Case 2. Suppose that y has more than one G-cover. Then each $x \in D$ has at least one G-cover. Let $K = \{x \in D\colon x$ has precisely one G-cover$\}$. We assert that K is not cofinal in D. For if K is cofinal, then for some $a \in G$ the set $K_a = \{x \in K\colon a$ is a G-cover of $x\}$ is cofinal in K and hence in D. But then $y = \text{l.u.b.}\, K_a$, so that a is a G-cover of y. If b is another G-cover of y, then b is incomparable with a. But $b \geqslant x$ for all $x \in K_a$, so that b is another G-cover for each $x \in K_a$: contradiction. Hence $D-K$ contains a residual subset E of D; and, as in Case 1, for all $x \in E$ we have $f_G(x) = f_G(y) = z$.

Case 3. Suppose that y has precisely one G-cover a. If $R = \{x \in D\colon$ a is the only G-cover of $x\}$, then a simple argument (similar to the cases above) shows that R contains a residual subset E of D. Also, for all $x \in E$ we have $f_G(x) = f_G(y) = f(a)$, by definition of f_G.

Thus in each of the above three cases we have a residual (and hence cofinal) subset E of D such that $f_G(y) = \text{l.u.b.}\, f_G[E]$. The continuity of f_G now follows by the lemma.

Sierpiński has shown in [11] that Theorem 15 does not remain valid, even when X and X' are *well ordered*, if "net" is replaced by "transfinite sequence".

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On the hyperspace of subcontinua of a finite graph, I

by

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§ 1. Introduction. Let X be a compact metric continuum with a metric ϱ. Throughout the paper $C(X)$ will denote the hyperspace of all non-empty subcontinua of X metrized by the Hausdorff metric ϱ^1 (shortly, the hyperspace for X):

$$\varrho^1(A, B) = \max[\sup_{a \in A} \varrho(a, B), \sup_{b \in B} \varrho(A, b)].$$

It has been known for a long time that $C(X)$ with the metric ϱ^1 is also a compact metric continuum, and some other properties of $C(X)$ have also been proved (cf. for instance Wojdysławski [10], Kelley [4], Duda [1], and Segal [6]). However, no characterization of spaces $C(X)$ has as yet appeared. Even what $C(X)$ is like is so far known for only a few and very simple continua X (after all, mainly in folk-lore).

The aim of the present paper is to inquire into the structure of spaces $C(X)$ in the case which seems to be natural to start with, that is in the case of spaces $C(X)$ which are locally connected and have finite dimension. The results obtained here uncover some features of their polyhedral structure and may eventually lead to their topological characterization (cf. remark following corollary 9.2).

As Vietoris [8] and Ważewski [9] have proved, continuum $C(X)$ is locally connected if and only if continuum X is locally connected, and it is fairly easy to show (cf. Kelley [4]) that the dimension of a locally connected continuum $C(X)$ is finite if and only if continuum X is a finite connected graph. Hence

1.1. *Continuum $C(X)$ is locally connected and of finite dimension if and only if continuum X is a finite connected graph.*

To gain our aims we shall proceed as follows. We start with a finite connected graph X dividing its hyperspace $C(X)$ into finitely many closed subsets $\mathfrak{M}_\mu$ which turn to be topological balls. Moreover, the decomposition of $C(X)$ into these balls (cells) is a good one (for X acyclic, cellular), and so in this way we come first to theorem 6.4 stating that $C(X)$ is a polyhedron if and only if X is a finite graph. This polyhedron is then subjected to an analysis resulting in formulas for its dimension (theorem 7.4). However, a most interesting and somewhat unexpected result is theorem 8.4, which states that polyhedron $C(X)$ determines in a natural way the set homeomorphic to the graph X (but with the exception of an arc and a simple closed curve, both having a 2-dimensional ball as hyperspace, see § 3) and this yields a sequence of immediate corollaries given in § 9. We infer hence the uniqueness of X for $C(X)$ (theorem 9.1: if a polyhedron P distinct from a 2-dimensional ball is a hyperspace, then it is a hyperspace for one X only); moreover, we obtain theorem 9.3, stating that for each $k \geqslant 2$ there are only finitely many k-dimensional polyhedra which are hyperspaces, etc. Another result worth mentioning is, perhaps, theorem 9.7, stating that in all but one case (a 2-dimensional ball) the polyhedron $C(X)$ is topologically prime, i.e. that it is not homeomorphic to a Cartesian product of two subspaces distinct from it.

§ 2. Topological lemma. The following simple lemma on combining homeomorphisms will be useful.

2.1. *Let a topological space Y be the union of finitely many compact sets Y_μ, where $\mu \in M$ and M is finite. If for any $\mu \in M$ there exists a homeomorphism $f_\mu\colon Y \to Z$ into a topological space Z satisfying the two conditions*

(i) $$f_\mu | Y_\mu \cap Y_\nu = f_\nu | Y_\mu \cap Y_\nu ,$$

(ii) $$f_\mu(Y_\mu \cap Y_\nu) = f_\mu(Y_\mu) \cap f_\nu(Y_\nu) = f_\nu(Y_\mu \cap Y_\nu)$$

for all indices $\mu \in M$ and $\nu \in M$, then the combined function $f\colon Y \to Z$ given by the formula

(1) $$f | Y_\mu = f_\mu \quad \textit{for} \quad \mu \in M ,$$

is also a homeomorphism.

Proof. By virtue of (i) the function f satisfying (1) does exist.

Function f is continuous. In fact, if F is any closed subset of Z, then it follows from the continuity of f_μ, $\mu \in M$, that $f_\mu^{-1}(F)$ is a closed subset of Y_μ. Since Y_μ is closed in Y, this implies that $f_\mu^{-1}(F)$ is a closed subset of Y. Since M is finite, then

$$f^{-1}(F) = \bigcup_{\mu \in M} f_\mu^{-1}(F)$$

is a closed subset of Y. Hence f is continuous.

Function f is one-to-one. For if $x \in Y$, $y \in Y$ and $f_\mu(x) = f_\nu(y)$, then by (ii) there exist points $x_0 \in Y_\mu \cap Y_\nu$ and $y_0 \in Y_\mu \cap Y_\nu$ such that

$$f_\mu(x_0) = f_\mu(x) = f_\nu(y) = f_\nu(y_0) .$$

But functions f_μ and f_ν are one-to-one by hypothesis, and so $x_0 = x$ and $y_0 = y$, and since function $f_\mu | Y_\mu \cap Y_\nu = f_\nu | Y_\mu \cap Y_\nu$ is one-to-one a fortiori, we have also $x_0 = y_0$. Hence $x = y$ and so f is one-to-one.

Being continuous and one-to-one on a compact space $Y = \bigcup_{\mu \in M} Y_\mu$, the function f is a homeomorphism.

§ 3. Two simple examples. Before considering the general case we shall examine here briefly hyperspace $C(X)$ where X is an arc or a simple closed curve.

1. X is an arc with end-points v and w. Without loss of generality we may assume that X is a segment $[0, 1]$ of the x-axis lying in the xy-plane and that $v = (0, 0)$ and $w = (1, 0)$. Under this assumption each subcontinuum C of X is determined by its middle point m_C and the diameter $\delta(C)$. The formula

$$g(C) = \big(m_C, \delta(C)\big) \quad \text{for} \quad C \in C(X)$$

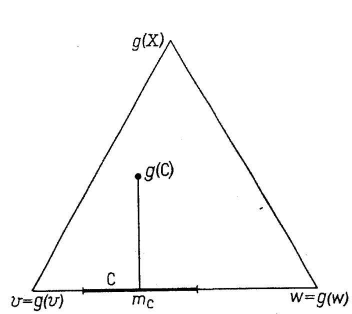

Fig. 1

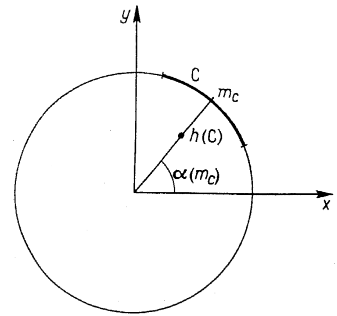

Fig. 2

then yields a one-to-one correspondence

$$g\colon C(X) \to T$$

from $C(X)$ onto the triangle T of vertices $g(v) = v$, $g(w) = w$ and $g(X) = (\frac{1}{2}, 1)$, and it is clearly obvious that g is continuous (fig. 1). Hence g is a homeomorphism between $C(X)$ and T.

Note that the edge of T joining $g(v)$ (resp. $g(w)$) to $g(X)$ (top vertex) represents subcontinua of X containing v (resp. w), and the edge joining $g(v)$ to $g(w)$ represents points of X (in the notation of the next section it is equal to $\varphi(X)$).

2. X is a simple closed curve. As before, we may assume that X is a unit circle

$$X = \{(x, y)\colon x^2 + y^2 = 1\} .$$

Under this assumption each proper subcontinuum C of X is, as a subarc of X, determined by its middle point m_C and the diameter $\delta(C)$, $0 \leqslant \delta(C) < 2\pi$.

Let D be the disc bounded by X. Introducing polar coordinates into the xy-plane, we may describe each point p of D by the pair $(\alpha(p), r(p))$, where $\alpha(p)$ is the angle under which p is seen from the pole $(0,0)$ if x-axis is the polar line and $r(p)$ denotes the distance between p and the pole.

The formulas

$$h(C) = (0,0) \quad \text{for} \quad C = X$$

and

$$h(C) = \big(\alpha(m_C), 1-\delta(C)/2\pi\big) \quad \text{for} \quad C \neq X\,, \ C \in C(X)\,,$$

yield a one-to-one correspondence

$$h\colon\ C(X) \to D$$

from $C(X)$ onto the disc D, and it is not difficult to see that h is also continuous and thus is a homeomorphism between $C(X)$ and D (fig. 2).

Note that in both examples considered here the hyperspace is topologically a 2-dimensional ball. It follows that one polyhedron may be a hyperspace for two topologically distinct graphs (in our case, for an arc and a simple closed curve). However, we shall proceed to show (cf. theorem 9.1) that if a polyhedron P distinct from a 2-dimensional ball is a hyperspace, then there exists one only continuum X (by theorem 6.4, it must be a finite connected graph) such that $P \underset{\text{top}}{=} C(X)$. Hence the case of a 2-dimensional ball is exceptional, and thus it is not surprising that this ball has also some other properties not shared by other hyperspaces (cf. 9.7).

§ 4. Preliminaries. From now on we assume that X is a finite connected graph, i.e. a point or a connected union of finitely many segments in which any two segments either are disjoint or meet at one or two of their end-points only. By a *segment* of X we shall always mean one of those segments (note, however, that two end-points of a segment of X may coincide, and then such a "segment" is topologically a simple closed curve), and by a *subgraph* of X—a graph contained in X and formed by some of those segments and their end-points. In particular, an end-point of a segment of X is a subgraph of X.

The end-points of segments of X are called *vertices* of X. For each vertex $v \in X$ we then have either $\operatorname{ord}_v X = 1$ if v is an end-point of X, or $\operatorname{ord}_v X \geqslant 2$ otherwise. If $\operatorname{ord}_v X \geqslant 3$, then v is called a *ramification point* of X.

A topology on X is the *identification topology* induced by embeddings of segments of X into X, and so by a metric on X we can mean a geodesic metric ϱ in which

(α) *each segment of X has length equal to* 1 *and the distance between any two of its points is equal to the length of the shortest arc joining them.*

However, it may happen under condition (α) that X contains an arc which consists of several segments of X and joins a ramification point to an end-point or to another ramification point (which may coincide with the first) without passing through any other ramification point. Wishing to have a metric in X as economical as possible we can then re-metrize X by replacing all such arcs by single segments and so—provided X is topologically distinct from a simple closed curve, which is the only case where this is not possible—we may also assume that (α) still holds true (for new segments) and the following condition (β) is also fulfilled:

(β) *each vertex of X is either an end-point or a ramification point of X.*

For some of our purposes, however, the metric in X satisfying both (α) and (β) may happen to be too coarse, and so occasionally we shall consider yet another metric in X yielded by introducing into X new vertices dividing some "bad" segments into two or even into three (as the case may be) and in a way that (α) holds and instead of (β) the following condition (γ) is satisfied:

(γ) *for any two vertices of X which are at a distance* 1 *from each other there exists only one segment of X joining them.*

Under condition (γ) we may denote the segment of end-points v and w by $\overline{vw}$.

Conditions (α), (β) and (γ) are compatible if X is acyclic, that is if X does not contain any simple closed curve. In general, however, (β) and (γ) are not compatible and so we shall have to make a choice between them in any case where a specification of a metric in X is needed. Generally speaking, in § 5 we assume condition (α) only, in § 6 conditions (α) and (γ), and in § 7, § 8 and § 9 conditions (α) and (β). We shall have a chance to recall this also later on.

Note that the fulfillment of conditions (β) or (γ) in addition to condition (α) often involves a change of graph structure in X. Nevertheless, it does not alter the topology which for us is the only important structure in X, and so from a topological point of view, it is irrelevant what kind of metric we use.

Let us recall here some notions and notations related to hyperspaces (cf. Kelley [4]).

For $A \subset X$ and $\eta \geqslant 0$ let $Q(A, \eta)$ denote a generalized solid sphere with centre A and radius η, i.e.

$$Q(A,\eta) = \{x \in X\colon\ \varrho(x, A) \leqslant \eta\}\,.$$

As is well known, the Hausdorff metric ϱ^1 can be equivalently defined by the formula

$$\varrho^1(A, B) = \inf\{\eta \geqslant 0\colon\ A \subset Q(B, \eta) \text{ and } B \subset Q(A, \eta)\}\ .$$

In accordance with the notation of the Hausdorff metric by ϱ^1 let $Q^1(A, \eta)$ denote a solid sphere in $C(X)$ with centre $A \in C(X)$ and diameter $\eta \geqslant 0$, i.e.

$$Q^1(A, \eta) = \{C \in C(X)\colon\ \varrho^1(C, A) \leqslant \eta\}\ .$$

For $A \subset X$ we define $\varphi(A)$ as the subset of $C(X)$ consisting of all points of A. In particular, $\varphi(X)$ is the set of all points (x) of X.

For $\mathfrak{A} \subset C(X)$ we define $\sigma(\mathfrak{A})$ to be the union of all A belonging to $\mathfrak{A}$. Clearly, we always have $\sigma(\mathfrak{A}) \subset X$ and sometimes even $\sigma(\mathfrak{A}) \in C(X)$.

The notions and notations not defined in this paper come from [1] and [5].

§ 5. Families $\mathfrak{M}_\mu$. In this section we shall define families $\mathfrak{M}_\mu$ and prove some of their properties. The metric ϱ is supposed here to satisfy condition (α) only.

Of special interest to us will be connected and non-empty subgraphs of X of the following two types: those which do not contain any end-point of X and those which do not contain any simple closed curve. We shall call them *internal* in the first case and *acyclic* in the second.

Crucial for us is the notion of a pair. Two connected subgraphs of X, $A \subset B$, will be said to form a *pair* in any of the following two cases:

1. $A = 0$ and B is a segment of X,

2. A is internal and B is the union of A and of some segments of X meeting A, i.e. A and B are connected subgraphs of X satisfying conditions $A \cap X^{[1]} = 0$ and $A \subset B \subset Q(A, 1)$.

In particular, $A \subset Q(A, 1)$ is a pair for any internal and acyclic A. Pair $A \subset B$, where $A = 0$ and B is a segment, and pairs $A \subset Q(A, 1)$, where A is internal and acyclic, will be called *fine*. Clearly, each fine pair is a pair and, conversely, in X acyclic each pair is fine.

Let $A \subset B$ be a pair. By $\mathfrak{M}_{A\subset B}$ we shall denote the family of subcontinua C of X which satisfy the condition $A \subset C \subset B$ and are such that if S is a segment of $\overline{B-A}$ with end-points $a \in A$ and $b \in B-A$, $S = \overline{ab}$, then $b \in C$ implies that the component of b in the set $C \cap Q(b, 1)$ is a subgraph of X. In other words, if $C \in \mathfrak{M}_{A\subset B}$ and M is a component of $\overline{C-A}$, then $\varrho^1(A \cap M, M) \leqslant 1$.

In particular, if X is acyclic, then the definition of $\mathfrak{M}_{A\subset B}$ can be written simply as

$$\mathfrak{M}_{A\subset B} = \{C \in C(X)\colon\ A \subset C \subset B\}\ .$$

5.1. *If $C \in C(X)$, then there exists a fine pair $A \subset B$ such that $C \in \mathfrak{M}_{A\subset B}$.*

Proof. Let $\mathfrak{A}$ be the family of all internal subgraphs of X contained in C.

If $\mathfrak{A} = 0$, then C is contained in a certain segment B of X and in that case $C \in \mathfrak{M}_{0\subset B}$.

And if $\mathfrak{A} \neq 0$, then the union $\sigma(\mathfrak{A})$ of all subgraphs of X belonging to $\mathfrak{A}$ is also a subgraph of X.

To prove that $\sigma(\mathfrak{A})$ is connected, it suffices to show that it is connected between any pair of its vertices. Let v and w be two vertices of $\sigma(\mathfrak{A})$. Since $\sigma(\mathfrak{A}) \subset C$ by the definition, the two vertices belong to C, and so in C there is an arc L consisting of segments of X and containing v and w. Clearly, $L \in \mathfrak{A}$.

Denote by A a connected and acyclic subgraph of $\sigma(\mathfrak{A})$ containing all vertices of X belonging to $\sigma(\mathfrak{A})$. Since, as a subgraph of internal $\sigma(\mathfrak{A})$, A is also internal, $A \subset Q(A, 1)$ is a fine pair. We shall show that this is the pair we want. But $A \subset \sigma(\mathfrak{A}) \subset C$ implies $A \subset C$ and so to complete the proof it remains to show that if M is a component of $\overline{C-M}$, then $\varrho^1(A \cap M, M) \leqslant 1$. Let x be any point of C and let S be a segment of X containing x. Since $x \in C$ and C is a continuum meeting internal vertices of X, C must contain an end-point u of S which is at the same time an internal vertex of X. But then $u \in \mathfrak{A}$ and so, by our definition of A, also $u \in A$. Hence $\varrho^1(A \cap M, M) \leqslant 1$.

Now we shall show an important lemma on the structure of sets $\mathfrak{M}_\mu$.

5.2. *Let $A \subset B$ be a pair of X. If $A = 0$, then $\mathfrak{M}_{A\subset B}$ is a 2-dimensional topological ball, and if $A \neq 0$, then $\mathfrak{M}_{A\subset B}$ is a topological ball of dimension $k+2l$, where k is the number of segments of $\overline{B-A}$ meeting A at one of their end-points only and l the number of segments of $\overline{B-A}$ meeting A at two of their end-points.*

Proof. Begin with the case $A = 0$. In this case, by the definition of a pair, B is a segment of X and so $\mathfrak{M}_{A\subset B} = C(B)$ is the family of all subcontinua of B. By the first example of § 3, $C(B)$ is a 2-dimensional topological ball.

Now, if $A \neq 0$, then B can be written in the form

$$B = A \cup \bigcup_{i=1}^{n} L_i\ ,$$

where each L_i is a segment of X. If $C \in \mathfrak{M}_{A\subset B}$, then

$$C = A \cup \bigcup_{i=1}^{n} C \cap L_i\ .$$

Let L_i be a segment with end-points a_i and b_i. If L_i meets A at one of its end-points only, say at a_i, then each continuum $C \in \mathfrak{M}_{A\subset B}$ meets L_i

along one arc $C \cap L_i = L_i(t_i)$ containing a_i, where $0 \leqslant t_i \leqslant 1$ is its length. The range T_i of numbers t_i is, in this case, a real segment $[0, 1]$.

And if the end-points a_i and b_i of L_i both belong to A_i, then each continuum $C \in \mathfrak{M}_{A\subset B}$ meets L_i along two arcs (virtually, they may cover L_i)

$$C \cap L_i = L_i(t_i') \cup L_i(t_i'') ,$$

where $0 \leqslant t_i'+t_i'' \leqslant 1$ and $L_i(t_i')$ denotes a subsegment of L_i containing a_i and having length t_i'. Similarly for $L_i(t_i'')$.

In this case the range T_i of pairs (t_i', t_i'') is homeomorphic to an isosceles triangle of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ in which the hypotenuse is contracted to a single point.

It should be obvious that the correspondence between $C \in \mathfrak{M}_{A\subset B}$ and the system of real numbers t_i or pairs of real numbers (t_i', t_i'') according to whether L_i meets A at one or two points, where $i = 1, 2, \ldots, n$, is one-to-one and continuous, and thus that it yields a homeomorphism between $\mathfrak{M}_{A\subset B}$ and the ball $\bigtimes_{i=1}^{n} T_i$.

Here is a simple corollary to lemma 5.2:

5.3. *Let $A \subset B$ be a pair of X. If $A \neq 0$, then* $\dim \mathfrak{M}_{A\subset B} = \operatorname{ord}_A B$. *In particular, if $B = Q(A, 1)$, then* $\dim \mathfrak{M}_{A\subset B} = \operatorname{ord}_A X$.

And the two lemmas 5.1 and 5.2 imply

5.4. *The union*

$$C(X) = \bigcup \mathfrak{M}_{A\subset B} ,$$

where $A \subset B$ runs over all (fine) pairs of X, is a decomposition of the hyperspace $C(X)$ into balls $\mathfrak{M}_{A\subset B}$.

§ 6. Decomposition of $C(X)$ into balls $\mathfrak{M}_\mu$. This section is devoted to the proof that if we look upon each ball $\mathfrak{M}_{A\subset B}$ as a cell and if the metric ϱ in X satisfies the two conditions (α) and (γ), then the union $C(X) = \bigcup \mathfrak{M}_{A\subset B}$, where $A \subset B$ runs over all pairs of X, is very close to a cellular decomposition (for X acyclic, it is exactly that).

The first lemma will be shown under the assumption of condition (α) only.

6.1. *Let $A \subset B$ and $D \subset E$ be two pairs of X such that $A \subset D \subset E \subset B$. If either $A \neq D$ or $E \neq B$, then $\mathfrak{M}_{D\subset E}$ is a ball lying on the surface of the ball $\mathfrak{M}_{A\subset B}$.*

Proof. We shall consider three cases.

I. The case $A = D = 0$ is impossible, because in this case B would be a segment and, in view of $E \subset B$ and $E \neq B$, E would be a vertex of B. But $D \subset E$ is not a pair, because "the empty set $\subset$ vertex" is not.

II. $A = 0 \neq D$. Here again B must be a segment and so $D \subset E$ is either a pair "vertex of $B \subset$ vertex of B" or a pair "vertex of $B \subset B$" or, finally, a pair $B \subset B$. But $\mathfrak{M}_{A\subset B} = C(B)$ is homeomorphic to the triangle T with basis 1 and height 1 (cf. § 3) and in this triangle the set $\mathfrak{M}_{D\subset E}$ would be, respectively, either a side vertex of T or an edge of T or a top vertex of T.

III. $A \neq 0$. In this case B can be written in the form

$$B = A \cup \bigcup_{i=1}^{n} L_i ,$$

where each L_i is a segment of X meeting A. Denoting the empty set by L_0 and re-ordering, if necessary, segments $L_1, \ldots, L_n$, we may write

$$D = A \cup \bigcup_{i=0}^{k} L_i \quad \text{and} \quad E = A \cup \bigcup_{i=0}^{l} L_i ,$$

where $0 \leqslant k \leqslant l \leqslant n$ and either $0 < k$ or $l < n$.

According to the notation of lemma 5.2, the set T_i is then either the segment $[0, 1]$ or the triangle of vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ with the hypotenuse contracted to a single point. Let us denote by $(0)_i$ the point 0 in the first case and the point $(0, 0)$ in the second, and by $(1)_i$ the point 1 in the first case and the point of contraction in the second. As we have shown in lemma 5.2, the family $\mathfrak{M}_{A\subset B}$ is homeomorphic to the ball $\bigtimes_{i=1}^{n} T_i$ and it is not difficult to see that $\mathfrak{M}_{D\subset E}$ is the ball

$$[(1)_1 \times \ldots \times (1)_k] \times \Big[\bigtimes_{i=k+1}^{l} T_i\Big] \times [(0)_{l+1} \times \ldots \times (0)_n] ,$$

where any of the three expressions in the square brackets [...] may vanish if, respectively, $k = 0$, $k = l$ or $l = n$. However, at least one of them remains, and so $\mathfrak{M}_{D\subset E}$ is a ball lying on the surface of the ball $\mathfrak{M}_{A\subset B}$.

6.2. *Let $A \subset B$ and $A' \subset B'$ be two distinct pairs of X. If*

$$\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} \neq 0 , \tag{1}$$

then there exist a finite number of pairs (for X acylic, only one pair) $D_k \subset E_k$ of X such that

$$A \cup A' \subset D_k \subset E_k \subset B \cap B' \quad \textit{for each } k \tag{2}$$

and

$$\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} = \bigcup_k \mathfrak{M}_{D_k\subset E_k} . \tag{3}$$

Proof. Hypothesis (1) implies that there exists a non-empty continuum C such that

$$A \subset C \subset B \quad \text{and} \quad A' \subset C \subset B' , \tag{4}$$

whence

(5) $$A \cup A' \subset C \subset B \cap B' .$$

In particular,

(6) $$A \cup A' \subset B \cap B' .$$

Therefore

(7) $$\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} \subset \{C \in C(X)\colon A \cup A' \subset C \subset B \cap B'\} .$$

We shall consider three cases.

I. $A = 0 = A'$. In this case B and B' are distinct segments and since, by (5), they meet, they have a common vertex v (by the assumption of (γ), one only) and $B \cap B' = (v)$. Clearly, (v) is the only subcontinuum of both B and B' and so

$$\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} = (v) .$$

On the other hand, however, $(v) \subset (v)$ is a pair of X and

$$(v) = \mathfrak{M}_{(v)\subset(v)} .$$

The two equalities imply our lemma in the case under consideration.

II. $A \cup A'$ is non-empty and connected.

Let

(8) $$S_1, S_2, \ldots, S_m$$

be the sequence of all segments $S = \overline{aa'} \subset B \cap B'$ such that $a \in A - A'$ and $a' \in A' - A$ (if there are no such segments, as in acyclic graphs, things become much easier).

In view of the connectedness of $A \cup A'$ and (6) there exists a component E' of $B \cap B'$ containing $A \cup A'$. Let E be a subgraph of E' obtained by removing from E' all segments (8) (but not their end-points). If X is acyclic, $E = E'$. It should be clear that E is a connected subgraph of X.

Consider now subgraphs D_k and E_k of X such that D_k is the union of $A \cup A'$ and of some segments from (8), and E_k—the union of E and of the same segments from (8)

(9) $$D_k = A \cup A' \cup S_{i_1} \cup \ldots \cup S_{i_l} , \qquad E_k = E \cup S_{i_1} \cup \ldots \cup S_{i_l} .$$

If X is acyclic, then there is only one such $D_k = A \cup A'$ and only one $E_k = E = E'$.

We shall show that each $D_k \subset E_k$ is a pair and that (3) holds true.

In fact, since a connected union $A \cup A'$ of two internal subgraphs of X (one of which, however, may be empty) is still internal, then, in view of the definition of D_k, D_k is also internal. Hence to prove that $D_k \subset E_k$ is a pair it remains to show that

(10) $$\varrho^1(D_k, E_k) \leqslant 1 .$$

But since $A \cup A' \subset D_k$ and $E_k \subset B \cap B'$ by definition, we have $\varrho^1(D_k, E_k) \leqslant \varrho^1(A \cup A', B \cap B')$, and so (10) will be proved if we show that

(11) $$\varrho^1(A \cup A' , B \cap B') \leqslant 1 .$$

Inequality (11) is nearly obvious if either A or A' is empty. Indeed, if $A' = 0$, then

$$\varrho^1(A, B \cap B') = \sup_{y \in B\cap B'} \varrho(A, y) \leqslant \sup_{y \in B} \varrho(A, y) = \varrho^1(A, B) \leqslant 1 ,$$

where the first equality follows from (6) and the last inequality from the assumption that $A \subset B$ is a pair and $A \neq 0$.

And if both A and A' are non-empty, then (11) results from the following sequence of inequalities:

$$\begin{aligned}\varrho^1(A \cup A' , B \cap B') &= \sup_{y \in B\cap B'} \varrho(A \cup A' , y) \\ &= \min\,[\sup_{y \in B\cap B'} \varrho(A, y) , \sup_{y \in B\cap B'} \varrho(A', y)] \\ &\leqslant \min[\sup_{y\in B} \varrho(A, y) , \sup_{y \in B'} \varrho(A', y)] \\ &= \min[\varrho^1(A, B) , \varrho^1(A', B')] \leqslant 1 .\end{aligned}$$

Thus we have shown that each $D_k \subset E_k$ is a pair.

To prove (3) take first $C \in \mathfrak{M}_{D_k\subset E_k}$, where $D_k \subset E_k$ is a pair such that (9). Clearly,

$$A \subset C \subset B \quad \text{and} \quad A' \subset C \subset B' .$$

In view of the symmetry, it suffices to show that $C \in \mathfrak{M}_{A\subset B}$. For that purpose consider a segment $T = \overline{ab}$ with $a \in A$ and $b \in B - A$. If $b \in A'$, then T either is one of the segments $S_{i_1}, \ldots, S_{i_l}$ or is not. If it is, then $T \subset C$, and if it is not, then $C \cap T = (a) \cup (b)$. We may then suppose that $b \in B' - A'$. But then $b \in E_k - D_k$, and so $C \cap Q(b, 1)$ is a subgraph of X, because $C \in \mathfrak{M}_{D_k\subset E_k}$ by hypothesis.

Before proceeding to the converse implication, let us consider first a segment S from (8). Let $S = \overline{aa'} \subset B \cap B'$, and $a \in A - A'$ and $a' \in A' - A$. We shall show that if $C \in \mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'}$, then

(12) $$\text{either } S \cap C = (a) \cup (a') \quad \text{or} \quad S \cap C = S .$$

Indeed, if $S \cap C$ contains a closed-open (in $S \cap C$) subset containing one of its end-points, say a, and distinct from both that end-point and the whole S, then $C \cap Q(a, 1)$ is not a subgraph of X, because $S \subset Q(a, 1)$ and $C \cap S$ is not a subgraph. Hence $C \notin \mathfrak{M}_{A'\subset B'}$.

Equalities (12) are realized by $C = A \cup A'$ and $C = E \cup S$, which both belong to $\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'}$, because $A \cup A' \subset A \cup A'$ and $A \cup A' \cup \cup S \subset E \cup S$ are both pairs $D_k \subset E_k$.

To complete the proof of (3) (and of the lemma) take $C \in \mathfrak{M}_{A\subset B} \cap \cap \mathfrak{M}_{A'\subset B'}$.

If $S_{i_1}, \ldots, S_{i_l}$ are all segments from (8) contained in C, then put (9). Hence $D_k \subset C \subset E_k$ and we shall show that $C \in \mathfrak{M}_{D_k\subset E_k}$. For that purpose take a segment $T = \overline{ab}$ such that $a \in D_k$ and $b \in E_k - D_k$. Then $a \in A \cup A'$, because each S_{i_j} has both end-points common with $A \cup A'$, and similarly, $b \in B \cap B' - (A \cup A')$. Now, if $a \in A$, then $b \in C$ implies that $C \cap Q(b, 1)$ is a subgraph of X, because $C \in \mathfrak{M}_{A\subset B}$ and $b \in B - A$. And, similarly, if $a \in A'$. Hence $C \in \mathfrak{M}_{D_k\subset E_k}$.

III. $A \cup A'$ is non-empty and not connected, i.e. $A \neq 0 \neq A'$ and $A \cap A' = 0$. If (1) holds true, then, as follows from the inclusions

$$A \cup A' \subset B \cap B' \subset Q(A, 1) \cap Q(A', 1),$$

we must also have $A \subset Q(A', 1)$ and $A' \subset Q(A, 1)$, i.e. $\varrho^1(A, A') \leqslant 1$. Since A and A' are disjoint, both A and A' must be vertices lying at a distance 1 from each other. By assumptions (α) and (γ), X must then contain a segment D joining them. But $D \subset D$ is clearly a pair and so, by virtue of (7), the proof will be completed if we show that $B \cap B' = D$ And this is obvious, because if S a segment distinct from D one vertex of which is, say, A, then, by assumptions (α) and (γ), for any point $x \in S - (A)$ we have $\varrho(x, A') \leqslant 1$. In other words, $x \notin B'$.

Remark. Assumption (γ) was actually used in the proof of cases I and III only. Hence (case II), if $A \cup A'$ is non-empty and connected, then our proposition holds true under the assumption of condition (α) only.

It is an immediate corollary to 5.2, 6.1 and 6.2 that

6.3. *If X is a finite connected and acyclic graph, and the metric in X satisfies conditions* (α) *and* (γ), *then*

$$C(X) = \bigcup \mathfrak{M}_{A\subset B},$$

where $A \subset B$ runs over all pairs of X, is a cellular complex.

6.4. *The hyperspace $C(X)$ is a polyhedron if and only if X is a finite connected graph.*

Indeed, if X is a finite connected graph, then by 5.2 and 6.2 its hyperspace $C(X)$ is a polyhedron. And if $C(X)$ is a polyhedron, then by Kelley's theorem ([4], theorem 5.4) X is a finite connected graph.

§ 7. Dimension of $C(X)$. From now on until the end of the paper we change the assumptions concerning the metric in X: we assume that it satisfies conditions (α) and (β).

In this section we shall uncover formulas for the dimension of the hyperspace $C(X)$ and find the structure of the set

$$(1) \qquad \{C \in C(X)\colon\ \dim_C C(X) = \dim C(X)\}.$$

However, before proceeding to do that we must make a more detailed examination of some subgraphs of X.

For the sake of brevity, an internal and acyclic subgraph of X containing all internal vertices of X will be called a *maximal fine subgraph* of X.

A simple induction argument leads to the following lemma:

7.1. *Let X be a graph topologically distinct from both an arc and a simple closed curve, and let A be a maximal fine subgraph of X. If B and D are two other connected subgraphs of X, both distinct from A and such that $B \subset A \subset D$, then*

$$\operatorname{ord}_B X < \operatorname{ord}_A X \quad \text{ and } \quad \operatorname{ord}_D X < \operatorname{ord}_A X.$$

Remark. If X is topologically an arc, then there is no maximal fine subgraph A, because there are no internal subgraphs at all. And if X is a simple closed curve, then condition (β) does not apply to X.

As a finite graph, X may also be treated as a 1-dimensional simplicial complex whose vertices are vertices of X and edges—segments of X. Denoting the number of vertices of X by α_0 and the number of segments of X by α_1 we then have the equality

$$d = 1 - \alpha_0 + \alpha_1,$$

where d is the degree of connectivity of the complex X, i.e. the number of edges which can be removed from X without violating its connectedness (cf. [7], p. 87).

Let e be the number of end-points of X. The following lemma should be obvious.

7.2. *Let X be a graph topologically distinst from both an arc and a simple closed curve. If A is a maximal fine subgraph of X, then*

$$\operatorname{ord}_A X = 2d + e.$$

Finally, if C is a subcontinuum of X, then $\operatorname{ord}_C X$ is equal either to 2 if C is contained in the interior of some segment or to $\operatorname{ord}_E X$, where E is the maximal connected subgraph of X contained in C. Hence

7.3. *If X is a graph topologically distinct from an arc, then*

$$\sup_{C \in C(X)} \operatorname{ord}_C X = \max_E \operatorname{ord}_E X,$$

where E runs over all connected subgraphs of X.

Since, by virtue of 5.4, the union $C(X) = \bigcup \mathfrak{M}_{A\subset B}$ is a decomposition of the hyperspace $C(X)$ into balls $\mathfrak{M}_{A\subset B}$, then the dimension of $C(X)$ is equal to the highest of the dimensions of balls $\mathfrak{M}_{A\subset B}$ (cf. [3], p. 30) and the set (1) is the union of the highest-dimensional balls $\mathfrak{M}_{A\subset B}$.

7.4. *If X is a finite connected graph, then*

$$\dim C(X) = \sup_{C\in C(X)} \operatorname{ord}_C X = 2d+e = 2+\sum_v (\operatorname{ord}_v X - 2)\,,$$

where v runs over all the ramification points of X.

In fact, if X is topologically either an arc or a simple closed curve, then $\dim C(X) = 2$ and all three equalities hold true (cf. § 3).

And if X is topologically distinct from both, then an immediate application of lemmas 5.2, 7.1, 7.2 and 7.3 yields the first two equalities, and the third (like the first, but see the remark below) is contained in Kelley's Theorem 5.5 from [4].

Remarks. Note on this occasion that the proof of the first equality given by Kelley contains a gap. Namely, he considers a collection $A_1, A_2, \ldots, A_n$ of connected subgraphs of X (which, as follows from his argument, are all non-empty) and claims that $C(X)$ is the union of families $\mathfrak{A}_i$ of continua for which A_i is the maximal subgraph. But this is not so, because X clearly contains also subcontinua which do not contain any non-empty subgraph of X.

The simple inductive proof of the equality

$$\sup_{C\in C(X)} \operatorname{ord}_C X = 2+\sum (\operatorname{ord}_v X - 2)$$

Kelley apparently leaves to the reader.

Now assuming X to be topologically an arc or a simple closed curve define q to be the number 1, $q = 1$, and for all other finite connected graphs let q be the number of all maximal fine subgraphs of X.

Clearly, if X is acyclic, then $q = 1$. In general, however, $q \geqslant 1$. For instance, if X is a simple closed curve with a diameter attached to it, then $q = 3$, and if X is a simple closed curve with two disjoint chords attached to it, then $q = 12$.

7.5. *If X is a finite connected graph, then the set* (1) *consists of q balls of dimension $2d+e$ each and such that any two of them meet along a common face containing X and proper for both.*

Proof. If X is topologically an arc or a simple closed curve, then — as follows from § 3 — our theorem holds true. Suppose then that X contains a ramification point v. As follows from the decomposition of $C(X)$ given in 5.4, the set (1) consists then of balls $\mathfrak{M}_{A\subset B}$ which have the highest dimension equal, by virtue of 7.4, to $2d+e$. And this number is attained, in view of 7.1 and 5.3, only if A is a maximal fine subgraph of X and $B = Q(A,1) = X$. The number of maximal fine subgraphs of X is q and so it remains to show that given two maximal subgraphs of X, A and A', the set $\mathfrak{M}_{A\subset X} \cap \mathfrak{M}_{A'\subset X}$ contains X and is a proper face of both $\mathfrak{M}_{A\subset X}$ and $\mathfrak{M}_{A'\subset X}$. However, $v \in A \cap A'$ by the definition of a maximal fine subgraph and so we have the inequality $A \cap A' \neq 0$, which implies that $A \cup A'$ is connected. Hence by 6.2 (cf. Remark following 6.2), $A \cup A' \subset X$ is a pair and

$$\mathfrak{M}_{A\subset X} \cap \mathfrak{M}_{A'\subset X} = \mathfrak{M}_{A\cup A'\subset X}\,,$$

because there is no segment $S = \overline{aa'}$ with $a \in A - A'$ and $a' \in A' - A$ (see 6.2, case II).

And by 6.1 (which was proved under assumption of condition (α) only and so can be applied here), $\mathfrak{M}_{A\cup A'\subset X}$ is a ball lying on the surfaces of both $\mathfrak{M}_{A\subset X}$ and $\mathfrak{M}_{A'\subset X}$, and, clearly, containing X.

§ 8. Analysis of the 2-dimensional part of $C(X)$. Now we shall show (under the assumption of conditions (α) and (β) on the metric in X) that the closure $D_{C(X)}$ of the 2-dimensional part of the polyhedron $C(X)$,

$$D_{C(X)} = \overline{\{C \in C(X)\colon \dim_C C(X) = 2\}}\,,$$

is equal to the union of all balls $\mathfrak{M}_{A\subset B}$ with $A = 0$, find its homeomorph and make some analysis of it.

8.1. *If X is a finite connected graph, then*

$$D_{C(X)} = \bigcup \mathfrak{M}_{A\subset B}\,,$$

where $A = 0$ and B runs over all segments of X.

Proof. If $C \in \mathfrak{M}_{A\subset B}$, where $A \subset B$ is a pair of X and $A \neq 0$, then, by the definition of a pair and in view of conditions (α) and (β), A is an internal subgraph of X containing a ramification point of X and B is a connected subgraph of X such that $A \subset B \subset Q(A,1)$. Let $A_1 \subset A$ be an acyclic subgraph of A containing all vertices of X belonging to A. Hence $Q(A_1,1) = Q(A,1)$ and so $A_1 \subset Q(A,1)$ is a pair. Clearly, $C \in \mathfrak{M}_{A_1\subset Q(A,1)}$ and since A_1, being acyclic, internal and containing a ramification point, must be of order $\operatorname{ord}_{A_1} X \geqslant 3$, then by 5.3 the ball $\mathfrak{M}_{A_1\subset Q(A,1)}$ has dimension $\dim \mathfrak{M}_{A_1\subset Q(A,1)} \geqslant 3$. Hence, in particular, $\dim_C C(X) \geqslant 3$, and so, by 5.4,

$$D_{C(X)} \subset \overline{C(X) - \bigcup_{0\neq A\subset B} \mathfrak{M}_{A\subset B}}\,. \tag{1}$$

On the other hand, however,

(2) if $A \subset B$ is a pair such that $A = 0$ and B is a segment of X, then $\mathfrak{M}_{A\subset B} \subset D_{C(X)}$.

In fact, $\mathfrak{M}_{A\subset B} = C(B)$ is topologically equivalent (see § 3, ex. 1) to a solid triangle T, and in this representation it is easy to observe that if C is a subcontinuum of B which contains none of the end-points of B, then $\dim_C X = 2$. But the set of all such C contains (in this representation) the interior of that triangle (actually, it is the union of the interior and of the edge $\varphi(B)$ but without its ends) and so from this follows (2).

From (1) and (2) follows our lemma.

To find a homeomorph of $D_{C(X)}$ start with a topological embedding of X into a 3-dimensional euclidean space E^3

$$h\colon X \to E^3\,,$$

and then, for each segment B, attach to the image $h(B)$ a 2-dimensional ball $\mathfrak{M}_{A\subset B}$ $(A = 0)$,

$$f_B\colon \mathfrak{M}_{A\subset B} \to E^3\,,$$

along an arc $\varphi(B)$ lying on its boundary and in such a way that

$$f_B|\varphi(B) = h|B\,.$$

Moreover, suppose that any two added balls $f_B(\mathfrak{M}_{A\subset B})$ and $f_{B'}(\mathfrak{M}_{A'\subset B'})$ $(A = A' = 0$ and B and B' are segments) meet if and only if segments B and B' meet and that in such a case the equality

$$f_B(\mathfrak{M}_{A\subset B}) \cap f_{B'}(\mathfrak{M}_{A'\subset B'}) = h(B) \cap h(B')$$

holds.

The formal proof that the function f defined by the formula

$$f|\mathfrak{M}_{A\subset B} = f_B \text{ for each pair } 0 = A \subset B$$

is a homeomorphism

$$f\colon D_{C(X)} \to E^3$$

easily follows from 8.1 and 2.1. Hence

8.2. *If X is a finite connected graph, then $D_{C(X)}$ is homeomorphic to X with a 2-dimensional ball attached to each segment B of X along an arc lying on the boundary of that ball in such a way that any two distinct balls are disjoint outside X.*

In particular,

8.3. *If v is a ramification point of X, then v locally separates $D_{C(X)}$.*

Let P be an arbitrary polyhedron. By E_P we shall denote the closure (in P) of a subset of P consisting of all points $p \in P$ for which there exists a closed neighbourhood V topologically equivalent to a disc (= 2-dimensional ball) and such that p lies on the boundary of that disc.

8.4. *If X is a finite connected graph containing a ramification point, then $E_{C(X)} \underset{\text{top}}{=} X$.*

Proof. Clearly, $E_{C(X)} \subset D_{C(X)}$ and so, by 8.1,

(1) $E_{C(X)} \subset \bigcup_{A\subset B} \mathfrak{M}_{A\subset B}$, where $A \subset B$ runs over all pairs of X such that $A = 0$.

Consider the set $\mathfrak{M}_{A\subset B}$, where $A = 0$ and B is a segment. As follows from § 3, ex. 1

(2) if $C \in \mathfrak{M}_{A\subset B}$ and C neither is a point of B nor contains any vertex of B (i.e. $h(C)$ does not lie on the boundary of T), then $C \notin E_{C(X)}$.

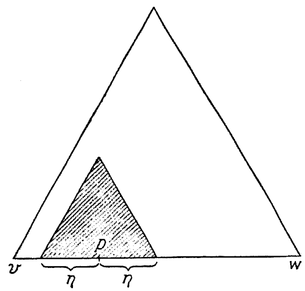

Fig. 3

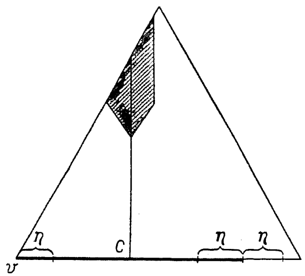

Fig. 4

Now, if $p \in B = \overline{vw}$ and $v \neq p \neq w$, then there exists an $\eta > 0$ such that $Q(p, \eta) \subset B$. Hence $Q^1\big((p), \eta\big) \subset \mathfrak{M}_{A\subset B}$. But, as can easily be seen in the triangle T (see § 3, ex. 1), the ball $Q^1\big((p), \eta\big)$ is homeomorphic under h to the triangle of base 2η and height 2η, on the boundary of which lies $h(p)$ (fig. 3). Therefore

$$\varphi(B) \subset E_{C(X)}\,, \tag{3}$$

because, moreover, no point of the set $\varphi(B)$, except for, perhaps, the end-points v and w, is common to any other $\mathfrak{M}_{A'\subset B'}$, where $A' \subset B'$ is a pair such that either $A \neq A'$ or $B \neq B'$.

We shall now show that

(4) $\{C \in C(X)\colon v \in C \subset B\} \subset E_{C(X)}$ if and only if v is an end-point of X.

In fact, if v is an end-point of X, then any ball $Q^1(C, \eta)$, where $v \in C \subset B$, $C \neq B$, and $0 < \eta < 1-\delta(C)$, is contained in $\mathfrak{M}_{A\subset B}$. And, as is not hard to observe in the triangle T, the ball $Q^1(C, \eta)$ is homeomorphic under h to a quadrangle on the boundary of which lies $h(C)$

(fig. 4). Hence $C \in E_{C(X)}$, and since $E_{C(X)}$ is closed and C has been any subcontinuum of B distinct from B itself and containing v, we have

$$\{C \in C(X)\colon\ v \in C \subset B\} \subset E_{C(X)}\,.$$

And if v is not an end-point of X, then v, being a vertex, must be a ramification point of X. Hence

$$Q(v,1) = \bigcup_{j=1}^{n} \overline{vb_j}\,,$$

where $n > 2$ and one of the vertices b_j, say b_1, is the other end of B, $b_1 = w$. Let C be a continuum distinct from both B and v and such that

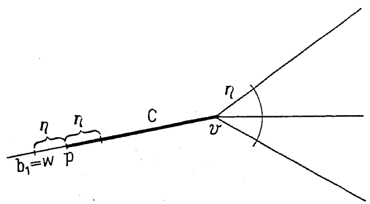

Fig. 5

$v \in C \subset B$ and $0 < \eta < \min[\delta(C), 1-\delta(C)]$, and let p be a point of $\overline{vb_1} = B$ such that $\overline{vp} = C$. By our assumptions on C and η, $Q(p,\eta) \subset \overline{vb_1}$. As is not difficult to see (cf. fig. 5), the common part

$$\mathfrak{M}_{(v)\subset Q(v,1)} \cap Q^1(C,\eta)$$

is an n-dimensional ball consisting of all continua D such that $v \in D \subset Q(v,1)$, $D \cap \overline{vb_1}$ is a subarc of $\overline{vb_1}$ containing v and of length $\delta(D \cap \overline{vb_1})$ satisfying $|\delta(D \cap \overline{vb_1}) - \delta(C)| \leqslant \eta$ (i.e. an arc whose other end is at a distance from p not greater than η), and $D \cap \overline{vb_j}$ for $j = 2, 3, \ldots, n$ is a subarc of $\overline{vb_j}$ containing v and of diameter not greater than η. Hence $C \notin E_{C(X)}$.

Therefore (4) is proved.

Put

$$Y_{A\subset B} = \begin{cases} \varphi(B) \text{ if both end-points of } B\,,\ v \text{ and } w\,, \text{ are ramification points}\,, \\ \varphi(B) \cup \{C \in C(X)\colon\ v \in C \subset B\} \text{ if } v \text{ is an end-point of } X\,. \end{cases}$$

It follows from (2), (3) and (4) that $Y_{A\subset B}$ is, in any case, topologically an arc and that

$$Y_{A\subset B} = E_{C(X)} \cap \mathfrak{M}_{A\subset B}\,. \tag{5}$$

Let $f_{A\subset B}$ be a homeomorphism

$$f_{A\subset B}\colon\ Y_{A\subset B} \to B \tag{6}$$

which maps $Y_{A\subset B}$ onto B in such a way that if an end-point w of B is a ramification point of X, then $f_{A\subset B}(w) = (w)$.

Since two sets $\mathfrak{M}_{A\subset B}$ and $\mathfrak{M}_{A'\subset B'}$, where $A = 0 = A'$ and both B and B' are segments, meet if and only if segments B and B' have vertices in common (one or two) and in that case the common part consists of those vertices (one or two), then, by our definition of homeomorphism (6),

$$f_{A\subset B}|\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} = f_{A'\subset B'}|\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'}$$

and

$$\begin{aligned} f_{A\subset B}(\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'}) &= f_{A\subset B}(\mathfrak{M}_{A\subset B}) \cap f_{A'\subset B'}(\mathfrak{M}_{A'\subset B'}) \\ &= f_{A'\subset B'}(\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'})\,. \end{aligned}$$

Hence, by lemma 2.1, the function

$$f\colon \bigcup_{A\subset B} Y_{A\subset B} \to \bigcup_{B} B\,,$$

where $A = 0$ and B runs over all segments of X, defined by the formula

$$f|Y_{A\subset B} = f_{A\subset B}$$

is a homeomorphism onto $\bigcup_B B = X$. And since, by (1) and (5), $\bigcup_{A\subset B} Y_{A\subset B} = E_{C(X)}$, then this is the required homeomorphism

$$f\colon\ E_{C(X)} \to X\,.$$

§ 9. Topological conclusions. This title is somewhat inadequate as some of the "topological conclusions" have already appeared before, to mention only theorem 6.4, stating that the hyperspace $C(X)$ is a polyhedron if and only if X is a finite connected graph, or the formulas for the dimension of polyhedron $C(X)$ and the structure of the highest dimensional part of $C(X)$ given in theorems 7.4 and 7.5. Nevertheless, we still have to draw some conclusions, topological in character and general in appearance, which form a natural end of this paper.

First we come to the following theorem on the uniqueness of the underlying continuum for the hyperspace $C(X)$.

9.1. *Let a connected polyhedron P of finite dimension be a hyperspace.*

If $\dim P = 0$, *then P is a point and $P = X = C(X)$.*

Case $\dim P = 1$ *is impossible.*

If $\dim P = 2$, *then P is a 2-dimensional ball and, as such, it is a hyperspace for both an arc and a simple closed curve.*

And if $\dim P \geqslant 3$, *then it is a hyperspace for one continuum only, which, by theorem 6.4, must be a finite connected graph.*

P r o o f. Indeed, if $\dim P = 0$, then P, being connected by hypothesis, must be a point, and then, clearly, $P = X = C(X)$.

If $\dim P > 0$ and P is a hyperspace for X, then X cannot be a point (see the case above). Hence, by theorem 6.4, X is a graph containing a segment B. The ball $\mathfrak{M}_{0\subset B} \subset C(X)$ is of dimension 2 (cf. lemma 5.2), and so $\dim P \geqslant 2$.

As follows from theorem 7.4, the only connected graph for which the hyperspace has dimension 2 can be, topologically, either an arc or a simple closed curve. The two cases were examined in § 3.

Finally, if polyhedron P is a hyperspace for X and $\dim P \geqslant 3$, then, by theorem 6.4, X must be a finite connected graph containing a ramification point (for the only connected graph without any ramification point is, topologically, either an arc or a simple closed curve, and for such a graph the hyperspace is of dimension 2). Therefore, by lemma 8.4, $X \underset{\text{top}}{=} E_P$ and so X must be unique.

The observation that if P is a 2-dimensional ball, then $C(E_P)$ is homeomorphic to P (cf. § 3, ex. 2), leads together with lemma 8.4 to the following observation concerning those connected polyhedra of finite dimension which are hyperspaces.

9.2. *Let P be a connected polyhedron of finite dimension. Then P is a hyperspace if and only if E_P is homeomorphic to a finite connected graph and $P \underset{\text{top}}{=} C(E_P)$.*

Remark. If we could describe the structure of the polyhedron $C(E_P)$ in topological terms and without involving functor C, then corollary 9.2 would turn into characterization of those locally connected continua of finite dimension which are hyperspaces $C(X)$ for some X, and so it may be considered as an attempt toward this end. However, an inspection of the structure of polyhedra which are hyperspaces, given in the present paper, is too cursory to make such a characterization possible. Therefore a somewhat deeper examination of the structure of polyhedra $C(X)$, where X is a finite connected graph, seems still to be desirable, and the aim of the next paper [2] will be to do this in the case of acyclic graphs.

9.3. *For each $k = 2, 3, \ldots$ there are only finitely many k-dimensional connected polyhedra which are hyperspaces.*

Proof. As follows from theorems 6.4 and 7.4, if a k-dimensional connected polyhedron P is a hyperspace $C(X)$ for a continuum X, then X must be a finite connected graph and the equality $k = 2d+e$, where d is the degree of connectivity of X and e the number of end-points of X, must hold. But, for a given k, there are only finitely many connected and topologically distinct graphs for which the equality $k = 2d+e$ holds true.

The number $\nu(k)$ of k-dimensional connected polyhedra which are hyperspaces rapidly increases as k tends towards infinity. For instance, $\nu(2) = 1$, $\nu(3) = 2$, $\nu(4) = 8$, $\nu(5) = 17$, $\nu(6) = 80$, etc.

What do the polyhedra which are hyperspaces look like? Deferring the answer to this question till further research, let us draw here only some immediate corollaries.

9.4. *If a finite graph X is connected and contains a ramification point (i.e., if X is not a single point and topologically it is neither an arc nor a simple closed curve), then the hyperspace $C(X)$ is not dimensionally homogeneous.*

In fact, in such a case the polyhedron $C(X)$ contains balls of different dimensions and non-empty interior (relative to $C(X)$). Such are, for instance, balls $\mathfrak{M}_{A\subset B}$, where $A = 0$ and B is a segment of X, and $\mathfrak{M}_{(v)\subset Q(v,1)}$, where v is a ramification point of X.

From 9.4 it follows, in particular (cf. [1]), that

9.5. *If a finite graph X is connected and contains a ramification point, then the polyhedron $C(X)$ cannot be convex in Euclidean metric.*

Another corollary follows from 9.4 and from examples analyzed in § 3.

9.6. *If a finite graph X is connected, then a (homological) interior of the polyhedron $C(X)$ is topologically homogeneous if and only if X is an arc or a simple closed curve.*

9.7. *If a finite graph X is connected and contains a ramification point, then the hyperspace $C(X)$ is topologically prime, i.e. $C(X)$ is not a Cartesian product of subspaces distinct from it.*

Proof. It suffices to show that the polyhedron $C(X)$ is not a Cartesian product of two subspaces distinct from it.

Suppose, a contrario, that $C(X) \underset{\text{top}}{=} W_1 \times W_2$, where W_1 and W_2 are two locally connected continua, both distinct from $C(X)$. Since $C(X)$ contains 2-dimensional balls of non-empty (relative to $C(X)$) interior, W_1 and W_2 must contain arcs of non-empty (relative to W_1 or W_2, respectively) interior. Let L_i be the union of arcs L contained in W_i, $i = 1$ or $i = 2$, and such that $L = \overline{\operatorname{Int}(L)}$. It is not hard to see that the 2-dimensional part of $C(X)$, i.e., by 8.1, the set $D_{C(X)} = \bigcup \mathfrak{M}_{A\subset B}$, where $A = 0$ and B runs over all segments of X, is homeomorphic to $L_1 \times L_2$,

$$D_{C(X)} \underset{\text{top}}{=} L_1 \times L_2 .$$

Now, in view of connectedness and local connectedness of $D_{C(X)}$ cf. 8.2), both L_1 and L_2 must be connected and locally connected continua. However, Cartesian product of two connected and locally connected continua is not separated by any of its points, and the set $D_{C(X)}$ contains, by 8.3, points of local separation.

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Note on metrization

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1. Introduction. In [1] Alexandroff has proved the following theorem:

A T_1-space is metrizable if and only if it is paracompact and has a uniform base.

A base $\mathcal{B}$ for a topological space X is called *uniform* if for each $x \in X$ and each neighbourhood U of x at most finitely many members of $\mathcal{B}$ contain x and intersect $X \setminus U$. The theorem quoted above contrasts other metrization theorems in the fact that it requires neither a decomposition of the base into countably many subfamilies nor the existence of a sequence of open covers with "nice" properties; cf. the theorems of Bing, Nagata-Smirnov ([5], p. 127), Arhangel'skiĭ, Morita, Stone ([4], p. 196), and Alexandroff–Urysohn ([2]). On the other hand, it invokes the explicit requirement of paracompactness. In Section 3 of the present paper we shall prove that *a T_1-space is metrizable if and only if it has a base which is locally finite outside closed sets.* (The necessary definitions are given in Section 2). Bases that are locally finite outside closed sets generalize in a natural way the concept of a uniform base, and, as we shall see, no decomposition into countably many subfamilies is required in their definition.

Section 2 contains the necessary lemmas for the proof of the metrization theorem in Section 3. As corollaries we obtain new characterizations of metacompact and paracompact spaces. In Section 3 we also briefly discuss how the classical metrization theorems of Urysohn ([5], p. 125, [7], [8]) can be deduced from our theorem.

For notation not explained here the reader is referred to Kelley [5]. We recall that a topological space is called *metacompact* (or *pointwise paracompact*) if each open cover has a point-finite open refinement. Finally, if $\{\mathcal{A}_i\}_{i \in I}$ is a finite collection of covers of a space X, then $\bigwedge \{\mathcal{A}_i | \ i \in I\}$ is the cover consisting of all non-empty sets of the form $\bigcap \{A_i | \ i \in I\}$, $A_i \in \mathcal{A}_i$.