One-point compactifications of intuitionistic locally compact spaces

by

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1. Introduction. Brouwer introduced in his paper [1] the so-called *located compact spaces*; they were described axiomatically by Freudenthal in [4]. In [6] an axiomatic treatment for a more general class of spaces was given.

By specializing this treatment for locally compact spaces a non-metric axiomatization was obtained, analogous to Freudenthal's non-metric axiomatization of [4] for the located compact spaces.

For locally compact spaces, one can try to construct one-point compactifications. Professor J. de Groot suggested to the author the possibility of a simplified proof of the adequateness of the non-metric characterization of locally compact spaces in [6] by means of the one-point compactification and Freudenthal's axiomatic characterization of located compact spaces.

To this purpose the one-point compactification has to be constructed metrically. This construction is executed in this paper and is fairly complicated. Once the one-point compactification is available, the adequateness of the axiomatization is easily checked.

As a compensation for the lengthy construction of this compactification, the elaborate constructions of [6], 4.2.3. and 2.3.12 are no longer necessary for the derivation of the axiomatization; a fairly simple proof of [6], 4.4.1a is obtained as a byresult (theorem 4.6).

2. Notations and generalities.

2.1. Intuitionistic notions not explained in this paper can be found in [5]. For most topological notions, we can take the usual definitions. However, classically equivalent definitions need not to be so intuitionistically, therefore some of the more important notions in intuitionistic topology are explicitly defined in this section.

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2.2. Notations. N denotes the *natural numbers*, zero excluded. i, j, k, l, m, n always stand for elements of N. *Denumerably infinite sequences of objects* $a_1, a_2, \ldots$ are written as $\langle a_n\rangle_n$. $\{X_i: i \in I\}$ denotes a *species* of objects X_i, indexed by a species I. $\delta, \varepsilon, \eta$ always denote *positive real numbers*.

The *restriction* of a mapping f to a species V is denoted by $f|V$. A *spread* $\langle \Theta, \vartheta\rangle$ is a spread with spread law Θ (identified with the species of admissible sequences) and complementary law ϑ (cf. [5], 3.1.2).

2.3. Definitions. A *topology on a species* V (topologies will be denoted by $\mathfrak{T}, \mathfrak{T}'$, etc.) is a collection of subspecies of V which contains $\emptyset$ and V, and which is closed with respect to finite intersections and arbitrary unions.

A *topological space* (to be denoted by greek capitals Γ, Γ', Δ, etc.) is a pair $\langle V, \mathfrak{T}\rangle$, $\mathfrak{T}$ a topology on V. The elements of V are the points of the space. Speaking about a given space $\langle V, \mathfrak{T}\rangle$, the *complement* of $W \subset V$ (denoted by W^c) is the species $V - W$.

W is called *secured* if W contains a point.

The elements of $\mathfrak{T}$ are the open sets of the space.

We define for any space $\Gamma = \langle V, \mathfrak{T}\rangle$ and any $p \in V$ $\mathfrak{T}_p$:

$$W \in \mathfrak{T}_p \leftrightarrow p \in W \in \mathfrak{T}\,.$$

W is a *neighbourhood* of p if $(\exists U \in \mathfrak{T}_p)(U \subset W)$.

$$p \in W^- \leftrightarrow \forall U \in \mathfrak{T}_p \exists q (q \in U \cap W)\,.$$

W^- is called the *closure* of W, and the points of W^- are the *closure points* of W. W is *closed* if $W^- = W$. $V \mathbin{\underline{\cup}} W = (V \cup W)^-$.

Interior W (or Int W) denotes the maximal open pointspecies contained in W.

A mapping f from Γ into Δ is *continuous* if the counterimage of an open species of Δ is always open in Γ.

2.4. Definitions. $\{V_i: i \in I\}$ *covers* W if $W \subset \bigcup \{V_i: i \in I\}$. $\{V_i: i \in N\}$ is called a *star-finite covering* if $\mathfrak{A}_j = \{i: V_i \cap V_j \neq \emptyset\}$ is a subspecies of a finite species for every $j \in N$.

2.5. Definitions. A *metric space* is a pair $\langle V, \varrho\rangle$ consisting of a species V with an apartness relation $\#$ and a mapping ϱ from $V \times V$ into the real numbers such that $(p, q, r \in V, W \subset V)$

(a) $\varrho(p, q) > 0 \leftrightarrow p \# q$,

(b) $\varrho(p, q) \not< 0$,

(c) $\varrho(p, q) = \varrho(q, p)$,

(d) $\varrho(p, q) \not> \varrho(p, r) + \varrho(r, q)$.

We put

$$U(\varepsilon, W) = \{q: \exists p \in W(\varrho(p, q) < \varepsilon)\}\,, \qquad U(\varepsilon, p) = U(\varepsilon, \{p\})\,.$$

$U(\varepsilon, W)$ is said to be an *ε-neighbourhood* of W.

The topology associated with ϱ in the usual way is denoted by $\mathfrak{T}(\varrho)$. To the metric space $\langle V, \varrho\rangle$ corresponds therefore the topological space $\langle V, \mathfrak{T}(\varrho)\rangle$. In many places where confusion is not to be expected, we simply identify $\langle V, \varrho\rangle$ and $\langle V, \mathfrak{T}(\varrho)\rangle$.

2.6. Definitions. $\Gamma = \langle V, \mathfrak{T}\rangle$. Let $W \subset V$. Then $\Gamma' = \langle W, \mathfrak{T}'\rangle$, $\mathfrak{T}' = \{W': \exists X\, W' = W \cap X \,\&\, X \in \mathfrak{T}\}$ is a *topological space*, and $\mathfrak{T}'$ is the topology on W relative to Γ, or the *relative topology*.

Speaking about a given space Γ, "the topological space W" for $W \subset V$ will denote Γ'.

2.7. Remark. Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$, $\Gamma' = \langle V', \mathfrak{T}(\varrho')\rangle$, and let f be a continuous mapping from V into V'. Then $\varrho'(fx, fy) > 0 \to \varrho(x, y) > 0$. Or, if we put $\varrho'(x, y) > 0 \leftrightarrow x \#' y$, $\varrho(x, y) > 0 \leftrightarrow x \# y$, we obtain

$$fx \#' fy \to x \# y\,.$$

(This is seen as follows. Let $\varrho'(fx, fy) > \varepsilon$. $f^{-1}U(\varepsilon, fx)$ is open, hence $U(\delta, x) \subset f^{-1}U(\varepsilon, fx)$ for some δ. So $y \notin U(\delta, x)$, i.e. $\varrho(y, x) \not< \delta$.)

2.8. Remarks. The topological product of a finite number of spaces is defined as usual. Especially

$$\langle V, \mathfrak{T}(\varrho)\rangle \times \langle V', \mathfrak{T}(\varrho')\rangle = \langle V \times V', \mathfrak{T}(\varrho'')\rangle\,,$$

where $\varrho''(\langle x, x'\rangle, \langle y, y'\rangle) = \sup\{\varrho(x, y), \varrho'(x', y')\}$.

2.9. Definition. Let $\Gamma = \langle V, \mathfrak{T}\rangle$. We call $W \subset V$ *located in* Γ (notation $W \in \mathfrak{L}(\Gamma)$) if

$$\forall p \in V \forall X \in \mathfrak{T}_p\big(\exists q (q \in X \cap W) \vee \exists Y \in \mathfrak{T}_p(Y \cap W = \emptyset)\big)\,.$$

We call W, W' *relatively located* $(\langle W, W'\rangle \in \mathfrak{L}^2(\Gamma))$ if

$$\forall p \in V \forall X \in \mathfrak{T}_p \exists Y \in \mathfrak{T}_p\big(\exists q (q \in Y \cap W) \,\&\, \exists q (q \in Y \cap W') \to \\ \to \exists q (q \in X \cap W \cap W')\big)\,.$$

2.10. Theorem. $\Gamma = \langle U, \mathfrak{T}\rangle$.

(a) $V, W \in \mathfrak{L}(\Gamma) \to V \cup W \in \mathfrak{L}(\Gamma)$,

(b) $V \in \mathfrak{L}(\Gamma) \to V^- \in \mathfrak{L}(\Gamma)$,

(c) $V, W \in \mathfrak{L}(\Gamma) \,\&\, \langle V, W\rangle \in \mathfrak{L}^2(\Gamma) \to V \cap W \in \mathfrak{L}(\Gamma)$,

(d) $\langle V, W\rangle \in \mathfrak{L}^2(\Gamma) \to \langle V^-, W\rangle, \langle V, W^-\rangle, \langle V^-, W^-\rangle \in \mathfrak{L}^2(\Gamma)$,

(e) $V^- \in \mathfrak{L}(\Gamma) \,\&\, \neg\neg\, x \in V^- \to x \in V^-$,

(f) $V, V', V'' \in \mathfrak{L}(\Gamma)$ & $\langle V, V''\rangle, \langle V', V''\rangle \in \mathfrak{L}^2(\Gamma)$ & $V'' = V''^{-} \to$
$$\to (V \mathbin{\underline{\cup}} V') \cap V'' = (V \cap V'') \mathbin{\underline{\cup}} (V' \cap V''),$$

(g) $V, V' \in \mathfrak{L}(\Gamma)$ & $\langle V, V'\rangle \in \mathfrak{L}^2(\Gamma) \to (V \cap V') \mathbin{\underline{\cup}} V'' = (V \mathbin{\underline{\cup}} V'') \cap$
$$\cap (V' \mathbin{\underline{\cup}} V''),$$

(h) $V, V', V'' \in \mathfrak{L}(\Gamma)$ & $\langle V, V'\rangle, \langle V, V''\rangle \in \mathfrak{L}^2(\Gamma) \to \langle V, V' \cup V''\rangle \in \mathfrak{L}^2(\Gamma)$.

Proofs. See [6], Chapter I, 4; all proofs are straighforward.

2.11. DEFINITION. Let $\langle V, \varrho\rangle$ be a metric space. $W \subset V$ is called *metrically located in* $\langle V, \varrho\rangle$ if $\varrho(p, W) = \inf\{\varrho(p, q): q \in W\}$ exists for every $p \in V$.

2.12. Remark. Every metrically located pointspecies is located. If W is metrically located, then W^{-} too, since $\varrho(p, W) = \varrho(p, W^{-})$ for every p.

2.13. THEOREM. *Let $\langle V, \varrho\rangle$ be a metric space, and let $\langle W_n\rangle_n$ be a sequence of metrically located pointspecies such that*

$$\forall i\big(W_i \subset W_{i+1} \subset U(\varepsilon_i, W_i)\big), \qquad \sum_{i=1}^{\infty} \varepsilon_i < \infty.$$

Then $W = \bigcup_{i=1}^{\infty} W_i$ is again a metrically located pointspecies.

Proof. Straightforward ([6], 1.4.12).

2.14. DEFINITION. $\Gamma = \langle V, \mathfrak{T}\rangle$. We define $\Subset$ by

$$W \Subset W' \leftrightarrow \forall p \in V \exists W'' \in \mathfrak{T}_p (W'' \cap W = \emptyset \vee W'' \subset W').$$

2.15. THEOREM. *Let $\Gamma = \langle V, \mathfrak{T}\rangle$ be a topological space, $U, U', W, W' \subset V$.*

(a) $W \Subset W' \to W'^{c} \Subset W^{c}$,

(b) $U \Subset U'$ & $W \Subset W' \to U \cup W \Subset U' \cup W'$.

Proof. Trivial.

3. Separable metric spaces.

3.1. DEFINITION. A topological space $\Gamma = \langle V, \mathfrak{T}\rangle$ is called *separable* if there is a sequence $\langle p_n\rangle_n \subset V$ such that $\langle p_n\rangle_n^{-} = V$. $\langle p_n\rangle_n$ is called a *basic pointspecies* for Γ.

3.2. DEFINITION. *Fundamental sequences* (*Cauchy sequences*) are defined as usual. A metric space $\langle V, \varrho\rangle$ is called *metrically complete* if every sequence which is fundamental with respect to ϱ is a convergent sequence. $\Gamma = \langle V, \mathfrak{T}\rangle$ is called (*topologically*) *complete* if for some metric ϱ $\mathfrak{T}(\varrho) = \mathfrak{T}$, and $\langle V, \varrho\rangle$ is metrically complete.

3.3. DEFINITION. Let $\langle V, \varrho\rangle$ be a metric space. We say that $\langle V, \varrho\rangle$ *has a point representation* if there is a sequence $\langle p_n\rangle_n$ (the *basis* of the representation) and a spread with a defining pair $\langle \Theta, \vartheta\rangle$ such that

(a) $\langle i_1, \ldots, i_k\rangle \in \Theta \to \vartheta\langle i_1, \ldots, i_k\rangle = \langle p_{i_1}, \ldots, p_{i_k}\rangle$,

(b) Every spread element converges to a point of V,

(c) For every $p \in V$ there exists a spread element converging to p.

3.4. Remark. Suppose that $\langle V, \varrho\rangle$ has a point representation and $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho')$. Then $\langle V, \varrho'\rangle$ has a point representation (as is easily verified). Hence we can say that the topological space $\langle V, \mathfrak{T}(\varrho)\rangle$ has a point representation.

3.5. THEOREM. *Let $\Gamma = \langle V, \mathfrak{T}\rangle$ be a separable complete space and let $I \subset N$. Then*

(a) *Γ possesses a point representation,*

(b) $V \subset \bigcup\{W_i: i \in I\} \to V \subset \bigcup\{\operatorname{Int} W_i: i \in I\}$,

(c) $X \in \mathfrak{L}(\Gamma) \to (X \Subset Y \leftrightarrow X^{-} \subset \operatorname{Int} Y)$,

(d) $X \Subset Y \leftrightarrow X^{c} \cup Y = V$,

(e) $V \subset \bigcup\{U_a: a \in A\}$, *$U_a$ open for $a \in A$. Then there is an enumerable subcovering $\langle U_{a(n)}\rangle_n$ of V.* (Intuitionistic analogue to Lindelöf's theorem.)

(f) *A mapping of Γ into a separable metric space is always continuous.*

Proof. (b)-(f) are proved in [7], [8]. (a) is proved in [6], 2.5. We describe the construction here.

Let $\langle p_n\rangle_n$ be a basic pointspecies for $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$, $\langle V, \varrho\rangle$ complete. We construct $\langle \Theta, \vartheta\rangle$ such that

(A) $$\langle \emptyset\rangle \in \Theta;\ i \in N \to \langle i\rangle \in \Theta,$$

(B) $$\langle i_1, \ldots, i_{k+1}\rangle \in \Theta \to \varrho(p_{i_{k+1}}, p_{i_k}) < 3.2^{-k},$$

(C) $$\langle i_1, \ldots, i_k\rangle \in \Theta \;\&\; \varrho(p_j, p_{i_k}) < 2^{-k+1} \to \langle i_1, \ldots, i_k, j\rangle \in \Theta,$$

(D) $$\vartheta\langle i_1, \ldots, i_k\rangle = \langle p_{i_1}, \ldots, p_{i_k}\rangle.$$

The verification that $\langle \Theta, \vartheta\rangle$ is a point representation is straightforward.

3.6. THEOREM. *A located subspecies of a separable complete space which contains at least one point (is secured) is separable.*

Proof. Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ be a separable topological space, $\langle V, \varrho\rangle$ complete. Suppose $q \in W$, $W \in \mathfrak{L}(\Gamma)$.

$$\mathfrak{C}_k = \Big\{U(\varepsilon, p):\ \varepsilon < 2^{-1}k^{-1} \;\&\; \big(U(\varepsilon, p) \cap W = \emptyset \vee (\exists r)\big(r \in U(\varepsilon, p) \cap W\big)\big)\Big\}$$

is an open covering, hence there exists (3.5(e)) an enumerable subcovering $\langle U(\varepsilon_{k,n}, p_{k,n})\rangle_n$ such that

$$\forall k, n\big(U(\varepsilon_{k,n}, p_{k,n}) \cap W = \emptyset \vee U(\varepsilon_{k,n}, p_{k,n}) \cap W \neq \emptyset\big).$$

We construct $q_{k,n}$ for every k, n such that

$$q_{k,n} \in U(\varepsilon_{k,n}, p_{k,n}) \cap W \vee \big(U(\varepsilon_{k,n}, p_{k,n}) \cap W = \emptyset \;\&\; q_{k,n} = q\big).$$

Take a $p \in W$. For every k there exists an n such that $p \in U(\varepsilon_{k,n}, p_{k,n})$, hence $\varrho(q_{k,n}, p) < k^{-1}$. Therefore $W \subset \{q_{k,n}\colon k, n \in N\}^{-}$.

The following lemma is useful on some occasions.

3.7. Lemma. *Let $\Gamma = \langle V, \varrho\rangle$ be a complete separable metric space. If $\varrho'(x, y)$ satisfies requirements* (b)-(d) *for a metric and $\varrho'(x, x) = 0$ for every x, then*

$$\forall x \forall \varepsilon \exists \delta \forall y \big(\varrho(x, y) < \delta \to \varrho'(x, y) < \varepsilon\big)\,.$$

Proof. ϱ' is defined on $V \times V$ and is therefore continuous on the complete separable metric space $\Gamma \times \Gamma$. Hence for every pair $\langle x, y\rangle \in V \times V$ and every ε there exists a δ such that

$$\varrho(x, x') < \delta \;\&\; \varrho(y, y') < \delta \to |\varrho'(x, y) - \varrho'(x', y')| < \varepsilon\,.$$

Take $x = y = y'$. Then

$$\varrho(x, x') < \delta \;\&\; \varrho(x, x) < \delta \to |\varrho'(x, x) - \varrho'(x', x)| < \varepsilon\,.$$

Therefore

$$\varrho(x, x') < \delta \to \varrho'(x', x) < \varepsilon\,.$$

3.8. Definition. A *located compact space* is a separable complete space with a finitary point representation.

3.9. Definition. Let $\langle V, \varrho\rangle$ be a metric space. A species $\{p_1, \ldots, p_n\} \subset V$ is called an ε-*net* for $\langle V, \varrho\rangle$ if

$$\forall x \in V \exists i (1 \leqslant i \leqslant n \;\&\; \varrho(p_i, x) < \varepsilon)\,.$$

3.10. Definition. A space is called *compact* if every open covering possesses a finite subcovering.

3.11. Theorem. $\Gamma = \langle V, \mathfrak{T}\rangle$. *The following conditions are equivalent*:

(a) *Γ is located compact.*

(b) *For a certain ϱ, $\mathfrak{T} = \mathfrak{T}(\varrho)$, $\langle V, \varrho\rangle$ is a complete metric space, and possesses an ε-net for every ε.*

(c) *For a certain ϱ, $\mathfrak{T} = \mathfrak{T}(\varrho)$, $\langle V, \varrho\rangle$ is a complete metric space and Γ is compact.*

Proof. (a)$\to$(c) was proved in [2], with a different definition of the "located compact space", but we can use essentially the same proof.

(c)$\to$(b). $\{U(\varepsilon, p)\colon p \in V\}$ is an open covering of V, hence there is a subcovering $\{U(\varepsilon, p_1), \ldots, U(\varepsilon, p_n)\}$ of V. Clearly $\{p_1, \ldots, p_n\}$ is an ε-net for $\langle V, \varrho\rangle$.

(b)$\to$(a). Let, for every k, $\langle p_1^k, \ldots, p_{n(k)}^k\rangle$ be a 2^{-k}-net. We consider the sequence $\langle p_n\rangle_n = p_1^1, \ldots, p_{n(1)}^1, p_1^2, \ldots, p_{n(2)}^2, p_1^3, \ldots$, and we put $m(k) = n(1) + \ldots + n(k)$. We change the construction of the point representation in 3.5 by substituting (C′) for (C):

(C′) $\langle i_1, \ldots, i_k\rangle \in \Theta \;\&\; \varrho(p_j, p_{i_k}) < 2^{-k+1} \;\&\; j \leqslant m(k) \to \langle i_1, \ldots, i_k, j\rangle \in \Theta$.

3.12. Theorem. *Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ be a located compact space, and let $\langle V, \varrho\rangle$ be complete. Then*:

(a) *Every mapping from Γ into a separable metric space is uniformly continuous.*

(b) *Let $P(\varepsilon, p)$ be a property such that $\delta < \varepsilon \;\&\; P(\varepsilon, p) \to P(\delta, p)$, $\forall p \in V \exists \varepsilon P(\varepsilon, p)$. Then $\exists \delta \forall p \in V P(\delta, p)$.*

(c) *If $f(x) > 0$ for all $x \in V$, then $\inf\{f(x)\colon x \in V\} > 0$.*

Proof. The assertions of this theorem are easily proved using the fan theorem with respect to a finitary point representation for Γ. For (a), see e.g. [3].

3.13. Remark. From 3.12(a) it is easily seen that, if $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ is a located compact space, then $\langle V, \varrho\rangle$ is complete.

3.14. Theorem. *Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ be a located compact space. Then*

(a) *(W is secured)$\to$($W \in \mathfrak{L}(\Gamma) \leftrightarrow W$ is metrically located),*

(b) *$\langle W, W'\rangle \in \mathfrak{L}^2(\Gamma) \leftrightarrow \forall \varepsilon \exists \delta \big(U(\delta, W) \cap U(\delta, W') \subset U(\varepsilon, W \cap W')\big)$.*

Proof. (a). Let W be secured. $W \in \mathfrak{L}(\Gamma)$,

$$\mathfrak{C}_k = \big\{U(\varepsilon, p)\colon \varepsilon < 2^{-k} \;\&\; \big(U(\varepsilon, p) \cap W = \emptyset \vee \exists q (q \in U(\varepsilon, p) \cap W)\big)\big\}$$

is an open covering, hence there is a finite subcovering (3.11(c))

$$\{U(\varepsilon_1, p_1), \ldots, U(\varepsilon_n, p_n)\} \subset \mathfrak{C}_k\,.$$

We suppose $q_i \in U(\varepsilon_i, p_i) \cap W$ for $1 \leqslant i \leqslant m$, $U(\varepsilon_i, p_i) \cap W = \emptyset$ for $m < i \leqslant n$. $\{U(\varepsilon_1, p_1), \ldots, U(\varepsilon_m, p_m)\}$ covers W.

If $q \in W$, it follows that

$$\inf\{\varrho(p, q_i)\colon 1 \leqslant i \leqslant m\} < \varrho(q, p) + 2^{-k+1}\,.$$

This proves that $\varrho(p, W)$ can be determined within 2^{-k+1} for every k. Remark that $q_1, \ldots, q_m$ is a 2^{-k+1}-net for W.

If W is metrically located, we obtain immediately that $W \in \mathfrak{L}(\Gamma)$. (Remark. We could also have proved (a) using 3.12(b), but from this proof we can obtain useful conclusions.)

(b) Let $\langle W, W'\rangle \in \mathfrak{L}^2(\Gamma)$. Consider the property $P_\varepsilon(\eta, p)$:

$$p \in U(\eta, W) \cap U(\eta, W') \to p \in U(\varepsilon, W \cap W')\,.$$

We apply 3.12(b) and obtain a δ such that

$$\forall p \big(p \in U(\delta, W) \cap U(\delta, W') \to p \in U(\varepsilon, W \cap W')\big)\,,$$

hence

$$U(\delta, W) \cap U(\delta, W') \subset U(\varepsilon, W \cap W')\,.$$

The converse of (b) is trivial.

3.15. Remarks. (a) From the proof of 3.14(a) we can draw the conclusion: *W is closed located in a located compact space $\langle V, \mathfrak{T}(\varrho)\rangle$ iff W is a located compact space in the relative topology* (with 3.11(b)).

(b) *If $W \in \mathfrak{L}(\Gamma)$, Γ located compact, then W is empty or secured* (see proof of 3.14(a)).

(c) *If X, Y are located, compact subspecies in a metric space $\langle V, \varrho\rangle$, then $\varrho(X, Y)$ is defined, and if $\varrho(x, y) > 0$ for all $x \in X$, $y \in Y$, then $\varrho(X, Y) > 0$.* (3.11(c), 3.12(c).)

3.16. Theorem (compare [4], 7.10). *Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ be a located compact space. Then*

(a) *For every ε, there exists a finite ε-covering, consisting of located pointspecies.*

(b) *Let $\delta < \varepsilon$, $W \in \mathfrak{L}(\Gamma)$. Then we can find a W', W' open, $W' \in \mathfrak{L}(\Gamma)$, such that $U(\delta, W) \subset W' \subset W'^{-} \subset U(\varepsilon, W)$, and a located W^* such that $U(\varepsilon, W)^c \subset W^* \subset W^{*-} \subset U(\delta, W)^c$.*

Proof. (a). Let $\langle \Theta, \vartheta\rangle$ be the finitary point representation according to the description in the proof of 3.11, (b)$\rightarrow$(a). Let $\langle \sigma_1, \ldots, \sigma_m\rangle$ be an enumeration (without repetitions) of all sequences of length $k+4$ in Θ. With every σ_i we associate a spread $\langle \Theta_i, \vartheta_i\rangle$ (a subspread of $\langle \Theta, \vartheta\rangle$) as follows. Θ_i consists of σ_i and all descendants and ascendants of σ_i in Θ. $\vartheta_i = \vartheta|\Theta_i$. The point species represented by $\langle \Theta_i, \vartheta_i\rangle$ we denote by W_i. Then, clearly, $\{W_1, \ldots, W_m\}$ covers Γ. If we have chosen k such that $2^{-k} < \varepsilon$, we also have diameter $W_i < \varepsilon$ for every i, $1 \leqslant i \leqslant m$.

(b). We use (a). We first prove the existence of a $W'' \in \mathfrak{L}(\Gamma)$, such that $U(\delta, W) \subset W'' \subset U(\varepsilon, W)$, for any pair δ, ε such that $\delta < \varepsilon$. Let $\eta = 2^{-1}(\varepsilon - \delta)$. Let $\{W_1, \ldots, W_m\}$ be an η-covering consisting of located pointspecies; we may suppose the W_i to be closed. We select a subspecies, say $\{W_1, \ldots, W_r\}$, such that

$$\varrho(W, W_i) < \delta \rightarrow i \leqslant r, \quad i \leqslant r \rightarrow \varrho(W, W_i) < \varepsilon - \eta .$$

(Since for every $i \leqslant m$ $\varrho(W, W_i) < \varepsilon - \eta \vee \varrho(W, W_i) > \delta$, the selection of this subspecies is always possible.)

We take $W'' = W_1 \cup \ldots \cup W_r$.

If $x \in U(\delta, W)$, then $x \in W_i$ for a certain i, $1 \leqslant i \leqslant m$; but since $\varrho(W, W_i) < \delta$, it follows that $i \leqslant r$, and $x \in W_i \subset W''$.

Let $x \in W''$. Then $x \in W_i$, $i \leqslant r$ for some i. Since $\varrho(W, W_i) < \varepsilon - \eta$, there exist $y \in W$, $z \in W_i$, $\varrho(y, z) < \varepsilon - \eta$; and since diameter $W_i < \eta$, $\varrho(x, z) < \eta$, hence $\varrho(y, x) < \varepsilon$; therefore $x \in U(\varepsilon, W)$.

Now let $\langle \eta_n\rangle_n$ be a sequence such that $\eta' = \sum_{n=1}^{\infty} \eta_n < \varepsilon - \delta$. We construct a sequence $\langle W_n\rangle_n$ such that $W = W_1$,

$$U(2^{-1}\eta_n, W_n) \subset W_{n+1} \subset U(\eta_n, W_n) .$$

$W' = \bigcup_{n=1}^{\infty} W_n$ is open, metrically located (2.12; hence W' is located), and $U(\delta, W) \subset W' \subset W'^{-} \subset U(\varepsilon, W)$.

The construction of W^* is effected as follows. We select a subspecies of $\{W_1, \ldots, W_m\}$, say $\{W_s, W_{s+1}, \ldots, W_m\}$, such that

$$\varrho(W, W_i) > \varepsilon - \eta \rightarrow i \geqslant s , \quad i \geqslant s \rightarrow \varrho(W, W_i) > \delta .$$

Put $W^* = W_s \cup \ldots \cup W_m$. We remark that $\sup\{\operatorname{diam} W_i \colon 1 \leqslant i \leqslant m\} = \eta'' < \eta$.

$x \notin U(\varepsilon, W) \rightarrow \varrho(x, W) \nless \varepsilon$, so $x \in W_i$ for some i. For arbitrary $y \in W_i$, $z \in W$ we have

$$\varrho(x, y) < \eta'' < \eta , \quad \varrho(y, z) \nless |\varrho(x, z) - \varrho(x, y)| .$$

Hence $\varrho(y, z) < \varepsilon - \eta''$, hence $\varrho(y, W) \ngtr \varepsilon - \eta''$, and therefore $\varrho(W_i, W) \ngtr \varepsilon - \eta'' < \varepsilon - \eta$, so $i \geqslant s$ and $x \in W^*$.

Let conversely $x \in W^{*-}$. Let $\varrho(W, W_i) > \delta + \eta'''$ for $s \leqslant i$. Then there is an $y \in W_i$, $i \geqslant s$, $\varrho(x, y) < \eta'''$. $\varrho(W, y) > \delta + \eta'''$, hence $\varrho(W, x) > \delta$.

3.17. Theorem. *Let $\langle V, \varrho\rangle$ be a metric space, and let X be located compact, $X, Y \subset V$. Then $X \Subset Y \leftrightarrow U(\varepsilon, X) \subset Y$ for some ε.*

Proof. $U(\varepsilon, X) \subset Y$ implies $X^{-} \subset \text{Interior } Y$, hence $X \Subset Y$ (3.5(c)). Suppose $X \Subset Y$. $\forall x \in X \exists \varepsilon (U(\varepsilon, x) \subset Y)$. If we take in 3.12(b): $P(\varepsilon, x) \leftrightarrow$ $\leftrightarrow U(\varepsilon, x) \subset Y$, we see that $\exists \delta \forall x \in X (U(\delta, x) \subset Y)$, hence $U(\delta, X) \subset Y$.

3.18. Theorem. *Let $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ be a located compact space.*

(a) *Let $\{W_1, \ldots, W_n\}$ be a covering of Γ. Then there exists an $\eta > 0$, such that for every $x \in V$ an i, $1 \leqslant i \leqslant n$ can be found such that $U(\eta, x) \subset W_i$.*

(b) *Suppose $X \cup Y = V$. Then there are closed located X^*, Y^*, such that $X^* \subset X$, $Y^* \subset Y$, $X^* \cup Y^* = V$.*

Proof. (a). $\{\text{Interior } W_i \colon 1 \leqslant i \leqslant n\}$ is a covering of Γ (3.5(b)), hence $\forall x \in V \exists \varepsilon \exists i (1 \leqslant i \leqslant n \;\&\; U(\varepsilon, x) \subset W_i)$. We apply 3.12(b) to obtain the η we looked for.

(b). Let $\forall x \in V (U(\eta, x) \subset X \vee U(\eta, x) \subset Y)$. Let $\{W_1, \ldots, W_n\}$ be an $2^{-1}\eta$-covering of Γ by located species (3.16(a)). For every i, $1 \leqslant i \leqslant n$, $U(2^{-1}\eta, W_i) \subset X \vee U(2^{-1}\eta, W_i) \subset Y$ (for we may suppose W_i to be secured). Let $x \in W_i$. Then $U(\eta, x) \subset X \vee U(\eta, x) \subset Y$; since diameter $W_i < 2^{-1}\eta$, it follows that $U(2^{-1}\eta, W_i) \subset X$ or $U(2^{-1}\eta, W_i) \subset Y$. So we are able to divide $\{W_1, \ldots, W_n\}$ into two groups, say $\{W_1, \ldots, W_r\}$ and $\{W_{r+1}, \ldots, W_n\}$, such that $U(2^{-1}\eta, W_i) \subset X$ for $i \leqslant r$, $U(2^{-1}\eta, W_i) \subset Y$ for $i > r$. Hence we can put $X^* = W_1 \underline{\cup} \ldots \underline{\cup} W_r$, $Y^* = W_{r+1} \underline{\cup} \ldots \underline{\cup} W_n$. Then $X^* \cup Y^* = V$, $X^* \subset X$, $Y^* \subset Y$.

3.19. Definition. A space $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$ is called *locally compact*, if Γ is complete, separable, and if every point of V possesses a located compact neighbourhood.

3.20. Remark. Equivalently in 3.19 we can require that every point x of V possesses an open neighbourhood U_x such that U_x^- is located compact (hence U_x is located). This is seen as follows. Let W_x be a located compact neighbourhood of x. Then $U(\delta, x) \subset W_x$ for some δ. Then we construct an open located neighbourhood U_x (3.16) such that $U(2^{-1}\delta, x) \subset U_x \subset U(\delta, x)$, $U_x^- \subset W_x$.

4. Compactifications.

4.1. Definitions. A located compact space $\Gamma' = \langle V', \mathfrak{T}(\varrho')\rangle$ is said to be a *minimal compactification* of the locally compact space Γ if Γ is homeomorphic to $\langle V, \mathfrak{T}(\varrho)\rangle$, $V \subset V'$, $\varrho = \varrho'|V \times V$ and

(a) $V^- = V'$,

(b) $\varrho'(x, y) > 0 \to x \in V \vee y \in V$.

Γ' is said to be a *one-point compactification* of Γ if Γ is homeomorphic to $\langle V, \mathfrak{T}(\varrho)\rangle$, $V \subset V'$, $\varrho = \varrho'|V \times V$ and

(a) $V' - V = \{x_0\}$,

(b) $x \in V \to \varrho'(x, x_0) > 0$.

Our aim is the construction of one-point compactifications. To achieve this we have to prove a number of lemmas and a theorem first.

4.2. Lemma. *Let $\langle V, \varrho\rangle$ be complete, separable, metric, and let $W \cup W' = V$. Then W and W' are relatively located.*

Proof. $\mathrm{Int}\, W \cup \mathrm{Int}\, W' = V$ (3.5(b)). Let $x \in V$. We can find an η such that $U(\eta, x) \subset W \vee U(\eta, x) \subset W'$. Take a $\delta < \eta$, and suppose that

$$y \in U(\delta, x) \cap W\,, \qquad z \in U(\delta, x) \cap W'\,.$$

$$U(\eta, x) \subset W \to z \in W \cap W'; \qquad U(\eta, x) \subset W' \to y \in W \cap W'\,.$$

In both cases $x \in U(\delta, W \cap W')$; this proves relative locatedness.

4.3. Lemma. *Let $\langle V, \varrho\rangle$ be complete, separable, metric, and let $W \Subset W' \Subset W'' \subset V$, $Z \subset V$, W'' and Z closed, W'' located and secured, $W \cap Z = \emptyset$, $W' \cup Z = W''$. Then*

(a) $W'^c \cup Z = W'^c \underline{\cup} Z$,

(b) $(W'^c \cup Z) \cap W'' = Z$,

(c) *If W and Z are relatively located in W'', then W and $W'^c \cup Z$ are relatively located in V,*

(d) *If Z is located in W'', then $Z \cup W'^c$ is located in V.*

Proof. (a). Let $x \in W'^c \underline{\cup} Z$. For every n there exists a point y_n such that $y_n \in U(n^{-1}, x) \cap (W'^c \cup Z)$.

Since $W' \Subset W''$, there exists an η such that $U(\eta, x) \subset W'^c \vee U(\eta, x) \subset W''$. If $U(\eta, x) \subset W'^c$, then $x \in W'^c \cup Z$. Suppose therefore $U(\eta, x) \subset W''$.

Since W'' is closed, located and secured, W'' is complete, separable, metric, hence (3.6) $(\mathrm{Int}\, W' \cap W'') \cup (\mathrm{Int}\, Z \cap W'') = W''$.

So, for some $\eta' < \eta$,

$$U(\eta', x) \subset W' \vee U(\eta', x) \subset Z\,.$$

If $U(\eta', x) \subset Z$, then $x \in W'^c \cup Z$. If $U(\eta', x) \subset W'$, then for an n such that $\varrho(y_n, x) < n^{-1} < \eta'$, we have $m \geq n \to y_m \in Z$. Z is closed, hence $x = \lim_{m\to\infty} y_m \in Z$, so $x \in W'^c \cup Z$.

The proof of (b) is simple, and the verification of (c) and (d) is routine, using the existence of an η for every x such that

$$U(\eta, x) \subset W'' \vee U(\eta, x) \subset W'^c\,.$$

4.4. Lemma. *Let $\langle V, \mathfrak{T}(\varrho)\rangle$ be a locally compact space. Then there exist sequences $\langle V_n\rangle_n$, $\langle W_n\rangle_n$ such that the V_n are located compact spaces, the W_n are closed located, and V_n, W_m relatively located for every pair n, m, $V = \bigcup_{n=1}^{\infty} V_n$; $V_n \cap W_n = \emptyset$, $V_{n+1} \cup W_n = V$, $V_n \Subset V_{n+1}$, $W_{n+1} \Subset W_n$ for all n.*

Proof. $\mathfrak{C} = \{W\colon \exists p (p \in W)\ \&\ W$ open $\&\ W^-$ is a located compact space$\}$ is an open covering of Γ, and contains therefore (3.5(e)) an enumerable subcovering $\langle U_n\rangle_n$. We construct a sequence $\langle n(i)\rangle_i$, $n(1) = 1$, $\forall i\,(n(i+1) > n(i))$, such that $\forall i\,(U_1 \underline{\cup} \ldots \underline{\cup} U_{n(i)} \subset U_1 \cup \ldots \cup U_{n(i+1)})$, and we put $V_1 = U_1^-$, $V_i = U_1 \underline{\cup} \ldots \underline{\cup} U_{n(i)}$. This construction is carried out by induction.

Suppose V_k to be constructed. Since V_k is compact, $\langle U_n\rangle_n$ must contain a finite subspecies which covers V_k. Therefore we can find an $n(k+1) > n(k)$ such that $V_k \subset U_1 \cup \ldots \cup U_{n(k+1)}$.

Since $\{U_1, \ldots, U_{n(k+1)}\}$ is an open covering of V_k, we have

$$\forall p \in V_k \exists \varepsilon \big(U(\varepsilon, p) \subset U_1 \cup \ldots \cup U_{n(k+1)}\big)\,.$$

Applying 3.5(c) we conclude to $V_k \Subset V_{k+1}$.

We now proceed with the construction of W_n.

Let (3.17) $U(\varepsilon, V_k) \subset V_{k+1}$. We can construct (3.16) a located V'_k such that $U(3^{-1}\varepsilon, V_k) \subset V'_k \subset U(2.3^{-1}\varepsilon, V_k)$. This proves $V_k \Subset V'_k \subset V_{k+1}$ (3.17), and if we take $3^{-1}\varepsilon = \delta$, we conclude

$$U(\delta, V_k) \subset V'_k \quad \text{and} \quad U(\delta, V'_k) \subset V_{k+1}\,.$$

We construct a closed located Z (3.16) such that

$$V_{k+1} \cap U(2^{-1}\delta, V_k)^c \subset Z \subset V_{k+1} \cap U(4^{-1}\delta, V_k)^c\,.$$

Remark that $V_k \cap Z = \emptyset$, V_k and Z relatively located in V_{k+1}.

$$x \in V_{k+1} \to \varrho(x, V_k) > 2^{-1}\delta \vee \varrho(x, V_k) < \delta\,,$$

hence $x \in Z \vee x \in V'_k$, so $V'_k \cup Z = V_{k+1}$.

Eventually we put $W_k = Z \cup V_k'^{c}$. Then $W_k \cap V_k = \emptyset$, $V_{k+1} \cup W_k = V$.

Applying 4.2, 4.3, we obtain that W_k is located and W_k, V_n relatively located for all n, k. From the method of construction we also see that $W_{k+1} \subset V_{k+1}'^{c} \Subset V_k'^{c} \subset W_k$, hence $W_{k+1} \Subset W_k$.

4.5. Lemma. *Let $\langle V, \varrho\rangle$ be complete, separable, metric. Let W, W' be non-empty located compact spaces, and let $W \Subset W'$.*

Then a continuous mapping f can be found such that

$$x \in W \to f(x) = 1\,, \quad x \notin W' \to f(x) = 0\,, \quad \forall x\,(0 \not> f(x) \not> 1)\,.$$

Proof. Let (3.17) $U(\varepsilon, W) \subset W'$. Take $f(x) = \sup\{1-\varrho(x, W)\varepsilon^{-1}, 0\}$. f is everywhere defined (3.14(a)) and satisfies the conditions. The continuity follows from 3.5(f).

4.6. Theorem. *Let $\langle V, \varrho\rangle$ be locally compact. Then there exists a metric ϱ' such that*

(a) $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho')$,

(b) *$\langle V, \varrho'\rangle$ is metrically complete,*

(c) *every secured located $W \subset V$ is metrically located with respect to ϱ',*

(d) *ϱ' is bounded by* 1.

Proof. Let $\langle V_n\rangle_n$, $\langle W_n\rangle_n$ be the sequences constructed in Lemma 4.4. We put

$$X_1 = V_2\,, \quad X_n = V_{n+1} \cap W_{n-1} \quad \text{for} \quad n > 1\,,$$

$$X_1' = X_1 \cup X_2\,, \quad X_n' = X_{n-1} \cup X_n \cup X_{n+1} \quad \text{for} \quad n > 1\,.$$

$\bigcup_{i=1}^{\infty} X_i = V$, since $\bigcup_{i=1}^{n} X_i = V_{n+1}$ can be proved by induction.

We remark that

$$X_i \cap X_j \neq \emptyset \to |i-j| \leqslant 1; \quad X_i' \cap X_j' \neq \emptyset \to |i-j| \leqslant 3\,.$$

Hence $\langle X_n\rangle_n$ and $\langle X_n'\rangle_n$ are star-finite coverings of V.

X_j, X_j' are located compact spaces and $X_j \Subset X_j'$ for every j. (In fact, $X_j' = V_{j+2} \cap W_{j-2}$, $W_{j-1} \Subset W_{j-2}$, $V_{j+1} \Subset V_{j+2}$, so $X_j \Subset X_j'$ (2.4(b)).) We apply Lemma 4.5 to obtain f_j, $0 \not> f_j(x) \not> 1$,

$$X_j = \emptyset \to f_j(x) = 0 \quad \text{for all } x\,.$$

$$X_j \neq \emptyset \to \big(x \in X_j \to f_j(x) = 1 \;\&\; x \notin X_j' \to f_j(x) = 0\big)\,.$$

We define a metric:

$$\varrho''(x, y) = \varrho(x, y) + \sum_{n=1}^{\infty} |f_n(x) - f_n(y)|\,.$$

Convergence is proved thus:

Let $x, y \in V_n$. Then

$$|f_m(x) - f_m(y)| = 0 \quad \text{for} \quad m \geqslant n+2$$

since $X_m' \cap V_n = \emptyset$ for $m \geqslant n+2$.

$\varrho''(x, y) = \varrho''(y, x)$, $\varrho''(x, y) \not> \varrho''(x, z) + \varrho''(z, y)$, $\varrho''(x, y) \not< 0$, and $\varrho''(x, x) = 0$ are immediate.

By 3.7 we obtain

$$\forall x\, \forall \varepsilon\, \exists \delta\, \forall y\, \big(\varrho(x, y) < \delta \to \varrho''(x, y) < \varepsilon\big)\,.$$

Since, on the other hand, $\varrho''(x, y) < \delta \to \varrho(x, y) < \delta$, it follows that ϱ'' is a metric and $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho'')$.

Put

$$\varrho'(x, y) = \inf\{1, \varrho''(x, y)\}\,.$$

ϱ' is again a metric and $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho')$.

Let $x \in X_n$, and let W be located and secured, say $p \in W$. Then

$$\mathfrak{C}_k = \big\{U(\varepsilon, y):\ \varepsilon < k^{-1} \;\&\; \big[U(\varepsilon, y) \cap W = \emptyset \vee \exists q\, \big(q \in U(\varepsilon, y) \cap W\big)\big]\big\}$$

is an open covering of V. Since X_n' is compact, there exists a finite subspecies of $\mathfrak{C}_k$ which covers X_n' (3.10(c)), say $\{U(\varepsilon_i, y_i):\ 1 \leqslant i \leqslant m\}$. Let $U(\varepsilon_i, y_i) \cap W = \emptyset \leftrightarrow i > t$, and let $q_i \in U(\varepsilon_i, y_i) \cap W$ for $1 \leqslant i \leqslant t$.

We put $\inf\{\varrho'(q_i, x), \varrho'(p, x):\ 1 \leqslant i \leqslant t\} = d_k$. We want to prove that $\langle d_k\rangle_k$ converges and that the limit d represents $\varrho'(x, W)$. Suppose $q \in W$. If $q \in X_n'$, then $q \in U(\varepsilon_i, y_i)$ for some i, $1 \leqslant i \leqslant t$. In that case $\varrho'(q, q_i) < 2k^{-1}$. Hence since $\varrho'(x, q_i) \not> \varrho'(x, q) + \varrho'(q_i, q)$,

$$\varrho'(x, q_i) < \varrho'(x, q) + 2k^{-1}\,.$$

We conclude to $d_k < \varrho'(x, q) + 2k^{-1}$, so $d_k - 2k^{-i} < \varrho'(x, q)$.

If $q \notin X_n'$, it follows that $\varrho'(x, q) = 1$, hence $d_k - 2k^{-1} < \varrho'(x, q)$. We obtain therefore:

$$q \in X_n' \vee \neg q \in X_n' \to d_k - 2k^{-1} < \varrho'(x, q)\,,$$

and hence

$$\neg\neg(q \in X_n' \vee \neg q \in X_n') \to d_k - 2k^{-1} \not> \varrho'(x, q)\,.$$

Since the premiss is valid, we have

$$\forall x \in V\, \forall q \in W\, \big(d_k - 2k^{-1} \not> \varrho'(x, q)\big)\,.$$

At the same time it is clear that

$$\forall x \in V\, \exists q \in W\, \big(\varrho'(x, q) < d_k + k^{-1}\big)\,.$$

Now consider a fixed x, and let q_k, q_{k+p} be chosen such that

$$\varrho'(x, q_k) < d_k + k^{-1}\,, \quad \varrho'(x, q_{k+p}) < d_{k+p} + (k+p)^{-1}\,.$$

At the same time we have

$$\varrho'(x, q_{k+p}) \not< d_k - 2k^{-1}, \qquad \varrho'(x, q_k) \not< d_{k+p} - 2(k+p)^{-1}.$$

Hence

$$d_k - 2k^{-1} \not> \varrho'(x, q_{k+p}) < d_{k+p} + (k+p)^{-1},$$

$$d_{k+p} - 2(k+p)^{-1} \not> \varrho'(x, q_k) < d_k + k^{-1}.$$

Therefore

$$d_k < d_{k+p} + 3k^{-1}, \qquad d_{k+p} < d_k + 3k^{-1}, \qquad \text{so} \qquad |d_k - d_{k+p}| < 3k^{-1}$$

for every p. This proves the convergence of $\langle d_n\rangle_n$. That $d = \lim_{n\to\infty} d_n = \varrho'(W, x)$ follows immediately.

ϱ' is also a complete metric. For let $\langle y_n\rangle_n$ be a fundamental sequence. Let, for example, $n, n' \geqslant m \to \varrho'(y_n, y_{n'}) < 2^{-1}$. Then $n \geqslant m \to \varrho'(y_n, y_m) < 2^{-1}$. Let $y_m \in X_s$. If $y_n \notin X'_s$, then $\varrho'(y_m, y_n) = 1$, hence for $n \geqslant m$ $\neg\neg y_n \in X'_s$. Since X'_s is located and closed, we have $\neg\neg y_n \in X'_s \to y_n \in X'_s$ (2.9(e)). Therefore $\langle y_{n+m}\rangle_n$ is a fundamental sequence in X'_s, so $\langle y_{n+m}\rangle_n$ converges to a point of X'_s since X'_s is complete.

4.7. Corollary to the proof of theorem 4.6. *Let $\langle V, \varrho\rangle$ be locally compact and let $\langle V_n\rangle_n$, $\langle W_n\rangle_n$ satisfy the conditions in 4.4. Put $X_1 = V_2$, $X_n = W_{n-1} \cap V_{n+1}$ for $n > 1$, $X'_1 = X_1 \cup X_2$, $X'_n = X_{n-1} \cup X_n \cup X_{n+1}$. Then there exists a metric ϱ' such that $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho')$, ϱ' complete, and $x \in X_n \,\&\, y \notin X'_n \to \varrho'(x, y) = 1$, ϱ' bounded by 1.*

4.8. Lemma. *Let $\langle V, \varrho\rangle$ be locally compact, and let $\langle V_n\rangle_n$, $\langle W_n\rangle_n$ satisfy the conditions of 4.4. Then there exists a function φ, continuous on V, such that $0 < \varphi(x) \not> 1$, and $x \in W_n \to \varphi(x) \not> 2^{-n}$ for every n.*

Proof. We construct for every k a function φ_k.

Let $\langle X_n\rangle_n$, $\langle X'_n\rangle_n$, ϱ' be defined as in 4.7. Suppose $x \in X_n$. We put

$$W_i \cap V_{n+2} = \emptyset \to \varphi_i(x) = 1,$$

$$W_i \cap V_{n+2} \neq \emptyset \to \varphi_i(x) = \varrho'(x, W_i \cap V_{n+2}).$$

φ_i is uniquely determined by this definition. In fact, if $x \in X_m \cap X_n$, then $|m-n| \leqslant 1$. Suppose $m = n+1$. If $W_i \cap V_{n+2} \neq \emptyset$, then $W_i \cap V_{n+3} \neq \emptyset$. $y \in V_{n+3} - V_{n+2}$ implies $y \in X_{n+2}$, so $\varrho'(x, y) = 1 \not< \varrho'(x, W_i \cap V_{n+3})$. Moreover,

$$y \in V_{n+2} \cap W_i \vee y \in V_{n+3} - V_{n+2} \to \varrho'(x, W_i \cap V_{n+2}) \not> \varrho'(x, y).$$

We have $y \in V_{n+3} \cap W_i \to \neg\neg(y \in V_{n+2} \cap W_i \vee y \in V_{n+3} - V_{n+2})$, hence

$$\varrho'(x, W_i \cap V_{n+3}) = \varrho'(x, W_i \cap V_{n+2}).$$

If $W_i \cap V_{n+2} = \emptyset$, $W_i \cap V_{n+3} \neq \emptyset$, then $\varrho'(x, W_i \cap V_{n+3}) = 1$. This proves that φ_i is uniquely determined.

Now we define

$$\varphi(x) = \sum_{i=1}^{\infty} 2^{-i-1}\varphi_i(x).$$

For $x \in X_n$, $W_{n+2} \cap V_{n+2} = \emptyset$, hence $\varphi_{n+2}(x) = 1$. So $\varphi(x) > 0$ for every $x \in V$.

Let again $x \in X_n$. Then $\varphi_i(x) = 0$ for $1 \leqslant i < n$. Hence $\varphi(x) \not> 2^{-n}$. $x \in W_n \to x \in X_m$ for some $m \geqslant n$. Hence $x \in W_n \to \varphi(x) \not> 2^{-n}$.

4.9. Lemma. *Let $\Gamma = \langle V, \mathfrak{T}(\varrho')\rangle$ be locally compact and let φ be a continuous function on Γ such that $0 < \varphi(x) \not> 1$ for all x. Suppose $\overline{\langle p_i\rangle_i} = V$. Then*

$$\varrho(x, y) = \sum_{i=1}^{\infty} 2^{-i}|\varphi(x)\varrho'(p_i, x) - \varphi(y)\varrho'(p_i, y)| + |\varphi(x) - \varphi(y)|$$

is a metric for which $\mathfrak{T}(\varrho) = \mathfrak{T}(\varrho')$.

Proof. The convergence of ϱ is immediate. $\varrho(x, y) = \varrho(y, x)$, $\varrho(x, y) \not> \varrho(x, z) + \varrho(z, y)$, $\varrho(x, y) \not< 0$, $\varrho(x, x) = 0$ are trivial. So (3.7) $\forall x\, \forall \varepsilon\, \exists \delta\, \forall y\, \big(\varrho'(x, y) < \delta \to \varrho(x, y) < \varepsilon\big)$ holds. Let $\varepsilon > 0$, $x \in V$. There exists an m such that $\varphi(x) > 2^{-m}$. So if $\varrho(x, y) < 2^{-m-1}$, then $\varphi(y) > 2^{-m-1}$.

Suppose $2^{-k} < \varepsilon$, and find a p_i such that $\varrho'(p_i, x) < 2^{-(k+m+3)}$. Then $\varphi(x)\varrho'(x, p_i) < 2^{-(k+m+3)}$.

Suppose $\varrho(x, y) < \delta = \inf\{2^{-n}, 2^{-(k+m+i+2)}\}$. Then

$$2^{-i}|\varphi(x)\varrho'(p_i, x) - \varphi(y)\varrho'(p_i, y)| < \varrho(x, y) < 2^{-(k+m+i+2)},$$

hence

$$|\varphi(x)\varrho'(p_i, x) - \varphi(y)\varrho'(p_i, y)| < 2^{-(k+m+2)},$$

so

$$\varphi(y)\varrho'(p_i, y) < 2^{-(k+m+2)} + \varphi(x)\varrho'(p_i, x) < 3.2^{-(k+m+3)}.$$

Since $\varphi(y) > 2^{-m-1}$, we obtain $\varrho'(p_i, y) < 3.2^{-(k+2)}$ and

$$\varrho'(x, y) \not> \varrho'(p_i, x) + \varrho'(p_i, y) < 2^{-(k+2)} + 3.2^{-(k+2)} = 2^{-k} < \varepsilon.$$

Therefore

$$\forall x\, \forall \varepsilon\, \exists \delta\, \forall y\, \big(\varrho(x, y) < \delta \to \varrho'(x, y) < \varepsilon\big).$$

4.10. Theorem. *Every locally compact space possesses a one-point-compactification.*

Proof. We consider a locally compact space $\Gamma = \langle V, \mathfrak{T}(\varrho)\rangle$; $\langle V_n\rangle_n$, $\langle W_n\rangle_n$ are constructed according to 4.4, φ is constructed according to 4.8, and ϱ has been constructed from some given metric ϱ' (bounded by 1) for Γ, according to 4.9.

We consider $V \cup \{x_0\}$, and extend the natural apartness relation on V $(x \# y \leftrightarrow \varrho(x, y) > 0$ by putting $x \# x_0$ for every $x \in V$. We extend ϱ to $(V \cup \{x_0\}) \times (V \cup \{x_0\})$ as follows.

At first we extend φ by putting $\varphi(x_0) = 0$. Then we define

$$\varrho(x, x_0) = \varrho(x_0, x) = \sum_{i=1}^{\infty} \varphi(x)\varrho'(p_i, x)2^{-i} + \varphi(x) .$$

We have to verify the properties for a metric after this extension.

$\varrho(x, y) = \varrho(y, x)$, $\varrho(x, y) \nless 0$ are trivial.

For the triangle inequality we must verify the special cases:

(1) $\varrho(x, y) \ngtr \varrho(x, x_0) + \varrho(x_0, y)$, $x, y \in V$;

(2) $\varrho(x_0, x) \ngtr \varrho(x_0, y) + \varrho(y, x)$,

which is routine.

Also, $x \# y \leftrightarrow \varrho(x, y) > 0$ must be valid.

$x_0 \# x \to x \in V$. Then $\varrho(x, x_0) \nless \varphi(x) > 0$.

$\varrho(x_0, x) > 0 \to \varphi(x) > 0$. This implies $x \in V$, so $x \# x_0$.

Finally we consider the metric closure V' of $V \cup \{x_0\}$ with respect to ϱ, which determines a topological space, say Γ'.

We shall prove that Γ' is compact, by proving that for every ε there exists an ε-net (3.11).

If $\langle p_n \rangle_n$ is a basic point species for Γ, then $\langle p_n \rangle_n \cup \{x_0\}$ is a basic point species for Γ'.

Suppose $2^{-m+1} < \varepsilon$, and let $\{q_1, \ldots, q_t\}$ be a 2^{-m}-net for V_{m+3} (3.11(b)). Then $\{q_1, \ldots, q_t, x_0\}$ is an ε-net for V'.

In fact, if $x \in V'$, then there is a $q \in \langle p_n \rangle_n \cup \{x_0\}$ such that $\varrho(x, q) < 2^{-m}$.

If $q = x_0$, then $\varrho(x, x_0) < 2^{-m} < \varepsilon$.

If $q = p_i$ for some i, then $p_i \in V_{m+3} \vee p_i \in W_{m+2}$. In the first case we have $\varrho(q_j, p_i) < 2^{-m}$ for some j, hence $\varrho(x, q_j) < 2^{-m+1} < \varepsilon$; in the second case, $\varphi(p_i) \ngtr 2^{-m-2} < 2^{-m-1}$, hence $\varrho(x_0, p_i) < 2^{-m}$, therefore $\varrho(x_0, x) < 2^{-m+1} < \varepsilon$.

Now we must show that $\varrho(x, x_0) > 0 \leftrightarrow x \in V$.

Let $\varrho(x, x_0) > 2^{-k+1}$. Since

$$\varrho(x, x_0) = \varphi(x)\Big(1 + \sum_{i=1}^{\infty} \varrho'(p_i, x)2^{-i}\Big) \ngtr 2\varphi(x) ,$$

it follows that $\varphi(x) > 2^{-k}$. $\varphi(x) > 2^{-k} \to x \notin W_k$, hence $x \in V_{k+1}$. If therefore $\varrho(x, x_0) > 2^{-k+1}$ and $\langle s_n \rangle_n \subset \langle p_n \rangle_n \cup \{x_0\}$ such that $\forall n \big(\varrho(s_n, x) < 2^{-k-n}\big)$, we see that $\langle s_n \rangle_n \subset \langle p_n \rangle_n$. Since now $\langle s_n \rangle_n \subset V$ and $\forall n \big(\varrho(s_n, x_0) > 2^{-k}\big)$, we have $\langle s_n \rangle_n \subset V_{k+1}$. V_{k+1} is compact, and closed with respect to every adequate metric (3.14), so $x \in V_{k+1} \subset V$. Therefore $\varrho(x, x_0) > 0 \to x \in V$.

The converse is trivial.

$x_0 \in V' - V$. Suppose also $y \in V' - V$. $\varrho(x_0, y) > 0 \to y \in V$, hence $\varrho(x_0, y) = 0$, which implies $y = x_0$. This proves $V' - V = \{x_0\}$.

4.11. Lemma. *Let $\Gamma = \langle V, \varrho \rangle$, $\Gamma' = \langle V', \varrho' \rangle$, and let f be a homeomorphism from Γ onto Γ'. Suppose ϱ, ϱ' to be uniformly equivalent with respect to f, i.e.*

$$\forall \varepsilon \exists \delta \Big(\big(\varrho(x, y) < \delta \to \varrho'(fx, fy) < \varepsilon\big) \;\&\; \big(\varrho'(fx, fy) < \delta \to \varrho(x, y) < \varepsilon\big)\Big) .$$

Then the completions of Γ, Γ' with respect to ϱ, ϱ' respectively are also homeomorphic (by an extension of f).

Proof. Trivial.

4.12. Theorem. *Let Γ be a locally compact space. Every two one-point compactifications of Γ are homeomorphic.*

Proof. Let $\Gamma = \langle V, \mathfrak{T}(\varrho_1) \rangle = \langle V, \mathfrak{T}(\varrho_2) \rangle$, and suppose $\Gamma' = \langle V', \mathfrak{T}(\varrho_1') \rangle$ and $\Gamma'' = \langle V'', \mathfrak{T}(\varrho_2') \rangle$ to be one-point compactifications of Γ, $\varrho_1' | V \times V = \varrho_1$, $\varrho_2' | V \times V = \varrho_2$, $V \subset V'$, $V \subset V''$.

Let $\langle V_n \rangle_n$ be as in 4.4, and suppose $\langle p_n \rangle_n$ to be a basic pointspecies for Γ. ε-neighbourhoods with respect to ϱ_1', ϱ_2' will be denoted by $U^1(\varepsilon, W)$, $U^2(\varepsilon, W)$, respectively.

Let $p_0 \in V' - V$, $p_0' \in V'' - V$; we put $U^1(4^{-1}\varepsilon, p_0) = W$. Then we have $V' = (V' - W) \cup U^1(2^{-1}\varepsilon, p_0)$. Since Γ' is a located compact space, we can find located closed W_1, W_2, (3.16) such that $W_1 \cup W_2 = V'$, $W_1 \subset W^c$, $W_2 \subset U^1(2^{-1}\varepsilon, p_0)$. As a consequence, $W_1 \subset V$, and since W_1 is compact, W_1 is covered by V_m for a certain m. So $V_m \cup U^1(2^{-1}\varepsilon, p_0) = V'$.

In the same manner we obtain for a certain n, $V_n \cup U^2(2^{-1}\varepsilon, p_0') = V''$. Taking $k = m + n$, we have $V_k \cup U^1(2^{-1}\varepsilon, p_0) = V'$, $V_k \cup U^2(2^{-1}\varepsilon, p_0') = V''$, hence $V'' - V_k \subset U^2(2^{-1}\varepsilon, p_0')$.

We introduce a mapping f from $\langle p_n \rangle_n \cup \{p_0\}$ onto $\langle p_n \rangle_n \cup \{p_0'\}$, defining $fp_n = p_n$, $fp_0 = p_0'$. We want to prove that ϱ_1', ϱ_2' are uniformly equivalent with respect to f on $\langle p_n \rangle_n \cup \{p_0\}$, $\langle p_n \rangle_n \cup \{p_0'\}$. We can find and η such that $U^1(\eta, V_k) \subset V_{k+1}$, $U^1(\eta, V_{k+1}) \subset V_{k+2}$.

$\langle V_{k+2}, \varrho_1' | V_{k+2} \times V_{k+2} \rangle$ and $\langle V_{k+2}, \varrho_2' | V_{k+2} \times V_{k+2} \rangle$ are homeomorphic. Therefore we can find an η' such that (3.11(a))

$$\forall x, y \in V_{k+2} \big(\varrho_1'(x, y) < \eta' \to \varrho_2'(x, y) < \varepsilon\big) .$$

Finally we put $\eta'' = \varrho'(p_0, V_k)$ $(\eta'' > 0$ since $\varrho_1'(p_0, x) > 0$ for every $x \in V_k$ (3.12(c))), and $\delta = \inf\{\eta, \eta', \eta''\}$.

For every pair $\{p, q\} \subset \langle p_n \rangle_n \cup \{p_0\}$ we want to show:

$$(*) \qquad \varrho_1'(p, q) < \delta \to \varrho_2'(p, q) < \varepsilon .$$

The proof is given by cases.

(1) $p, q \in \langle p_n \rangle_n$.

(a) $p, q \in V_{k+1}$ & $\varrho_1'(p, q) < \delta \to \varrho_2'(p, q) < \varepsilon$.

(b) $p, q \in V - V_k \to \varrho_2'(p, p_0') < 2^{-1}\varepsilon$ & $\varrho_2'(q, p_0') < 2^{-1}\varepsilon$, hence $\varrho_2'(p, q) < \varepsilon$.

(c) $p \in V_{k+1}$ & $q \in V - V_k$ & $\varrho_1'(p, q) < \delta \to q \in V_{k+2}$. Hence $p, q \in V_{k+2}$ & $\varrho_1'(p, q) < \delta$, so $\varrho_2'(p, q) < \varepsilon$.

(d) $p \in V - V_k$ & $q \in V_{k+1}$ is treated likewise.

(2) $p = p_0$, $q \in \langle p_n \rangle_n$.

$$\varrho_1'(p, q) < \delta \to \varrho_1'(p, q) < \eta'',$$

so $q \in V - V_k$, hence $q \in U^2(2^{-1}\varepsilon, p_0')$. This implies $\varrho_2'(p_0', q) < \varepsilon$.

(3) $p = q = p_0$ is trivial.

This proves (*).

In the same way we prove the existence of a δ' such that for

$$\{p, q\} \subset \langle p_n \rangle_n \cup \{p_0'\}, \qquad \varrho_2'(p, q) < \delta' \to \varrho_1'(p, q) < \varepsilon.$$

Together with Lemma 4.4 this proves our theorem.

4.13. We are not able to prove for every locally compact space that it possesses a minimal compactification, as is illustrated by the following example.

Let $\Pi(x)$ stand for: x is the number of the last decimal of the first sequence of ten consecutive numerals 7 in the decimal representation of π. $(\exists x)\Pi(x) \vee \neg(\exists x)\Pi(x)$ is an unsolved problem. We define a locally compact space $\Gamma = \langle V, \varrho \rangle$ by $V = \{x: x = 1 \vee \Pi(x)\}$, ϱ is the natural metric on N.

Classically, Γ is compact; intuitionistically, we are not able to prove this.

Now suppose $\Gamma' = \langle V', \varrho' \rangle$ to be a minimal compactification of Γ. Let f be the homeomorphism from $\langle V, \varrho \rangle$ onto $\langle fV, \varrho' | fV \rangle$, such that fV is dense in V'.

Since f is a homeomorphism, we have $\exists\delta\, \forall y \big(\varrho'(f1, fy) < \delta \to \varrho(1, y) < 1\big)$. $\varrho(1, y) < 1 \to y = 1$, so it follows that $\varrho'(f1, fy) < \delta \to fy = f1$.

We can determine

$$\operatorname{diam} V' = \sup\{\varrho'(x, y): x \in V' \ \& \ y \in V'\}.$$

$$\operatorname{diam} V' = \operatorname{diam} fV \quad \text{and} \quad \operatorname{diam} V' < \delta \vee \operatorname{diam} V' > 2^{-1}\delta.$$

In the first case $fV = f1$, hence $\neg(\exists x)\Pi(x)$.

In the second case there are $x, y \in V'$ such that $\varrho'(fx, fy) > 2^{-1}\delta$, hence $x \# y$, so $(\exists x)\Pi(x)$.

The assumption that Γ possesses a minimal compactification therefore leads to a proof of $(\exists x)\Pi(x) \vee \neg(\exists x)\Pi(x)$.

This implies that we are not able to prove the existence of a space Γ' as described.

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