) and (Y, Y_0) are *fundamentally equal*.

A fundamental class $[\underline{f}]$ from (X, X_0) to (Y, Y_0) is said to be a *fundamental equivalence* if there exists a fundamental class $[\underline{g}]$ from (Y, Y_0) to (X, X_0) such that the compositions $[\underline{f}][\underline{g}]$ and $[\underline{g}][\underline{f}]$ are both fundamental identity classes. If there exists a fundamental equivalence from (X, X_0) to (Y, Y_0), then we say that the pair (X, X_0) is *fundamentally equivalent* to the pair (Y, Y_0), and we write $(X, X_0) \underset{F}{\simeq} (Y, Y_0)$. It is evident that the relation of the fundamental equivalence is reflexive, symmetric and transitive. Thus the collection of all pairs (X, X_0) of compacta in H decomposes into disjoint sets of fundamentally equivalent pairs, called *fundamental types* (of pairs). Manifestly, two pairs of compacta of the same fundamental type are fundamentally equal.

Let us remark that in the case $X_0 = Y_0 = 0$, instead of relations between pairs (X, X_0) and (Y, Y_0) we obtain the corresponding relations between the compacta X and Y.

One says (following Whitehead [3], p. 1133) that (X, X_0) *homotopically dominates* (Y, Y_0) if there exist two maps $f: (X, X_0) \to (Y, Y_0)$ and $g: (Y, Y_0) \to (X, X_0)$ such that their composition $fg: (Y, Y_0) \to (Y, Y_0)$ is homotopic to the identity. Then it is evident that the fundamental class $[\underline{g}]$ generated by the map g is a right inverse of the fundamental class $[\underline{f}]$ generated by the map f. Consequently:

(8.3) *Homotopical domination implies fundamental domination.*

Two pairs (X, X_0) and (Y, Y_0) are said (following W. Hurewicz) to be *homotopically equivalent* provided there exist two maps $f: (X, X_0) \to (Y, Y_0)$ and $g: (Y, Y_0) \to (X, X_0)$ such that the compositions $fg: (Y, Y_0) \to (X, X_0)$ and $gf: (X, X_0) \to (X, X_0)$ are both homotopic to identities. In this case the compositions $[\underline{f}][\underline{g}]$ and $[\underline{g}][\underline{f}]$ of the fundamental classes $[\underline{f}]$ and $[\underline{g}]$, generated by f and g respectively, are both fundamental identity classes. Consequently:

(8.4) *Homotopy equivalence implies fundamental equivalence.*

In particular, *two homeomorphic pairs are always fundamentally equivalent.* It follows that

(8.5) *If (X, X_0) is homeomorphic to (X', X_0') and (Y, Y_0) is homeomorphic to (Y', Y_0'), then:*

$$(X, X_0) \underset{F}{\geqslant} (Y, Y_0) \textit{ implies } (X', X_0') \underset{F}{\geqslant} (Y', Y_0');$$

$$(X, X_0) \underset{F}{=} (Y, Y_0) \textit{ implies } (X', X_0') \underset{F}{=} (Y', Y_0');$$

$$(X, X_0) \underset{F}{\simeq} (Y, Y_0) \textit{ implies } (X', X_0') \underset{F}{\simeq} (Y', Y_0'),$$

i.e., all relations $\underset{F}{\geqslant}, \underset{F}{=}, \underset{F}{\simeq}$ are topological.

If X, Y are two fundamentally equivalent ANR's, then Theorem (5.1) implies that there exist two maps $f: X \to Y$ and $g: Y \to X$ generating the fundamental classes $[\underline{f}]$ from X to Y and $[\underline{g}]$ from Y to X respectively, such that $[\underline{f}][\underline{g}]$ and $[\underline{g}][\underline{f}]$ are fundamental identity classes. It follows by (2.1) and (4.3) that the maps $fg: Y \to Y$ and $gf: X \to X$ are homotopic to identities, whence X and Y are homotopically equivalent. Consequently:

(8.6) *If $X, Y \in$ ANR, then the relation $X \underset{F}{\simeq} Y$ coincides with homotopy equivalence.*

§ 9. Fundamental equivalence for plane continua.

The following example illustrates to some extent the sense of the relation of fundamental equivalence. Let E^2 denote the Euclidean plane which

we consider as identical with the subset of the Hilbert space H consisting of all points of the form $(x_1, x_2, 0, 0, \dots)$, and let p denote the projection of H onto E^2 given by the formula

$$p(x_1, x_2, x_3, \dots) = (x_1, x_2, 0, 0, \dots).$$

Let us prove the following

(9.1) THEOREM. *Two continua $X, Y \subset E^2$ decomposing E^2 into the same number of regions are fundamentally equivalent.*

Proof. Let us limit ourselves to the case where $E^2 - X$ and $E^2 - Y$ have an infinite number of components. If the number of components is finite, then the proof is similar but simpler.

Let us arrange the components of $E^2 - X$ in a sequence $A_0, A_1, \dots$ and the components of $E^2 - Y$ in a sequence $B_0, B_1, \dots$ such that A_0 and B_0 are unbounded. Then there exist, for every $k = 1, 2, \dots$, two open subsets U_k and V_k of E^2 satisfying the following conditions:

(9.2) *The sets U_k, V_k are both connected and $X \subset U_k$, $Y \subset V_k$. The boundary of U_k (in E^2) is the union of $k+1$ disjoint simple closed curves $C_0, C_1, \dots, C_k$ such that $C_i \subset A_i$ for $i = 0, 1, \dots, k$. The boundary of V_k (in E^2) is the union of $k+1$ disjoint simple closed curves $D_0, D_1, \dots, D_k$ such that $D_i \subset B_i$ for $i = 0, 1, \dots, k$.*

(9.3) *If $x \in A_i \cap U_k$ with $i = 0, 1, \dots, k$, then $\varrho(x, X) \leqslant 1/k$. If $y \in B_i \cap V_k$ with $i = 0, 1, \dots, k$, then $\varrho(y, Y) \leqslant 1/k$.*

(9.4) $$\overline{U}_{k+1} \subset U_k, \quad \overline{V}_{k+1} \subset V_k \quad \textit{for } k = 1, 2, \dots$$

Condition (9.2) implies that there exists a sequence of homeomorphisms

$$h_k: E^2 \to E^2, \quad k = 1, 2, \dots,$$

preserving the orientation of E^2 and satisfying the conditions

$$h_k(C_i) = D_i \text{ for } i = 0, 1, \dots, k \quad \text{and} \quad h_{k+1}|(E^2 - U_k) = h_k|(E^2 - U_k).$$

It follows hence, by an elementary argument, that

(9.5) $$h_k(U_k) = V_k,$$

(9.6) $h_k/U_k \simeq h_{k+n}/U_k$ in V_k for $k = 1, 2, \dots$ and $n = 0, 1, 2, \dots$

Let j denote the inclusion of E^2 into H. Setting

$$f_k(x) = jh_kp(x) \quad \text{for every point } x \in H,$$

one gets a sequence of maps $f_k: H \to H$. Now let us consider an arbitrary neighborhood V of Y. It follows by (9.3) that there exists an index k such that $V_k \subset V$. For this index k there exists a neighborhood U of X such that $p(U) \subset U_k$. Then

$$f_k/U = jh_kp/U, \quad f_{k+n}/U = jh_{k+n}p/U \quad \text{for every } k = 1, 2, \dots \text{ and } n = 0, 1, 2, \dots$$

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and we infer by $p(U) \subset U_k$ and (9.6) that

$$f_k/U \simeq f_{k+n}/U \text{ in } V_k \subset V \quad \text{for} \quad k = 1, 2, \ldots \text{ and } n = 0, 1, 2, \ldots$$

Thus we have shown that $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence.

By an analogous argument, one shows that setting $g_k(y) = jh_k^{-1}p(y)$ for every point $y \in H$, one obtains a fundamental sequence $\underline{g} = \{g_k, Y, X\}$. Then $\underline{gf} = \{g_k f_k, X, X\}$. But $g_k f_k(x) = jh_k^{-1}pjh_k p(x)$ for every point $x \in H$, where $h_k p(x) \in E^2$, whence $pjh_k p(x) = h_k p(x)$ and consequently $g_k f_k(x) = ih_k^{-1}h_k p(x) = p(x)$ for every point $x \in H$. If we recall that $p(x) = x$ for every point $x \in X$, we infer that $\{g_k f_k, X, X\}$ is a representative of the identity class. Hence the fundamental class $[\underline{f}]$ is a right inverse of the fundamental class $[\underline{g}]$. By an analogous argument one shows that the fundamental class $\underline{g}$ is a right inverse of the fundamental class $[\underline{f}]$. Hence the continua X and Y are fundamentally equivalent.

By a similar argument one shows that

(9.7) *If X, Y are continua in E^2 and the number of components of $E^2 - X$ is not less than the number of components of $E^2 - Y$, then X fundamentally dominates Y.*

Let T_0 denote the segment (in E^2) with endpoints $(0, 0)$ and $(0, 1)$ and let T_k (for $k = 1, 2, \ldots$) denote the boundary of the triangle with vertices $(0, 1)$, $(1/4k, 0)$, $(1/(4k+1), 0)$. Manifestly, the set $P_n = \bigcup_{k=0}^{n} T_k$ is a polyhedron decomposing E^2 into $n+1$ regions, and the set $P_\infty = \bigcup_{k=0}^{\infty} T_k$ is a continuum and $E^2 - P_\infty$ has an infinite number of components. Finally, let us set $P_0 = 0$. Since, for $m \neq n$, the polyhedrons P_m, P_n are homotopically not equivalent, we infer by (8.6) that P_m, P_n are fundamentally not equivalent. Moreover, (8.3) implies that P_∞ fundamentally dominates P_n for every $n = 0, 1, 2, \ldots$ If we had $P_\infty \underset{F}{\simeq} P_n$ for some n, then P_n would fundamentally dominate P_{n+1}, and consequently also P_n would homotopically dominate P_{n+1}, which is not true. Hence all continua $P_\infty, P_0, P_1, P_2, \ldots$ are of different fundamental types, and it follows from (9.1) that every plane continuum is fundamentally equivalent to one of them. Hence

(9.8) *The collection of all fundamental types of plane continua is countable.*

Remark. Let X be the union of the closure F of the diagram of the function $y = \sin(1/x)$, where $0 < x < 1/\pi$, and of an arc $L \subset E^2$ with the ends $(0, 0)$ and $(1/\pi, 0)$ and the interior lying in $E^2 - F$, and let Y denote a simple closed curve lying in E^2. By Theorem (9.1) there exist a fundamental sequence $\underline{f}$ from X to Y and a fundamental sequence $\underline{g}$ from Y to X such that $\underline{fg}$ is homotopic to a fundamental sequence generated by the identity map i: $Y \to Y$. By Theorem (5.1), the fundamental sequence $\underline{f}$ is generated by a map f: $X \to Y$. If the fundamental sequence $\underline{g}$ were generated also by a map g: $Y \to X$, then the map fg: $Y \to Y$ would be homotopic to the identity i: $Y \to Y$, which is evidently impossible. Thus the fundamental sequence $\underline{g}$ is not generated by any map.

§ 10. Fundamental equivalence for plane compacta.

As we have seen, the collection of all fundamental types of plane continua is only countable. Now let us show that the situation is different for plane compacta. First let us prove the following

(10.1) Lemma. *If X, Y are two compacta of the same fundamental type, then there exists a one-to-one correspondence Λ between their components such that the corresponding components have the same fundamental types.*

Proof. Let $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, X\}$ be two fundamental sequences such that the compositions $\underline{fg}$ and $\underline{gf}$ belong to fundamental identity classes. Consider a component A of X and let $a \in A$. Then $\lim_{k=\infty} \varrho(f_k(a), Y) = 0$ and since Y is compact, there exists an increasing sequence $\{k_n\}$ of indices and a point $b \in Y$ such that $\lim_{n=\infty} f_{k_n}(a) = b$. Let B denote the component of Y containing the point b. Then for every neighborhood V of B there exists a neighborhood V_0 of Y such that the component of the set V_0 containing B lies in V. Since $\underline{f}$ is a fundamental sequence, there exists a neighborhood U_0 of X such that $f_k/U_0 \simeq f_{k+1}/U_0$ in V_0 for almost all k. If U denotes the component of U_0 containing A, we infer easily that $f_k/U \simeq f_{k+1}/U$ in V for almost all k. Thus we have shown that

(10.2) *For every component A of X there exists a component B of Y such that for every neighborhood V of B there exists a neighborhood U of A with the property $f_k/U \simeq f_{k+1}/U$ in V for almost all k.*

It is clear that for every component A of X there is only one component B of Y satisfying (10.2). Setting $\Lambda(A) = B$, we get a function Λ assigning to every component A of X a component $\Lambda(A)$ of Y. Moreover, we infer by (10.2) that $\underline{f_A} = \{f_k, A, B\}$ is a fundamental sequence.

By an analogous argument we infer that there exists a function Λ' assigning to every component B of Y a component $A' = \Lambda'(B)$ of X such that

(10.3) *For every neighborhood U' of A' there exists a neighborhood V' of B such that $g_k/V' \simeq g_{k+1}/V'$ in U' for almost all k.*

We infer by (10.3) that $\underline{g_B} = \{g_k, B, A'\}$ is a fundamental sequence.

Consider now a neighborhood U' of A'. By (10.3), there is a neighborhood V' of B such that

$$g_k(V') \subset U' \quad \text{for almost all } k.$$

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Moreover, (10.2) implies that $f_k(A) \subset V'$ for almost all k. Consequently

(10.4) $$g_k f_k(A) \subset U' \quad \text{for almost all } k.$$

On the other hand, for every neighborhood U of A there is a neighborhood $\hat{U}_0$ of X such that the component $\hat{U}$ of $\hat{U}_0$ containing A lies in U. Since $\underline{gf} \simeq \underline{i}_X$, we infer that

(10.5) $$g_k f_k(A) \subset \hat{U} \subset U \quad \text{for almost all } k.$$

It follows by (10.4) and (10.5) that every neighborhood U' of the component A' intersects every neighborhood U of the component A. Hence $A = A'$, that is $\Lambda'\Lambda(A) = A$ for every component A of X. By an analogous argument one shows that $\Lambda\Lambda'(B) = B$ for every component B of Y. Hence Λ is one-to-one.

Moreover, keeping the notations U and $\hat{U}_0$, we infer by the relation $\underline{gf} \simeq \underline{i}_X$ that there exists a neighborhood U_1 of X such that

$$g_k f_k/U_1 \simeq i/U_1 \text{ in } \hat{U}_0 \text{ for almost all } k.$$

It follows that $g_k f_k(A) \subset U$ for almost all k, and we infer that the component W of U_1 containing the set A is a neighborhood of A satisfying the condition

$$g_k f_k/W \simeq i/W \text{ in } U \text{ for almost all } k.$$

Thus we have shown that $\underline{g_B}\underline{f_A} = \{g_k f_k, A, A\} \simeq \underline{i}_A$.

By an analogous argument one shows that $\underline{f_A}\underline{g_B} = \{f_k g_k, B, B\} \simeq \underline{i}_B$. Thus $A \underset{F}{\simeq} B = \Lambda(A)$ and the proof of Lemma (10.1) is finished.

Now let us prove the following

(10.7) THEOREM. *The family of all fundamental types of plane compacta is of the power* $2^{\aleph_0}$.

Proof. Since the power of the class of all compacta is $2^{\aleph_0}$, it suffices to prove that there exists a function assigning to every real number t a plane compactum X_t so that for $t < t'$ the compacta $X_t, X_{t'}$ are not fundamentally equivalent.

Let $\{w_n\}$ be a sequence of all rational numbers such that $n \neq m$ implies $w_n \neq w_m$. To every real number t, let us assign the increasing sequence $\{k_j(t)\}$ of all indices k for which $w_k < t$. Evidently for $t < t'$ the sequence $\{k_j(t')\}$ contains an infinity of natural numbers not appearing in the sequence $\{k_j(t)\}$.

Now let Q_j denote, for $j = 1, 2, \ldots$, the disk in E^2 defined as the set of all points (x, y) with

$$\left(x - \frac{1}{j}\right)^2 + y^2 \leqslant \frac{1}{4j^2(j+1)^2}.$$

It is evident that the disks Q_j are disjoint and that they converge to the set Q_0, consisting only of the point $(0, 0)$. Now let us assign to every real number t a set X_t defined as follows:

Let P_n denote the polyhedron $\bigcup_{i=1}^{n} T_k$, where T_k is the boundary of the triangle with vertices $(0, 1)$, $(1/4k, 0)$, $(1/(4k+1), 0)$, and let h_j denote a homeomorphism mapping $P_{k_j(t)}$ onto a subset $P'_{k(t)} \subset Q_j$. It is evident that the set

$$X_t = Q_0 \cup \bigcup_{j=1}^{\infty} P'_{k_j(t)}$$

is a one-dimensional compactum in E^2 with components Q_0 and $P'_{k_j(t)}$. If $t < t'$, then $X_{t'}$ is not fundamentally equivalent to X_t, because otherwise every component $P'_{k_{j'}(t')}$ would be—by Lemma (10.1)—fundamentally equivalent to a component $P'_{k_j(t)}$. Then $P_{k_{j'}(t')}$ would be fundamentally equivalent to $P_{k_j(t)}$, which implies $k_{j'}(t') = k_j(t)$. But this is impossible, because not every number $k_{j'}(t')$ belongs to the sequence $\{k_j(t)\}$. Thus for $t < t'$ the compacta X_t and $X_{t'}$ are not fundamentally equivalent and Theorem (10.7) is proved.

§ 11. Homomorphisms of homology groups induced by fundamental classes. Let us recall the basic notions of the Vietoris homology theory in a slightly modified form, appropriate to our aims. By an *n-dimensional chain over an Abelian group* $\mathfrak{A}$ we understand a linear form

$$\varkappa = a_1\sigma_1 + a_2\sigma_2 + \ldots + a_k\sigma_k,$$

where the coefficients $a_1, a_2, \ldots, a_k$ are elements of $\mathfrak{A}$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are n-dimensional simplexes in H, i.e., systems of $n+1$ points (*vertices*) $x_0, x_1, \ldots, x_n$ of H given in a definite order modulo an even permutation. If the diameter of the set of vertices of a simplex σ is less than ε, then σ is said to be an *ε-simplex*. If every simplex of $\varkappa$ is an ε-simplex and if, moreover, all its vertices lie at the distance $< \varepsilon$ from a compactum $X \subset H$, then $\varkappa$ is said to be an *ε-chain in X over* $\mathfrak{A}$. The n-dimensional ε-chains in X over $\mathfrak{A}$ constitute a group which we denote by $C_n(X^\varepsilon; \mathfrak{A})$, If X_0 is a closed subset of X, then $C_n(X^\varepsilon; \mathfrak{A})$ contains the subgroup $Z_n(X^\varepsilon, X_0^\varepsilon; \mathfrak{A})$ consisting of all n-dimensional *ε-cycles in X modulo X_0 over* $\mathfrak{A}$, i.e., of chains $\varkappa \in C_n(X^\varepsilon; \mathfrak{A})$ with the boundary $\partial\varkappa \in C_{n-1}(X_0^\varepsilon; \mathfrak{A})$. Two n-dimensional ε-cycles γ_1, γ_2 in X modulo X_0 over $\mathfrak{A}$ are said to be *η-homologous in X modulo X_0* (notation: $\gamma_1 \underset{\eta}{\sim} \gamma_2$ in X mod X_0) if there exists a chain $\varkappa \in C_{n+1}(X^\eta; \mathfrak{A})$ such that $\partial\varkappa = \gamma_1 - \gamma_2 + \lambda$, where $\lambda \in C_n(X_0^\eta; \mathfrak{A})$.

By a *true n-dimensional cycle in X modulo X_0 over* $\mathfrak{A}$ we understand a sequence $\underline{\gamma} = \{\gamma_i\}$ such that there exists a sequence $\{\varepsilon_i\}$ of positive numbers converging to zero such that $\gamma_i \in Z_n(X^{\varepsilon_i}, X_0^{\varepsilon_i}; \mathfrak{A})$ and $\gamma_i \underset{\varepsilon_i}{\sim} \gamma_{i+1}$

in X mod X_0 for every $i = 1, 2, \ldots$ The sequence $\{\varepsilon_i\}$ is called a *majorant of the true cycle* $\underline{\gamma}$. The n-dimensional true cycles in X modulo X_0 over $\mathfrak{A}$ constitute a group $Z_n(X, X_0; \mathfrak{A})$ with the group operation defined by the formula

$$\{\gamma_i\} + \{\gamma'_i\} = \{\gamma_i + \gamma'_i\}.$$

Two true cycles $\underline{\gamma} = \{\gamma_i\}$, $\underline{\gamma}' = \{\gamma'_i\} \in Z_n(X, X_0; \mathfrak{A})$ are said to be *homologous in* X *modulo* X_0 (notation: $\underline{\gamma} \sim \underline{\gamma}'$ in X mod X_0) if there exists a sequence $\{\eta_i\}$ of positive numbers converging to zero and such that $\gamma_i \underset{\eta_i}{\sim} \gamma'_i$ in X mod X_0 for $i = 1, 2, \ldots$ In particular, the set of all n-dimensional true cycles in X modulo X_0 homologous in X modulo X_0 to zero (i.e., to the true cycle $\{\gamma_i\}$ with $\gamma_i = 0$ for $i = 1, 2, \ldots$) constitutes a subgroup $B_n(X, X_0; \mathfrak{A})$ of the group $Z_n(X, X_0; \mathfrak{A})$. The factor group

$$H_n(X, X_0; \mathfrak{A}) = Z_n(X, X_0; \mathfrak{A})/B_n(X, X_0; \mathfrak{A})$$

is called the *n-dimensional homology group of the pair* (X, X_0) *over* $\mathfrak{A}$. Its elements are homology classes of n-dimensional true cycles in X modulo X_0 over $\mathfrak{A}$, i.e., the classes of all such true cycles homologous to one another in X modulo X_0.

One can easily see that if $\underline{\gamma} = \{\gamma_i\}$ is a true cycle in X modulo X_0 over $\mathfrak{A}$ and $\{i_k\}$ is a sequence of indices with $\lim_{k=\infty} i_k = \infty$, then the sequence $\underline{\gamma}' = \{\gamma_{i_k}\}$ is a true cycle homologous to $\underline{\gamma}$ in X modulo X_0.

Now let $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$ be a fundamental sequence and let $\underline{\gamma} = \{\gamma_i\}$ be a true cycle in X modulo X_0 over $\mathfrak{A}$ with a majorant $\{\varepsilon_i\}$. Then:

(11.1) *There exists an increasing sequence $\{i_k\}$ of indices such that for every sequence of indices $\{j_k\}$ satisfying the inequality $j_k \geqslant i_k$ for $k = 1, 2, \ldots$ the sequence $\{f_k(\gamma_{j_k})\}$ is a true cycle in Y modulo Y_0 over $\mathfrak{A}$.*

Proof. Let (V, V_0) be a neighborhood of (Y, Y_0) and α a positive number. Since $\underline{f}$ is a fundamental sequence from (X, X_0) to (Y, Y_0), there exists a neighborhood (U, U_0) of (X, X_0) such that for almost all indices k the conditions

$$f_k/(U, U_0) \simeq f_{k+1}/(U, U_0) \text{ in } (V, V_0)$$

and

$$\operatorname{Max}\left[\sup_{x \in U} \varrho\big(f_k(x), Y\big), \sup_{x \in U} \varrho\big(f_k(x), Y_0\big)\right] < \alpha$$

are both satisfied. It follows that there exist two sequences $\{\alpha_k\}$, $\{\beta_k\}$ of positive numbers convergent to zero and such that:

If $\gamma \in Z_n(X^{\beta_k}, X_0^{\beta_k}; \mathfrak{A})$, then $f_k(\gamma), f_{k+1}(\gamma) \in Z_k(Y^{\alpha_k}, Y_0^{\alpha_k}; \mathfrak{A})$ and $f_k(\gamma) \underset{\alpha_k}{\sim} f_{k+1}(\gamma)$ in Y mod Y_0.

If $\gamma, \gamma' \in Z_n(X^{\beta_k}, X_0^{\beta_k}; \mathfrak{A})$ and $\gamma \underset{\beta_k}{\sim} \gamma'$ in X mod X_0, then $f_k(\gamma) \underset{\alpha_k}{\sim} f_k(\gamma')$ in Y mod Y_0.

Now let us fix an increasing sequence $\{i_k\}$ of indices such that

$$\varepsilon_j \leqslant \beta_k \quad \text{for every } j \geqslant i_k.$$

Then, if a sequence of indices $\{j_k\}$ satisfies the inequality $j_k \geqslant i_k$ for $k = 1, 2, \ldots$, then γ_{j_k} and $\gamma_{j_{k+1}}$ are β_k-cycles in X modulo X_0 such that $\gamma_{j_k} \underset{\beta_k}{\sim} \gamma_{j_{k+1}}$ in X mod X_0 and consequently

$$f_k(\gamma_{j_{k+1}}) \underset{\alpha_k}{\sim} f_{k+1}(\gamma_{j_{k+1}}) \quad \text{in } Y \text{ mod } Y_0$$

and

$$f_k(\gamma_{j_k}) \underset{\alpha_k}{\sim} f_k(\gamma_{j_{k+1}}) \quad \text{in } Y \text{ mod } Y_0,$$

whence

$$f_k(\gamma_{j_k}) \underset{\alpha_k}{\sim} f_{k+1}(\gamma_{j_{k+1}}) \quad \text{in } Y \text{ mod } Y_0.$$

Thus we have shown that the sequence $\{f_k(\gamma_{j_k})\}$ is a true cycle in Y modulo Y_0 over $\mathfrak{A}$ and (11.1) is proved.

It is evident that the homology class of the cycle $\{f_k(\gamma_{j_k})\}$ does not depend on the choice of the sequence $\{j_k\}$ satisfying the condition $j_k \geqslant i_k$ for $k = 1, 2, \ldots$ Moreover, if $\underline{\gamma}' = \{\gamma'_i\}$ is another true cycle homologous to $\underline{\gamma}$ in X modulo X_0, then the sequence of indices $\{j_k\}$ can be selected so that the sequences $\{f_k(\gamma_{j_k})\}$ and $\{f_k(\gamma'_{j_k})\}$ are both true cycles homologous in Y modulo Y_0. It follows that if we assign to the homology class with the representative $\underline{\gamma} = \{\gamma_i\} \in Z_n(X, X_0; \mathfrak{A})$ the holomogy class of the true cycle $\{f_k(\gamma_{j_k})\} \in Z_n(Y, Y_0; \mathfrak{A})$, we get a function with the range $H_n(X, X_0; \mathfrak{A})$ and values in $H_n(Y, Y_0; \mathfrak{A})$. It is evident that this function is additive, whence it is a homomorphism of the group $H_n(X, X_0; \mathfrak{A})$ into the group $H_n(Y, Y_0; \mathfrak{A})$. We say that this homomorphism is *induced* by the fundamental sequence $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$.

If $\underline{f}' = \{f'_k, (X, X_0), (Y, Y_0)\}$ is another fundamental sequence homotopic to the fundamental sequence $\underline{f} = \{f_k, (X, X_0), (Y, Y_0)\}$, then to every neighborhood (V, V_0) of (Y, Y_0) there exists a neighborhood (U, U_0) of (X, X_0) such that

$$f_k/(U, U_0) \simeq f'_k/(U, U_0) \text{ in } (V, V_0) \quad \text{for almost all } k.$$

Then one shows, as before, that for every true cycle $\underline{\gamma} = \{\gamma_i\}$ in X modulo X_0 there exists a sequence of indices $\{i_k\}$ with $\lim_{k=\infty} i_k = \infty$ such that the sequences $\{f_k(\gamma_{i_k})\}$ and $\{f'_k(\gamma_{i_k})\}$ are both true cycles homologous in Y modulo Y_0. Consequently, the homomorphisms of the group $H_n(X, X_0; \mathfrak{A})$ into the group $H_n(Y, Y_0; \mathfrak{A})$ induced by both fundamental sequences are the same. Thus the homomorphism of $H_n(X, X_0; \mathfrak{A})$ into $H_n(Y, Y_0; \mathfrak{A})$ induced by a fundamental sequence $\underline{f}$ from (X, X_0) to

(Y, Y_0) depends only on the fundamental class $[f]$. We shall denote it by

(11.2) $$[f]_*: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A}),$$

and we shall say that this homomorphism is *induced* by the fundamental class $[f]$.

Let us observe that in particular:

(11.3) *If the fundamental class $[f]$ from (X, X_0) to (Y, Y_0) is generated by a map $\varphi: (X, X_0) \to (Y, Y_0)$, then the homomorphism $[f]_*: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$ induced by $[f]$ is the same as the homomorphism $\varphi_*: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$ induced by φ.*

Proof. We can select as a representative of the fundamental class $[f]$ a fundamental sequence $f = \{f_k, (X, X_0), (Y, Y_0)\}$ such that $f_k(x) = \varphi(x)$ for every point $x \in X$. Moreover, it is clear that for every element of the group $H_n(X, X_0; \mathfrak{A})$ there exists a representative $\gamma = \{\gamma_k\} \in Z_n(X, X_0; \mathfrak{A})$ such that all the vertices of γ_k belong to X for every $k = 1, 2, \ldots$ Then $f_k(\gamma_{j_k}) = \varphi(\gamma_{j_k})$, and since the true cycle $\{\varphi(\gamma_{j_k})\}$ is homologous in Y modulo Y_0 to the true cycle $\varphi(\gamma) = \{\varphi(\gamma_k)\}$, we infer that the homomorphisms f_* and φ_* both assign to every element of the group $H_n(X, X_0; \mathfrak{A})$ the same element of the group $H_n(Y, Y_0; \mathfrak{A})$.

In particular:

(11.4) *Homomorphisms induced by fundamental identity classes are identities.*

Moreover, it is clear that if $[f]$ is a fundamental class from (X, X_0) to (Y, Y_0) and $[g]$ is a fundamental class from (Y, Y_0) to (Z, Z_0), then the homomorphism induced by the composition $[g][f]$ is the same as the composition of the homomorphisms induced by $[f]$ and $[g]$. Hence

(11.5) $$([g][f])_* = [g]_*[f]_*.$$

By (11.2), (11.4) and (11.5) we obtain the following

(11.6) Theorem. *If one assigns to each fundamental class $[f]$ from (X, X_0) to (Y, Y_0) the induced homomorphism $[f]: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$, then one gets a covariant functor H_n from the fundamental category $\mathfrak{F}$ to the category $\mathfrak{G}_A$ of Abelian groups.*

(11.7) Corollary. *If the fundamental class $[g]$ from (Y, Y_0) to (X, X_0) is a right inverse of the fundamental class $[f]$ from (X, X_0) to (Y, Y_0), then for each homology group $H_n(X, X_0; \mathfrak{A})$ of (X, X_0) the homomorphism $[g]_*: H_n(Y, Y_0; \mathfrak{A}) \to H_n(X, X_0; \mathfrak{A})$ induced by $[g]$ is a right inverse of the homomorphism $[f]_*: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$ induced by $[f]$.*

If we recall ([1], p. 34) that the existence of a right invertible homomorphism of an Abelian group $\mathfrak{G}$ into another Abelian group $\mathfrak{H}$ implies that $\mathfrak{H}$ is a divisor of $\mathfrak{G}$, we get the following

(11.8) Corollary. *If (X, X_0) fundamentally dominates over (Y, Y_0), then $H_n(Y, Y_0; \mathfrak{A})$ is a divisor of the group $H_n(X, X_0; \mathfrak{A})$.*

(11.9) Corollary. *A fundamental equivalence $[f]$ from (X, X_0) to (Y, Y_0) induces an isomorphism $[f]_*: H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$.*

§ 12. Pointed sequences. Let x_0 be a point of a compactum $X \subset H$ and y_0 a point of a compactum $Y \subset H$. A sequence of maps

$$f_k: (H, x_0) \to (H, y_0)$$

will be said to be a *pointed sequence from the set X* (pointed by x_0) *to the set Y* (pointed by y_0) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/(U, x_0) \simeq f_{k+1}/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k.$$

This pointed sequence will be denoted by $f = \{f_k, (X, x_0), (Y, y_0)\}$. If $g = \{g_k, (X, x_0), (Y, y_0)\}$ is another pointed sequence, then f and g will be said to be *homotopic* (in symbols: $f \simeq g$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/(U, x_0) \simeq g_k/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k.$$

It is clear that the homotopy of pointed sequences is a reflexive, symmetric and transitive relation. Consequently, the collection of all pointed sequences from X (pointed by x_0) to Y (pointed by y_0) decomposes into mutually exclusive sets of homotopic pointed sequences, called *pointed fundamental classes*. The pointed fundamental class with a representative f will be denoted by $[f]$.

If $\varphi: (X, x_0) \to (Y, y_0)$ is a map, then there exists a map $\bar\varphi: (H, x_0) \to (H, y_0)$ such that $\bar\varphi(x) = \varphi(x)$ for every point $x \in X$. Setting $f_k = \bar\varphi$ for every $k = 1, 2, \ldots$, we get a pointed sequence $\{f_k, (X, x_0), (Y, y_0)\}$ which is said to be *generated* by the map φ.

By a slight modification of the argument given in § 4, one shows that:

(12.1) *If $\varphi, \psi: (X, x_0) \to (Y, y_0)$ are maps such that for every neighborhood V of Y the relation $\varphi \simeq \psi$ in (V, y_0) holds, and if the maps $\bar\varphi, \bar\psi: (H, x_0) \to (H, y_0)$ satisfy the condition $\bar\varphi(x) = \varphi(x)$, $\bar\psi(x) = \psi(x)$ for every point $x \in X$, then setting $f_k = \bar\varphi$, $g_k = \bar\psi$ for $k = 1, 2, \ldots$, one gets two homotopic pointed sequences $\{f_k, (X, x_0), (Y, y_0)\}$, $\{g_k, (X, x_0), (Y, y_0)\}$.*

We infer that for every map $\varphi: (X, x_0) \to (Y, y_0)$ and for every map $\bar\varphi: (H, x_0) \to (H, y_0)$ satisfying the condition $\bar\varphi(x) = \varphi(x)$ for every point $x \in X$, the pointed fundamental class $[f]$ with the representative $\{f_k, X, Y, x_0, y_0\}$, where $f_k = \varphi$ for $k = 1, 2, \ldots$, does not depend on the choice of the map $\bar{\varphi}$. Thus we can say that the pointed fundamental class $[\underline{f}]$ is *generated* by the map φ. In particular, the pointed fundamental class generated by the identity map i: $(X, x_0) \to (X, x_0)$ is said to be the *pointed fundamental identity class for* (X, x_0).

Moreover, (12.1) implies that the pointed fundamental class $[\underline{f}]$ generated by a map φ: $(X, x_0) \to (Y, y_0)$ remains fixed if one replaces φ by another map ψ: $(X, x_0) \to (Y, y_0)$ such that $\varphi \simeq \psi$ in (V, y_0) for every neighborhood V of Y. By an argument given in § 2, one can see that in the case of $Y \in \mathrm{ANR}$ the condition $\varphi \simeq \psi$ in (V, y_0) for every neighborhood V of Y can be replaced by the equivalent condition $\varphi \simeq \psi$ in (Y, y_0).

Repeating the proof of (4.3), we get the following proposition:

(12.2) *If two maps* φ, ψ: $(X, x_0) \to (Y, y_0)$ *generate the same pointed fundamental class, then* $\varphi \simeq \psi$ *in* (V, y_0) *for every neighborhood* V *of* Y.

Let us observe that:

(12.3) *If* $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ *is a pointed sequence and* $\{i_k\}$ *is a sequence of indices with* $\lim_{k=\infty} i_k = \infty$, *then setting* $f'_k = f_{i_k}$ *for* $k = 1, 2, \ldots$, *one gets a pointed sequence* $\underline{f}' = \{f'_k, (X, x_0), (Y, y_0)\}$ *homotopic to* $\underline{f}$.

(12.4) *If* $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ *and* $\underline{g} = \{g_k, (Y, y_0), (Z, z_0)\}$ *are two pointed sequences, then their composition* $\underline{gf} = \{g_k f_k, (X, x_0), (Z, z_0)\}$ *is a pointed sequence.*

Let $\{f_k\}$ and $\{f'_k\}$ be two sequences of maps of (H, x_0) into (H, y_0). We say that $\{f'_k\}$ is obtained from $\{f_k\}$ by a *translation* which is *infinitesimal on the set* X if for every $\varepsilon > 0$ there exists a neighborhood U of X such that the inequality

(12.5) $$\varrho\big(f_k(x), f'_k(x)\big) < \varepsilon \quad \text{for every point } x \in U$$

holds for almost all k.

Let us show that:

(12.6) *If* $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ *is a pointed sequence and if the sequence of maps* $\{f'_k\}$ *of* (H, x_0) *into* (H, y_0) *is obtained from* $\{f_k\}$ *by an infinitesimal translation on* X, *then* $\{f'_k, (X, x_0), (Y, y_0)\}$ *is a pointed sequence homotopic to* $\underline{f}$.

Let V be a neighborhood of Y. Then there exists a positive number ε such that $\varrho(y, Y) < 2\varepsilon$ implies $y \in V$. Let V_ε denote the neighborhood of Y consisting of all points $y \in H$ with $\varrho(y, Y) < \varepsilon$. Then there exist a neighborhood U_ε of X and an index k_0 such that

$$f_k(U_\varepsilon) \subset V_\varepsilon \quad \text{for every } k \geqslant k_0 .$$

Since $\{f'_k\}$ is obtained from $\{f_k\}$ by an infinitesimal translation on X, there exist a neighborhood $U \subset U_\varepsilon$ of X and an index $k_1 \geqslant k_0$ such that (12.5) holds for every $k \geqslant k_1$. Setting

$$\varphi_k(x, t) = (1-t)\cdot f_k(x) + t\cdot f'_k(x) \quad \text{for} \quad (x, t) \in H \times \langle 0, 1\rangle ,$$

we get a homotopy joining f_k with f'_k. If $x \in U$ and $k \geqslant k_1$, then (12.5) implies that $\varrho\big(\varphi_k(x, t), f_k(x)\big) < \varepsilon$ for every $0 \leqslant t \leqslant 1$. Since $f_k(x) \in V_\varepsilon$, we infer that

$$\varrho\big(\varphi_k(x, t), Y\big) \leqslant \varrho\big(\varphi_k(x, t), f_k(x)\big) + \varrho\big(f_k(x), Y\big) < 2\varepsilon ,$$

and consequently $\varphi_k(x, t) \in V$. Moreover, $\varphi_k(x_0, t) = y_0$ for every $0 \leqslant t \leqslant 1$. This implies

$$f_k/(U, x_0) \simeq f'_k/(U, x_0) \text{ in } (V, y_0) \quad \text{for every } k \geqslant k_1 ,$$

and the proof of (12.6) is finished.

Moreover:

(12.7) *If* $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$, $\underline{f}' = \{f'_k, (X, x_0)\}, (Y, y_0)\}$, *and* $\underline{g} = \{g_k, (Y, y_0), (Z, z_0)\}$, $\underline{g}' = \{g'_k, (Y, y_0), (Z, z_0)\}$ *are pointed sequences such that* $\underline{f} \simeq \underline{f}'$ *and* $\underline{g} \simeq \underline{g}'$, *then* $\underline{gf} \simeq \underline{g'f'}$.

In fact, since $\underline{g} \simeq \underline{g}'$, there exists for every neighborhood W of Z a neighborhood V of Y such that

$$g_k/(V, y_0) \simeq g'_k/(V, y_0) \text{ in } (W, z_0) \quad \text{for almost all } k .$$

Moreover, $\underline{f} \simeq \underline{f}'$ implies that there is a neighborhood U of X such that

$$f_k/(U, x_0) \simeq f'_k/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k .$$

We infer that

$$g_k f_k/(U, x_0) \simeq g'_k f'_k/(U, x_0) \text{ in } (W, z_0) \quad \text{for almost all } k ,$$

i.e., the pointed sequences $\underline{gf}$ and $\underline{g'f'}$ are homotopic.

It follows that if $[\underline{f}]$ is a pointed fundamental class from X pointed by x_0 to Y pointed by y_0, and if $[\underline{g}]$ is a pointed fundamental class from Y pointed by y_0 to Z pointed by z_0, then all compositions $\underline{gf}$, where $\underline{f}$ is a representative of $[\underline{f}]$ and $\underline{g}$ is a representative of $[\underline{g}]$, belong to one pointed fundamental class (from X pointed by x_0 to Z pointed by z_0). This pointed fundamental class will be called the *composition* of the pointed fundamental classes $[\underline{f}]$ and $[\underline{g}]$. It will be denoted by $[\underline{g}][\underline{f}]$.

It is clear that one gets a category if one considers the pointed compacta lying in H as objects and the pointed fundamental classes as morphisms. The product of morphisms is defined as the composition of pointed fundamental classes and the identities—as pointed fundamental identity classes. Let us call this category the *pointed fundamental category*.

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Moreover, the notion of fundamental domination and the notion of fundamental equivalence introduced in § 8 may be extended to the case of pointed compacta as follows:

The pointed compactum (X, x_0) *fundamentally dominates* the pointed compactum (Y, y_0) provided there exists a pointed sequence $\underline{f}$ from (X, x_0) to (Y, y_0) for which there exists a right inverse, i.e., a pointed sequence $\underline{g}$ from (Y, y_0) to (X, x_0) such that $[\underline{f}][\underline{g}]$ is the pointed fundamental identity class for (Y, y_0).

In the case where also $[\underline{g}][\underline{f}]$ is a pointed fundamental identity class (for (X, x_0)), then the pointed compacta (X, x_0) and (Y, y_0) are said to be *fundamentally equivalent*.

Now let us prove the following

(12.8) THEOREM. *If $Y \in$ ANR, then every pointed fundamental class from X (pointed by x_0) to Y (pointed by y_0) is generated by a map.*

Proof. Since $Y \in$ ANR, there exist a neighborhood W of Y and a retraction $r\colon W \to Y$. Manifestly, there is a map $s\colon H \to H$ such that $s(y) = r(y)$ for every point $y \in W$. Consider now a pointed sequence $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$. Setting $f'_k(x) = sf_k(x)$ for every point $x \in H$, one gets a sequence of maps $f'_k\colon (H, x_0) \to (H, y_0)$ obtained from the sequence $\{f_k\}$ by an infinitesimal translation on X. It follows by (12.6) that $\underline{f}' = \{f'_k, (X, x_0,) (Y, y_0)\}$ is a pointed sequence homotopic to $\underline{f}$. Thus there exist a neighborhood U of X and an index k_0 such that

$$f'_k/(U, x_0) \simeq f'_{k_0}/(U, x_0) \text{ in } (W, y_0) \quad \text{for every } k \geqslant k_0 .$$

Since $r(W) = Y$ and $r(y_0) = y_0$, we infer that

$$rf'_k/(U, x_0) \simeq rf'_{k_0}/(U, x_0) \text{ in } (Y, y_0) \quad \text{for every } k \geqslant k_0 . \tag{12.9}$$

But the map s coincides in W with the retraction r. Hence $rf'_k/(U, x_0) = f'_k/(U, x_0)$ for every $k \geqslant k_0$ and we infer by (12.9) that

$$f'_k/(U, x_0) \simeq f'_{k_0}/(U, x_0) \text{ in } (Y, y_0) \quad \text{for every } k \geqslant k_0 . \tag{12.10}$$

Setting $\varphi(x) = f'_{k_0}(x)$ for every point $x \in X$, we get a map $\varphi\colon (X, x_0) \to (Y, y_0)$ and the homotopy (12.10) implies that the pointed sequence generated by the map φ is homotopic to $\underline{f}'$, whence also to $\underline{f}$. Thus the proof of (12.8) is completed.

§ 13. Approximative maps. Let $x_0 \in X$, $y_0 \in Y$. A sequence of maps $\xi_k\colon (X, x_0) \to (H, y_0)$ will be said to be an *approximative map of (X, x_0) towards (Y, y_0)* if

(13.1) *For every neighborhood V of Y the homotopy $\xi_k \simeq \xi_{k+1}$ in (V, y_0) holds for almost all k.*

Let us denote this approximative map by $\{\xi_k, (X, x_0) \to (Y, y_0)\}$ or, more briefly, by $\underline{\xi}$. It is evident that if $f\colon (X, x_0) \to (Y, y_0)$ is a map, then

setting $\xi_k(x) = f(x)$ for every point $x \in X$ and $k = 1, 2, \ldots$, one gets a sequence $\xi_k\colon (X, x_0) \to (H, y_0)$ being an approximative map $\{\xi_k, (X, x_0) \to (Y, y_0)\}$. We shall say that this approximative map is *generated* by the map f.

The notion of an approximative map is strictly related to the notion of a "*mapping towards a space*" introduced by D. E. Christie ([2], p. 289). The author supposes that the notion of the n-dimensional fundamental group, introduced in § 14 of this paper and based on the notion of approximative maps, differs (for compacta) only formally from the notion of the "n-th weak homotopy group" introduced by D. E. Christie ([2], p. 297).

Let $\{\xi_k, (X, x_0) \to (Y, y_0)\}$ be an approximative map and let $\{\xi'_k\}$ be a sequence of maps $\xi'_k\colon (X, x_0) \to (H, y_0)$ such that for every neighborhood V of Y

$$\xi_k \simeq \xi'_k \text{ in } (V, y_0) \text{ for almost all indices } k . \tag{13.2}$$

We infer by (13.1) that then $\xi'_k \simeq \xi'_{k+1}$ in (V, y_0) for almost all k, i.e., the maps ξ'_k constitute an approximative map $\underline{\xi}' = \{\xi'_k, (X, x_0) \to (Y, y_0)\}$. We say that the approximative map $\underline{\xi}'$ satisfying (13.2) is *homotopic* to $\underline{\xi}$.

It is clear that the homotopy of approximative maps is a reflexive, symmetric and transitive relation. Consequently, the collection of all approximative maps of (X, x_0) towards (Y, y_0) decomposes into mutually exclusive sets of homotopic approximative maps, called *approximative classes from (X, x_0) towards (Y, y_0)*. The approximative class with a representative $\underline{\xi}$ will be denoted by $[\underline{\xi}]$. If an approximative class has a representative generated by a map f, then we say that this approximative class is *generated* by the map f. The approximative class generated by the identity $i\colon (X, x_0) \to (X, x_0)$ is said to be the *approximative identity class for* (X, x_0).

Let us observe that

(13.3) *If $\underline{\xi} = \{\xi_k, (X, x_0) \to (Y, y_0)\}$ is an approximative map and $\{i_k\}$ is a sequence of indices with $\lim_{k=\infty} i_k = \infty$, then setting $\xi'_k = \xi_{i_k}$ for every $k = 1, 2, \ldots$ one gets an approximative map $\underline{\xi}' = \{\xi'_k, (X, x_0) \to (Y, y_0)\}$ homotopic to $\underline{\xi}$.*

Let $\{\xi_k\}$ and $\{\xi'_k\}$ be two sequences of maps (X, x_0) into (H, y_0). We say that $\{\xi'_k\}$ is obtained from $\{\xi_k\}$ by an *infinitesimal translation* if there exists a sequence $\{\varepsilon_k\}$ of positive numbers converging to zero and such that

$$\varrho(\xi_k(x), \xi'_k(x)) < \varepsilon_k \quad \text{for every point } x \in X \text{ and } k = 1, 2, \ldots$$

It is clear that

(13.4) *If* $\underline{\xi} = \{\xi_k, (X, x_0) \to (Y, y_0)\}$ *is an approximative map, then every sequence* $\{\xi'_k\}$ *obtained from* $\{\xi_k\}$ *by an infinitesimal translation is an approximative map homotopic to* $\underline{\xi}$.

(13.5) *If* $\underline{\xi} = \{\xi_k, (X, x_0) \to (Y, y_0)\}$ *is an approximative map and* $\underline{f} = \{f_k, (Y, y_0), (Z, z_0)\}$ *is a pointed sequence, then the maps* $f_k \xi_k: (X, x_0) \to (H, y_0)$ *constitute an approximative map of* (X, x_0) *towards* (Z, z_0).

The approximative map $\{f_k \xi_k, (X, x_0) \to (Z, z_0)\}$ will be denoted by $\underline{f\xi}$ and called the *composition* of the approximative map $\underline{\xi}$ and of the pointed sequence $\underline{f}$.

Let us prove the following proposition:

(13.6) *If* $\underline{\xi} = \{\xi_k, (X, x_0) \to (Y, y_0)\}$ *and* $\underline{\eta} = \{\eta_k, (X, x_0) \to (Y, y_0)\}$ *are two homotopic approximative maps and if* $\underline{f} = \{f_k, (Y, y_0), (Z, z_0)\}$ *and* $\underline{g} = \{g_k, (Y, y_0), (Z, z_0)\}$ *are two homotopic pointed sequences, then the approximative maps* $\underline{f\xi}$ *and* $\underline{g\eta}$ *are homotopic.*

In fact, for every neighborhood W of Z there exists a neighborhood V of Y such that

$$f_k/(V, y_0) \simeq g_k/(V, y_0) \text{ in } (W, z_0) \quad \text{for almost all } k.$$

Moreover, the homology $\xi_k \simeq \eta_k$ in (V, y_0) holds for almost all k. We infer that

$$f_k \xi_k \simeq g_k \eta_k \text{ in } (W, z_0) \quad \text{for almost all } k,$$

i.e., the approximative maps $\underline{f\xi}$ and $\underline{g\eta}$ are homotopic.

It follows by (13.5) and (13.6) that for every approximative class from (X, x_0) towards (Y, y_0) and for every pointed fundamental class $[\underline{f}]$ from (Y, y_0) to (Z, z_0) all compositions $\underline{f\xi}$, where $\underline{\xi}$ is a representative of $[\underline{\xi}]$, and $\underline{f}$ is a representative of $[\underline{f}]$, belong to one approximative class called the *composition* of the approximative class $[\underline{\xi}]$ and of the pointed fundamental class $[\underline{f}]$. We shall denote it by $[\underline{f}][\underline{\xi}]$.

Now let us prove the following

(13.7) THEOREM. *If* $Y \in$ ANR, *then every approximative class from* (X, x_0) *towards* (Y, y_0) *is generated by a map.*

Proof. Since $Y \in$ ANR, there exists a closed neighborhood W of Y and a retraction $r: W \to Y$. Manifestly, there is a map $s: (H, y_0) \to (H, y_0)$ such that $s(y) = r(y)$ for every point $y \in W$.

Consider now an approximative map $\underline{\xi} = \{\xi_k, (X, x_0) \to (Y, y_0)\}$. Setting $\xi'_k(x) = s\xi_k(x)$ for every point $x \in X$, one gets a sequence of maps $\xi'_k: (X, x_0) \to (H, y_0)$ obtained from the sequence $\{\xi_k\}$ by an infinitesimal translation. It follows by (13.4) that $\underline{\xi}' = \{\xi'_k, (X, x_0) \to (Y, y_0)\}$ is an approximative map homotopic to $\underline{\xi}$. Thus there exists an index k_0 such that

$$\xi'_k \simeq \xi'_{k_0} \text{ in } (W, y_0) \quad \text{for every } k \geqslant k_0.$$

Since $r(W) = Y$ and $r(y_0) = y_0$, we infer that

(13.8) $$r\xi'_k \simeq r\xi'_{k_0} \text{ in } (Y, y_0) \quad \text{for every } k \geqslant k_0.$$

But the map s coincides in W with retraction r. Hence $r\xi'_k = \xi'_k$ for every $k \geqslant k_0$ and we infer by (13.8) that

13.9) $$\xi'_k \simeq \xi'_{k_0} \text{ in } (Y, y_0) \quad \text{for every } k \geqslant k_0.$$

Setting $\varphi(x) = \xi'_{k_0}(x)$ for every point $x \in X$, we get a map $\varphi: (X, x_0) \to (Y, y_0)$ and the homotopy (13.9) implies that the approximative map generated by the map φ is homotopic to $\underline{\xi}'$, whence also to $\underline{\xi}$. Thus the proof of (13.7) is completed.

§ 14. Approximative maps of spheres. Let $S = S^n$ denote the n-dimensional sphere in H consisting of all points $(x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots) \in H$ such that $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$. Let c be a point of S. By an *n-dimensional ball* on S with *centre* c and *radius* ε we understand the set P consisting of all points $x \in S$ such that $\varrho(x, c) \leqslant \varepsilon$. By $\mathring{P}$ we shall denote the *interior* of P, i.e., the set consisting of all points $x \in P$ with $\varrho(x, c) < \varepsilon$.

Let a be a point of S and y_0 a point of a compactum $Y \subset H$, and let $\underline{\xi} = \{\xi_k, (S, a) \to (Y, y_0)\}$ be an approximative map. Consider an n-dimensional ball P on S with radius $\varepsilon < 1$ such that $a \in S - \mathring{P}$. Manifestly, there exists a homotopy $\alpha: S \times \langle 0, 1\rangle \to S$ such that

(14.1) $$\begin{cases} \alpha(x, 0) = x & \text{for every point } x \in S, \\ \alpha(a, t) = a & \text{for every } 0 \leqslant t \leqslant 1, \\ \alpha(S - \mathring{P}, 1) = (a). \end{cases}$$

Setting

(14.2) $$\bar{\varphi}_k(x, t) = \xi_k \alpha(x, t) \quad \text{for every } (x, t) \in S \times \langle 0, 1\rangle,$$

we get a homotopy

$$\bar{\varphi}_k: (S \times \langle 0, 1\rangle, a \times \langle 0, 1\rangle) \to (\xi_k(S), y_0)$$

joining in $(\xi_k(S), y_0)$ the map $\xi_k: (S, a) \to (\xi_k(S), y_0)$ with the map $\hat{\xi}_k: (S, a) \to (\xi_k(S), y_0)$ given by the formula

(14.3) $$\hat{\xi}_k(x) = \bar{\varphi}_k(x, 1) \quad \text{for every point } x \in S.$$

It follows by (14.1) and (14.3) that

(14.4) $$\hat{\xi}_k(S - \mathring{P}) = (y_0).$$

Now let $\underline{\eta} = \{\eta_k: (S, a) \to (Y, y_0)\}$ be another approximative map and let Q be another n-dimensional ball on S having the interior $\mathring{Q} \subset S - \mathring{P}$

and such that $a \in S - \mathring{Q}$. By an analogous argument we infer that for every $k = 1, 2, \ldots$ there exists a map $\hat{\eta}_k\colon (S, a) \to (\eta_k(S), y_0)$ homotopic in $(\eta_k(S), y_0)$ to η_k and such that

$$\hat{\eta}_k(S - \mathring{Q}) = (y_0). \tag{14.5}$$

It follows by (14.4) and (14.5) that the formula

$$\zeta_k(x) = \begin{cases} \hat{\xi}_k(x) & \text{for every point } x \in S - \mathring{Q}, \\ \hat{\eta}_k(x) & \text{for every point } x \in S - \mathring{P} \end{cases}$$

defines a map $\zeta_k\colon (S, a) \to (H, H)$ satisfying the condition $\zeta_k(a) = y_0$. This map is called the *join* of the maps $\hat{\xi}_k$ and $\hat{\eta}_k$.

Now let V be a neighborhood of Y. Then there exists an index k_0 such that

$$\xi_k \simeq \xi_{k+1} \text{ in } (V, y_0), \qquad \eta_k \simeq \eta_{k+1} \text{ in } (V, y_0) \qquad \text{for every } k \geqslant k_0.$$

It is known ([1], p. 46) that then

$$\zeta_k \simeq \zeta_{k+1} \text{ in } (V, y_0) \qquad \text{for every } k \geqslant k_0.$$

Thus the sequence of joins ζ_k of maps $\hat{\xi}_k$ and $\hat{\eta}_k$ is an approximative map $\underline{\zeta} = \{\zeta_k, (S, a) \to (Y, y_0)\}$. This approximative map is said to be a *join* of the approximative maps $\underline{\xi}$ and $\underline{\eta}$.

If $\underline{\xi}' = \{\xi'_k, (X, x_0) \to (Y, y_0)\}$ and $\underline{\eta}' = \{\eta'_k, (X, x_0) \to (Y, y_0)\}$ are approximative maps homotopic to $\underline{\xi}$ and $\underline{\eta}$ respectively, and if $\hat{\xi}'_k$ and $\hat{\eta}'_k$ are maps obtained from ξ'_k and η'_k in the way in which the maps $\hat{\xi}_k$ and $\hat{\eta}_k$ have been obtained from ξ_k and η_k, then there exists an index k_1 such that $\hat{\xi}'_k \simeq \hat{\xi}_k$ in (V, y_0) and $\hat{\eta}'_k \simeq \hat{\eta}_k$ in (V, y_0) for every $k \geqslant k_1$. One can easily see (comp. [1], p. 46) that the join ζ'_k of $\hat{\xi}'_k$ and $\hat{\eta}'_k$ is homotopic to the join ζ_k of $\hat{\xi}_k$ and $\hat{\eta}_k$ in (V, y_0) for every $k \geqslant k_1$. It follows that the approximative map $\underline{\zeta}' = \{\zeta'_k, (S, a) \to (Y, y_0)\}$ is homotopic to $\underline{\zeta}$. Hence the approximative class $[\underline{\zeta}]$ does not depend on the choice of the representatives $\underline{\xi}$ and $\underline{\eta}$ of the approximative classes $[\underline{\xi}]$ and $[\underline{\eta}]$.

Moreover, it is known ([1], p. 47) that in the case of $n > 1$ if one replaces the balls P, Q and the homotopy a by balls P^*, Q^* and by a homotopy a^* satisfying analogous conditions, then one obtains instead of the approximative $\underline{\zeta}$ another approximative map $\underline{\zeta}^* = \{\zeta^*_k, (S, a) \to (Y, y_0)\}$ satisfying for every neighborhood V of Y the condition

$$\zeta_k \simeq \zeta^*_k \text{ in } (V, y_0) \qquad \text{for almost all } k.$$

Hence the approximative maps $\underline{\zeta}$ and $\underline{\zeta}^*$ are both representatives of the same approximative class. This class is said to be the *join* of the approximative classes $[\underline{\xi}]$ and $[\underline{\eta}]$.

In the case of $n = 1$, the last relation requires the hypothesis (comp. [1], p. 47) that the orientation on S of the triple a, b, c, where $b \in \mathring{P}$ and $c \in Q$, is the same as the orientation on S of the triple a, b^*, c^*, where $b^* \in \mathring{P}^*$ and $c^* \in \mathring{Q}^*$.

Since for homotopy classes of maps of (S, a) into (V, y_0) the operation of the join is associative ([1], p. 48), and for $n > 1$, commutative, we infer that the operation of the join, defined for approximative classes from (S, a) towards (Y, y_0), is also associative, and for $n > 1$, commutative. Manifestly, the approximative class generated by the constant map f_0 assigning to every point $x \in S$ the point y_0 is the module for the operation of the join. In order to define the inverse for a given approximative class with a representative $\underline{\xi} = \{\xi_k, (S, a) \to (Y, y_0)\}$ let P and Q be open half-spheres of S with a common boundary R containing the point a. Let $s(x)$ denote, for every point $x \in S$, a point symmetric to x relative to the n-dimensional hyperplane containing R. Consider, for every $k = 1, 2, \ldots$, the map $\hat{\xi}_k$ given by formula (14.3). Then $\hat{\underline{\xi}} = \{\hat{\xi}_k, (S, a) \to (Y, y_0)\}$ is an approximative map homotopic to $\underline{\xi}$ and satisfying condition (14.4). Setting

$$\bar{\xi}_k(x) = \hat{\xi}_k s(x) \qquad \text{for every point } x \in S,$$

one obtains a sequence of maps $\bar{\xi}_k\colon (S, a) \to (H, H)$ and it is easy to see that, for every neighborhood V of Y, the homotopy

$$\bar{\xi}_k \simeq \bar{\xi}_{k+1} \text{ in } (V, y_0)$$

holds for almost all k. Thus $\bar{\underline{\xi}} = \{\bar{\xi}_k, (S, a) \to (Y, y_0)\}$ is an approximative map. Moreover, it is known ([1], p. 49) that the join of $\hat{\xi}_k$ and $\bar{\xi}_k$ is homotopic in (V, y_0) to the constant map f_0. Hence the join of the approximative classes with representatives $\underline{\xi}$ and $\bar{\underline{\xi}}$ is the approximative class generated by f_0.

Thus we infer that the approximative classes from (S, a) towards (Y, y_0) with the operation of the join, constitute a group which is abelian for $n > 1$. Let us call this group the *n-th fundamental group of the pair* (Y, y_0) and let us denote it by $\underline{\pi}_n(Y, y_0)$.

Remark. In order to avoid a misunderstanding, let us explicitly notice that the fundamental group $\underline{\pi}_n(Y, y_0)$ differs in general from the n-dimensional homotopy group $\pi_n(Y, y_0)$. For instance, if Y denotes the n-dimensional continuum which we obtain from the curve X, considered in the Remark in § 9, by iterating $n-1$ times the operation of suspension, and if $y_0 \in Y$, then one can easily see that the n-th fundamental group $\underline{\pi}_n(Y, y_0)$ is infinite and cyclic and that the n-th homotopy group $\pi_n(Y, y_0)$ is trivial. Thus the classical term "*fundamental group*" as the synonym for the term "*1-dimensional homotopy group*" will be excluded here.

If $Y \in \mathrm{ANR}$, then we infer by Theorem (13.7) that each approximative class from (S, a) towards (Y, y_0) is generated by a map of (S, a) into (Y, y_0). One can easily see that the join of two approximative classes generated by two maps $f\colon (S, a) \to (Y, y_0)$ and $g\colon (S, a) \to (Y, y_0)$, respectively, is the approximative class generated by the homotopy join of the maps f and g. Since the collection of all homotopy classes of maps of (S, a) into (Y, y_0) with the operation of the homotopic join may be considered as the n-th homotopy group $\pi_n(Y, y_0)$ in the classical sense (comp. [1], p. 50), we infer that

(14.6) *If $Y \in$ ANR, then the n-th fundamental group $\underline{\pi}_n(Y, y_0)$ is isomorphic to the n-th homotopy group $\pi_n(Y, y_0)$.*

§ 15. Homomorphisms of fundamental groups induced by pointed fundamental classes. Let $[\underline{f}]$ be a pointed fundamental class and let $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ be a representative of it. Let a be a point of the n-dimensional sphere S and let $\underline{\xi} = \{\xi_k, (S, a) \to (X, x_0)\}$ be a representative of an approximative class $[\underline{\xi}]$. By (13.5) the maps $f_k\xi_k\colon (S, a) \to (H, y_0)$ constitute an approximative map $\underline{f}\underline{\xi} = \{f_k\xi_k, (S, a) \to (Y, y_0)\}$ and we infer by (13.6) that its approximative class does not change if we replace the pointed sequence $\underline{f}$ by another representative $\underline{f}' = \{f_k, (X, x_0), (Y, y_0)\}$ of the pointed fundamental class $[\underline{f}]$ and the approximative map $\underline{\xi}$ by another approximative map $\underline{\xi}' = \{\xi_k, (X, x_0) \to (Y, y_1)\}$ homotopic to it. Thus the approximative class with the representative $\{f_k\xi_k, (S, a) \to (Y, y_0)\}$ depends only on the pointed fundamental class $[\underline{f}]$ and on the approximative class $[\underline{\xi}]$.

Now let us assume that $S = P \cup Q$, where P, Q are n-dimensional balls on S with disjoint interiors $\mathring{P}$ and $\mathring{Q}$ and with $a \in R = S - \mathring{P} - \mathring{Q}$. Let

$$\underline{\xi} = \{\xi_k, (S, a) \to (X, x_0)\} \quad \text{and} \quad \underline{\eta} = \{\eta_k, (S, a) \to (X, x_0)\}$$

be two approximative maps such that

$$\xi_k(x) = x_0 \text{ for every } x \in S - \mathring{P} \quad \text{and} \quad \eta_k(x) = x_0 \text{ for every } x \in S - Q\,.$$

Then the join $\underline{\zeta} = \{\zeta_k, (S, a) \to (X, x_0)\}$ of $\underline{\xi}$ and $\underline{\vartheta}$ is defined by the formula

$$\zeta_k(x) = \begin{cases} \xi_k(x) & \text{for} \quad x \in S - \mathring{Q}\,, \\ \eta_k(x) & \text{for} \quad x \in S - \mathring{P}\,. \end{cases}$$

Now, if $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ is a pointed sequence, then it is clear that the map $f_k\zeta_k\colon (S, a) \to (H, y_0)$ is the join of the maps $f_k\xi_k\colon (S, a) \to (H, y_0)$ and $f_k\eta_k\colon (S, a) \to (H, y_0)$. Consequently, the approximative class of $\{f_k\zeta_k, (S, a) \to (Y, y_0)\}$ is the join of the approximative class of $\{f_k\xi_k, (S, a) \to (Y, y_0)\}$ and of the approximative class of $\{f_k\xi_k, (S, a) \to (Y, y_0)\}$, i.e., the operation of the composition of the approximative classes from (S, a) towards (X, x_0) and of the pointed fundamental classes from (X, x_0) to (Y, y_0) is commutative with the operation of the join of approximative classes. It follows that if we assign to every approximative class $[\underline{\xi}]$ from (S, a) towards (X, x_0), which may be considered as an element of the n-th fundamental group $\underline{\pi}_n(X, x_0)$, the approximative class $[\underline{f}][\underline{\xi}]$ from (S, a) towards (Y, y_0), which is an element of the group $\underline{\pi}_n(Y, y_0)$, we get a homomorphism of the group $\underline{\pi}_n(X, x_0)$ into the group $\underline{\pi}_n(Y, y_0)$. We say that this homomorphism is *induced by the pointed fundamental class* $[\underline{f}]$ and we denote it by $[\underline{f}]_*$. Thus:

(15.1) *Every pointed fundamental class $[\underline{f}]$ from (X, x_0) to (Y, y_0) induces a homomorphism $[\underline{f}]_*\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$.*

It is evident that:

(15.2) *The homomorphism of the fundamental group $\underline{\pi}_n(X, x_0)$ induced by the pointed fundamental identity class of (X, x_0) is the identity homomorphism.*

Moreover, it is clear that:

(15.3) *If $[\underline{f}]$ is a pointed fundamental class from (X, x_0) to (Y, y_0) and $[\underline{g}]$ is a pointed fundamental class from (Y, y_0) to (Z, z_0), then the homomorphism of $\underline{\pi}_n(X, x_0)$ into $\underline{\pi}_n(Z, z_0)$ induced by the composition $[\underline{g}][\underline{f}]$ of these pointed fundamental classes is the composition of the homomorphisms induced by these pointed fundamental classes, i.e., $([\underline{g}][\underline{f}])_* = [\underline{g}]_*[\underline{f}]_*$.*

By (15.1), (15.2) and (15.3) we obtain the following

(15.4) THEOREM. *If one assigns to each pointed fundamental class $[\underline{f}]$ from (X, x_0) to (Y, y_0) the induced homomorphism $[\underline{f}]_*\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$, then one gets a covariant functor Π_n from the fundamental category $\mathfrak{F}$ to the category $\mathfrak{A}$ of groups (abelian for $n > 1$).*

In the case where the pointed fundamental class $[\underline{f}]$ from (X, x_0) to (Y, y_0) is generated by a map $f\colon (X, x_0) \to (Y, y_0)$, then we denote the induced homeomorphism $[\underline{f}]_*\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$ also by f_* and we say that it is *induced by the map* f.

(15.5) COROLLARY. *If the pointed fundamental class $[\underline{g}]$ from (Y, y_0) to (X, x_0) is a right inverse of the pointed fundamental class $[\underline{f}]$ from (X, x_0) to (Y, y_0), then the homomorphism $[\underline{g}]_*\colon \underline{\pi}_n(Y, y_0) \to \underline{\pi}_n(X, x_0)$ induced by $[\underline{g}]$ is a right inverse of the homomorphism $[\underline{f}]_*\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$ induced by $[\underline{f}]$.*

If we recall that the existence of a right-invertible homomorphism of an abelian group $\mathfrak{G}$ into another abelian group $\mathfrak{H}$ implies that the group $\mathfrak{H}$ is a divisor of the group $\mathfrak{G}$ ([1], p. 34), we infer that

(15.6) *If (X, x_0) fundamentally dominates (Y, y_0), then the group $\underline{\pi}_n(Y, y_0)$ is a divisor of the group $\underline{\pi}_n(X, x_0)$ for every $n = 2, 3, \ldots$*

(15.7) *A fundamental equivalence $\underline{f}$ from (X, x_0) to (Y, y_0) induces an isomorphism $[\underline{f}]\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$ for every $n = 1, 2, \ldots$*

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The topology of a partially well ordered set*

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1. Introduction. If $(X, \leqslant)$ is a partially ordered set, there are many known ways of using the order properties of X to define certain natural or "intrinsic" topologies on X. In particular, we may define the well-known *interval topology* $\mathfrak{I}$ on X by taking all sets of the form $\{x \in X\colon x \leqslant a\}$ or $\{x \in X\colon x \geqslant b\}$ as a sub-base for the closed sets. We also define another topology $\mathfrak{D}$ on X, which we call its *Dedekind topology*, as follows. A subset A of X is said to be *up-directed* (*down-directed*) if and only if for all $x \in A$ and $y \in A$ there exists $z \in A$ with $x \leqslant z$, $y \leqslant z$ ($x \geqslant z$, $y \geqslant z$). A subset containing a greatest element is trivially up-directed, and dually. Following McShane [8], we call a subset K of X *Dedekind-closed* if and only if whenever A is an up-directed subset of K and $y = \text{l.u.b.}\, A$, or A is a down-directed subset of K and $y = \text{g.l.b.}\, A$, we also have $y \in K$. We then define $\mathfrak{D}$ as the topology whose closed sets are precisely the Dedekind-closed subsets of X. It is clear that $\mathfrak{I} \subseteq \mathfrak{D}$ for all partially ordered sets X. In [12], we called an arbitrary topology $\mathfrak{T}$ on X *order-compatible* if and only if $\mathfrak{I} \subseteq \mathfrak{T} \subseteq \mathfrak{D}$.

Let us say that a subset A of X is *totally unordered* if and only if x and y are incomparable (with respect to the order $\leqslant$) for all $x, y \in A$ with $x \neq y$. Naito [9] showed that if every totally unordered subset of X is finite, then X possesses a unique order-compatible topology (i.e., the topologies $\mathfrak{I}$ and $\mathfrak{D}$ coincide).

A partially ordered set X is called *partially well ordered* (pwo) if and only if all totally unordered subsets of X are finite and all chains in X are well ordered. The purpose of this paper is to study some of the properties of the unique order-compatible topology of a pwo-set X. We call this topology on X its *intrinsic topology*. Among our results, we characterize the convergent nets and the closure operation in this topology in "order-theoretic" terms. We show that the intrinsic topology is completely regular for any pwo-set X, and may be obtained from a certain natural proximity relation definable in terms of the ordering in X. The normal

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