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On a class of plane acyclic continua with the fixed point property

by

K. Sieklucki (Warszawa)

1. Introduction. Let X be a space and F a family of mappings of X into itself. We say that X *has the fixed point property* for the class F if for every $f \in F$ there exists an $x \in X$ such that $f(x) = x$. It is unknown whether a plane continuum which does not separate the plane has the fixed point property even for homeomorphisms. G. Choquet proved in [4] that a plane continuum which does not separate the plane has the fixed point property for homeomorphisms which are extensible to periodic homeomorphisms of the plane with period $\neq 2$. M. L. Cartwright and J. E. Littlewood proved in [3] that the same is true for homeomorphisms which are extensible to orientation preserving homeomorphisms of the plane.

Other partial results follow from some theorems of K. Borsuk. Namely, [1] implies that every Peano continuum which lies in the plane and does not separate the plane has the fixed point property for continuous mappings. Moreover, it follows from a theorem of [2] that any plane acyclic continuum which is a boundary set and is arcwise connected has the fixed point property for continuous mappings. Another class of planar acyclic continua with the fixed point property for continuous mappings is the class of snake-like continua [5]. More details concerning the problem can be found in the book of van der Walt [9].

O. H. Hamilton proved in [6] that if D is a bounded simply connected plane domain whose closure does not separate the plane and whose boundary is hereditarily decomposable, then $\bar{D}$ has the fixed point property for homeomorphisms. The purpose of the present paper is to give a generalization of the theorem of Hamilton. Namely, we prove that any plane continuum which does not separate the plane and whose boundary is hereditarily decomposable has the fixed point property for continuous mappings. More precisely, we shall prove the following

1.1. THEOREM. *If X is a plane continuum which does not separate the plane E_2 and $f\colon X \to E_2$ is a continuous mapping such that $f(\mathrm{Fr}\,X) \subset X$*

and $f(x) \neq x$ for every $x \in X$, then there exists an indecomposable continuum $X_0 \subset \operatorname{Fr} X$ such that $f(X_0) = X_0$ [1].

2. General definitions and notations. All sets considered in this paper are subsets of the Euclidean plane E_2. If $A \subset E_2$, then by $\bar{A}$, $\operatorname{Int} A$, and $\operatorname{Fr} A$ we shall understand the closure, the interior and the boundary of A, respectively, with respect to the plane E_2.

A *dendrite* is a 1-dimensional locally connected continuum which does not contain any simple closed curve. A dendrite which is the union of a finite number of simple arcs is said to be *finite*. If D is a dendrite, then by $\operatorname{ram}(d)$ we shall denote the order of ramification of D at a point $d \in D$. We also define $\operatorname{ram} D = \sup_{d \in D} \operatorname{ram}(d)$. By $\dot{D}$ we shall understand the set of those points of D for which $\operatorname{ram}(d) = 1$. Moreover, we write $\underset{\cdot}{D} = D - \dot{D}$.

Let D be a finite plane dendrite and let $d \in D$. A set $L \subset E_2$ is called a *local cut of D at the point d* provided L is the union of $\operatorname{ram}(d)$ simple closed arcs which are mutually disjoint and disjoint with D beyond their common origin d and L is such that for a sufficiently small neighbourhood U of d each component of $U - L$ contains exactly one component of $(U \cap D) - \{d\}$.

If α, β are real numbers (or the symbols $-\infty$, $+\infty$) and $\alpha < \beta$, then we write $[\alpha, \beta] = \{t\colon \alpha \leqslant t \leqslant \beta\}$, $(\alpha, \beta] = \{t\colon \alpha < t \leqslant \beta\}$, $[\alpha, \beta) = \{t\colon \alpha \leqslant t < \beta\}$, $(\alpha, \beta) = \{t\colon \alpha < t < \beta\}$. The use of the same symbol to denote an open interval and a point with two coordinates will not lead to any misunderstanding. If $a, b \in E_2$, then by $\overline{[a, b]}$, $\overline{(a, b]}$, $\overline{[a, b)}$, and $\overline{(a, b)}$ we shall denote the closed, half-closed, and open segment, respectively, with the end-points a, b. Any closed, half-closed, and open simple arc with the end-points a, b (which is obviously not uniquely determined by the ends) will be denoted by $[\widehat{a, b}]$, $(\widehat{a, b}]$, $[\widehat{a, b})$, and $(\widehat{a, b})$, respectively.

The unit interval $[0, 1]$ will be denoted by I. We write I^2 for the Cartesian square $I \times I$. The unit circle will be denoted by S^1. The vector from a point $a \in E_2$ to a point $b \in E_2$ will be denoted by $\overrightarrow{a, b}$. The open ball with radius r about a set A will be denoted by $B(A, r)$; the same symbol will be used when the set A reduces to a point. Any plane set homeomorphic to the closed unit disc is called a *topological disc*. A topological disc is said to be *ε-lanky* if it does not contain any geometric disc of diameter ε. If $A \neq 0 \neq B$, then we write $\operatorname{dist}(A, B) = \inf_{a \in A, b \in B} \varrho(a, b)$.

[1] After giving the paper to the editors I got from Prof. H. Bell a copy of the galley proofs of his paper: *On fixed point properties of plane continua*, to appear in the Transactions of Amer. Math. Soc., in which he obtained an equivalent result.

For any set $X \subset E_2$ let us define $\hat{X}$ to be the union of X and of all bounded components of $E_2 - X$. We shall prove some lemmas concerning this notion.

2.1. Lemma. *If $X \subset E_2$ is a continuum, then $\hat{X}$ is a continuum which does not separate the plane E_2.*

Proof. Each bounded component U of $E_2 - X$ is, in this case, homeomorphic to an open disc and $\operatorname{Fr} U \subset X$; hence $\hat{X}$ is connected. Moreover, it can easily be seen that $\operatorname{Fr}(\hat{X} - X) \subset X$ and $\operatorname{diam} \hat{X} = \operatorname{diam} X$, whence $\hat{X}$ is closed in E_2 and bounded, and thus compact. Finally, if U is a component of $E_2 - \hat{X}$, then $\operatorname{Fr} U \subset X$ and this implies that U must be unbounded. Thus, there exists only one component of $E_2 - \hat{X}$.

2.2. Lemma. *If $X_1 \subset X_2 \subset E_2$, then $\hat{X}_1 \subset \hat{X}_2$.*

Proof. If $p \in \hat{X}_1$, then either $p \in X_1 \subset X_2 \subset \hat{X}_2$, or $p \in U$, where U is a bounded component of $E_2 - X_1$. In the second case, $\operatorname{Fr} U$ separates p from ∞, and since $\operatorname{Fr} U \subset X_1 \subset X_2$, we have $p \in \hat{X}_2$.

2.3. Lemma. *If $X = \hat{X} \subset E_2$ and $C \subset \operatorname{Fr} X$ are continua, then $\operatorname{Fr} \hat{C} = C$.*

Proof. If $p \in \operatorname{Fr} \hat{C}$, then since Lemma 2.1 implies that $\hat{C}$ is compact, we have $p \in \hat{C}$. If $p \in U$, where U is a bounded component of $E_2 - C$, then $p \in \operatorname{Int} \hat{C}$, which is impossible. Thus $p \in C$.

Conversely, let us now suppose that $p \in C \subset \operatorname{Fr} X$. Then any ball about p meets both C and $E_2 - X$; hence it meets both $\hat{C}$ and $E_2 - X = E_2 - \hat{X} \subset E_2 - C$. This implies that $p \in \operatorname{Fr} \hat{C}$.

3. Trees. Let C_0 be the standart Cantor set in the interval $[0, 1]$ and let e_n^i $(i = 1, 2, \ldots, 2^{n-1})$ denote the centres of the open intervals removed in the nth step of the construction of C_0 $(n = 1, 2, \ldots)$. Let $c_0 = (1/2, 1) \in E_2$ and $c_n^i = (e_n^i, 2^{-n}) \in E_2$, where $i = 1, 2, \ldots, 2^{n-1}$; $n = 1, 2, \ldots$ Let C be the closure of the union

$$\overline{[c_0, c_1^1]} \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} \overline{[c_n^i, c_{n+1}^{2i-1}]} \cup \overline{[c_n^i, c_{n+1}^{2i}]}$$

(Fig. 1). It is easy to see that C is a dendrite, $\operatorname{ram} C = 3$, and $\dot{C} = C_0 \cup \{c_0\}$.

A bounded set $D \subset E_2$ is called a *tree with origin d_0* if there exist a dendrite $C' \subset C$ such that $c_0 \in \dot{C}' \subset C_0 \cup \{c_0\}$ and a homeomorphism $h\colon (C' - C_0) \xrightarrow{\text{onto}} D$ such that $h(c_0) = d_0$. If C' is a simple arc, then the tree D is said to be *simple*. If $d_n \in D$ for $n = 1, 2, \ldots$ and $\operatorname{dist}(h^{-1}(d_n), C_0) \to 0$, then we write conventionally $d_n \to \infty$. Let us also adopt the notation $\operatorname{Lim} D = \bar{D} - D$.

3.1. Lemma. *If D is a tree and a connected set $H \subset D$ satisfies the condition $\bar{H} \cap \operatorname{Lim} D \neq 0$, then there exists a simple tree $\tilde{D} \subset H$ such that $\operatorname{Lim} \tilde{D} \subset \operatorname{Lim} D$.*

Proof. Suppose that $h\colon (C'-C_0) \xrightarrow{\text{onto}} D$, where C' is a subdendrite of C and $c_0 \in C' \subset C_0 \cup \{c_0\}$, is the homeomorphism existing by the definition of the tree D. The set $F = h^{-1}(H)$ is a connected subset of $C'-C_0$. Since $\overline{H} \cap \operatorname{Lim} D \neq 0$, there exists a sequence of points $d_n \in H$ with $d_n \to \infty$, Hence, there exists a sequence of points $c_n \in F$ with $\operatorname{dist}(c_n, C_0) \to 0$, and this yields $\overline{F} \cap C_0 \neq 0$. There exists, therefore, a simple closed arc $[\widehat{a,b})$ $\subset \overline{F}$, where $a \in C'-C_0$, $b \in C_0$. Let us write $\widetilde{D} = h([\widehat{a,b}))$. Then evidently $\widetilde{D}$ is a simple tree and $\widetilde{D} \subset H$.

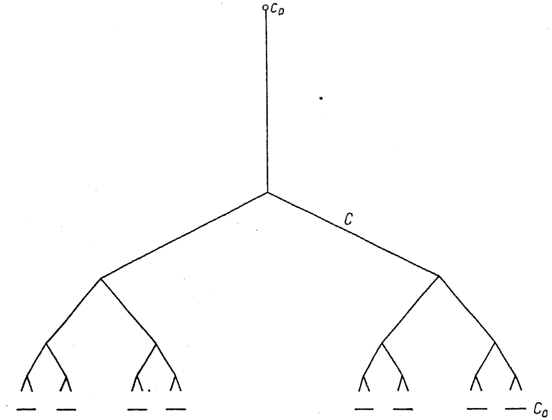

Fig. 1

Moreover, since $\widetilde{D} \subset H \subset D$, we have $(\overline{\widetilde{D}}) \subset \overline{D}$, and since $[\widehat{a,b})$ is closed in $C'-C_0$, $\widetilde{D}$ is closed in D. Thus if $p \in \operatorname{Lim}\widetilde{D} = (\overline{\widetilde{D}}) - \widetilde{D}$, then $p \in \overline{D}$; furthermore, $p \notin D$, for otherwise $\widetilde{D}$ would not be closed in D. Hence $p \in \operatorname{Lim} D$ and the proof has been concluded.

3.2. Lemma. *If D is a simple tree, then* $\operatorname{Lim} D$ *is a non-empty continuum.*

Proof. By the definition of a simple tree, we can assume that there exists a homeomorphism $h\colon [0,1) \xrightarrow{\text{onto}} D$. Let $D_n = h\left(\left[1-\frac{1}{n}, 1\right)\right)$ for $n = 1, 2, \ldots$ For each n the set D_n is non-empty and connected. Moreover, $D_{n+1} \subset D_n$ for $n = 1, 2, \ldots$ Hence $\{\overline{D}_n\}$ is a decreasing sequence of non-empty continua and consequently $\bigcap_{n=1}^{\infty} \overline{D}_n$ is a non-empty continuum. We shall prove that $\operatorname{Lim} D = \bigcap_{n=1}^{\infty} \overline{D}_n$.

Let us note that $D \cap \overline{D}_n = D_n$ for $n = 1, 2, \ldots$ Indeed, evidently $D_n \subset D \cap \overline{D}_n$. On the other hand, if $d = h(t) \in D \cap \overline{D}_n$, then there exists a sequence $d_i = h(t_i) \in D_n$ $(i = 1, 2, \ldots)$ such that $d_i \to d$, whence $t_i \to t$. Since $1-\frac{1}{n+1} \leqslant t_i < 1$, we infer that $1-\frac{1}{n+1} \leqslant t < 1$, thus $d \in D_n$. This proves the inclusion $D \cap \overline{D}_n \subset D_n$.

We now have

$$0 = \bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} (D \cap \overline{D}_n) = D \cap \bigcap_{n=1}^{\infty} \overline{D}_n .$$

On the other hand, the well-known formula

$$\overline{\bigcup_{n=1}^{\infty} F_n} = \bigcup_{n=1}^{\infty} \overline{F}_n \cup \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} F_m}$$

yields for $F_n = h\left(\left[1-\frac{1}{n}, 1-\frac{1}{n+1}\right)\right)$ the following equality

$$\overline{D} = D \cup \bigcap_{n=1}^{\infty} \overline{D}_n .$$

From what we have proved it follows that $\operatorname{Lim} D = \overline{D} - D = \bigcap_{n=1}^{\infty} \overline{D}_n$ and the proof of the lemma has been concluded.

4. Canals. Let $X \subset E_2$ be a continuum and let $D \subset E_2 - \hat{X}$ be a tree. A set L is said to be a *bridge from a point $d \in D$ to X* if L is a local cut of D at the point d and $\dot{L} \subset X$.

A tree $D \subset E_2 - \hat{X}$ is called a *canal in X* if for every $d \in D$ there exists a bridge $L(d)$ from d to X such that $\operatorname{diam} L(d) \to 0$ if $d \to \infty$. The canal D is said to be *simple* if the tree D is simple.

4.1. Lemma. *If D is a canal in X, then D is a canal in* $\operatorname{Fr} X$.

Proof. The lemma follows from the obvious remark that in the definition of a canal the ends of a bridge from $d \in D$ to X can be chosen in $\operatorname{Fr} X$.

4.2. Lemma. *If D is a canal in X, then* $\operatorname{Lim} D \subset \operatorname{Fr} X$.

Proof. If $p \in \operatorname{Lim} D$, then there exists a sequence $d_n \in D$ such that $d_n \to p$ and $d_n \to \infty$. Hence $\operatorname{diam} L(d_n) \to 0$, where $L(d)$ is a bridge from $d \in D$ to X. It follows that every ball with centre p meets $\operatorname{Fr} X$. Thus $p \in \operatorname{Fr} X$.

4.3. Lemma. *If D is a canal in X and $\widetilde{D} \subset D$ is a tree such that* $\operatorname{Lim}\widetilde{D} \subset \operatorname{Lim} D$, *then $\widetilde{D}$ is also a canal in X.*

Proof. It is easy to see that if $d \in \widetilde{D} \subset D$ and $L(d)$ is a bridge from d to X with respect to the tree D, then we can choose a bridge $\widetilde{L}(d) \subset L(d)$ from d to X with respect to the tree $\widetilde{D}$. Then evidently $\operatorname{diam}\widetilde{L}(d) \leqslant \operatorname{diam} L(d)$ and it remains to prove that if $d_n \to \infty$ in $\widetilde{D}$, then $d_n \to \infty$

in D. But if it were false, then there would exist a subsequence $d_{k_n} \to d \in D$, whence $d \in \operatorname{Lim}\widetilde{D} - \operatorname{Lim} D$, contrary to the assumption that $\operatorname{Lim}\widetilde{D} \subset \operatorname{Lim} D$.

4.4. Lemma. *If D is a simple canal in X and $\operatorname{Lim} D = X$, then X is indecomposable.*

Proof. By Lemma 3.2, X is a continuum. Let us suppose that it is decomposable and $X = A \cup B$, where A, B are continua different from X. The sets $A - B$ and $B - A$ are, therefore, non-empty and open in X. Let $a \in A - B$ and $b \in B - A$ be interior points with respect to X, and let us suppose that $B(a, r) \cap X \subset A - B$, $B(b, r) \cap X \subset B - A$, $B(a, r) \cap B(b, r) = 0$. By the definition of a simple tree, there exists a homeomorphism $h\colon [0, 1) \xrightarrow{\text{onto}} D$. By the definition of a canal, for every $d \in D$ there exists a bridge $L(d)$ from d to X such that $\operatorname{diam} L(d) \to 0$ if $h^{-1}(d) \to 1$.

There exists, therefore, a number $0 \leqslant t_0 < 1$ such that $\operatorname{diam} L(h(t)) < r/2$ for $t > t_0$. Since $\operatorname{Lim} D = X$, there exist numbers $t_0 < t'_a < t'_b < t''_a$ such that $d'_a = h(t'_a) \in B(a, r/2)$, $d''_a = h(t''_a) \in B(a, r/2)$, and $d'_b = h(t'_b) \in B(b, r/2)$. Let M be the union of the arc $[\widehat{d'_a, d''_a}] \subset D$ and the segment $[\overline{d''_a, d'_a}]$. Let N be a simple closed curve contained in M and containing d_b. Since $L(d_b)$ cuts D locally at the point d_b and since $\operatorname{diam} L(d_b) < r/2$, the two ends of $L(d_b)$ are contained in $B(b, r) \cap X \subset B - A$ and separated by N. This implies that N separates B, which in view of the evident equality $N \cap B = 0$ contradicts the connectedness of B.

5. Auxiliary constructions

5.1. Lemma. *If $X \subset E_2$ is a continuum which does not separate the plane E_2, then there exists a sequence of topological discs Q_n with polygonal boundaries B_n $(n = 0, 1, \ldots)$ such that*

(i) $X = \bigcap_{n=0}^{\infty} Q_n$;

(ii) $Q_{n+1} \subset \operatorname{Int} Q_n$ $(n = 0, 1, \ldots)$;

(iii) *For every $b \in B_n$ there exists an $x(b) \in \operatorname{Fr} X$ such that writing $I(b) = [\overline{b, x(b)}]$ we have $I(b) \cap B_n = \{b\}$, $\operatorname{diam} I(b) < 2^{-n}$, and $\dot{I}(b') \cap \dot{I}(b'') = 0$ for $b' \neq b''$ $(n = 0, 1, \ldots)$.*

Proof. Let $\mathfrak{T}_n$ $(n = 0, 1, \ldots)$ be the covering of the plane E_2 by the closed squares

$$|x_1 - k \cdot 2^{-n-1}| \leqslant 2^{-n-2}, \quad |x_2 - l \cdot 2^{-n-1}| \leqslant 2^{-n-2} \quad (k, l = 0, \pm 1, \ldots).$$

Let us suppose that $K_1, K_2, \ldots, K_m$ are those squares of $\mathfrak{T}_n$ which meet X. Let us define $Q_n = \widehat{\bigcup_{i=1}^{\infty} K_i}$. From the construction it follows that Q_n is a topological disc whose boundary B_n is a polygonal line.

To prove condition (i) it is sufficient to show that $\bigcap_{n=0}^{\infty} Q_n \subset X$, for the inverse inclusion is trivial. Let $a \in E_2 - X$ and let us consider an arc $L = [\widehat{a, c}]$ such that $c \in E_2 - Q_0$, $L \cap X = 0$. Let $\varepsilon = \operatorname{dist}(L, X)$ and let n be an integer number such that $2^{-n} < \varepsilon$. Then $a \in E_2 - Q_n$. Indeed, otherwise $L \cap B_n \neq 0$ and for a point $b \in L \cap B_n$ we would have $\operatorname{dist}(b, X) < 2^{-n} < \varepsilon$, contrary to the definition of ε. Thus $E_2 - X \subset \bigcup_{n=0}^{\infty} (E_2 - Q_n)$, whence $\bigcup_{n=0}^{\infty} Q_n \subset X$.

By virtue of Lemma 2.2, we have $Q_{n+1} \subset Q_n$ for $n = 0, 1, \ldots$ Since evidently $X \subset Q_n$ for $n = 0, 1, \ldots$, we can assume, choosing a suitable subsequence if necessary, that condition (ii) is satisfied.

It remains to verify (iii). Let $b \in B_n$ and let us suppose that $b \in A$, where A is a side of a square $K \in \mathfrak{T}_n$ such that $K \subset Q_n$. Then $K \cap X \neq 0$ and let $x(b) \in K \cap X$ be a point nearest to b. From this definition we infer that $I(b) = [\overline{b, x(b)}]$ meets B_n only at the point b and $\operatorname{diam} I(b) < 2^{-n}$. Let $b', b'' \in B_n$ and $b' \neq b''$. We are going to prove that $\dot{I}(b') \cap \dot{I}(b'') = 0$. This is evidently true if b' and b'' belong to sides of different squares, for each square is convex and contains the segment $I(b)$. Let us suppose that b', b'' belong to the boundary of the same square $K \in \mathfrak{T}_n$ such that $K \subset Q_n$. Then $x(b'), x(b'')$ belong to $K \cap X$. Let us assume that $c \in I(b') \cap I(b'')$. Then

$$\varrho(b', c) + \varrho(c, x(b')) = \varrho(b', x(b')) \leqslant \varrho(b', x(b'')) \leqslant \varrho(b', c) + \varrho(c, x(b'')),$$

whence $\varrho(c, x(b')) \leqslant \varrho(c, x(b''))$. By symmetry we get $\varrho(c, x(b')) = \varrho(c, x(b''))$. Furthermore, we infer that

$$\varrho(b', x(b'')) = \varrho(b', c) + \varrho(c, x(b''))$$

and

$$\varrho(b'', x(b')) = \varrho(b'', c) + \varrho(c, x(b')),$$

for otherwise, by the same reasoning as above, we would get either $\varrho(c, x(b')) < \varrho(c, x(b''))$ or $\varrho(c, x(b'')) < \varrho(c, x(b'))$, contrary to what we have proved. Thus $c \in [\overline{b', x(b'')}] \cap [\overline{b'', x(b')}]$. Since $b' \neq b''$, we infer that $c = x(b') = x(b'')$, but this implies $\dot{I}(b') \cap \dot{I}(b'') = 0$.

5.2. Lemma. *Let us suppose that $Q \subset E_2$ is a topological disc with the boundary M and that $\varepsilon > 0$ is a number such that Q is $\varepsilon/12$-lanky. Let $d^j \in M$ for $j = 0, 1, \ldots, k$ be points such that $\operatorname{diam}[\widehat{d^j, d^{j+1}}] < \varepsilon$ for $j = 0, 1, \ldots, k$; mod k. Then there exists a finite dendrite $D \subset Q$ such that $\operatorname{ram} D \leqslant 3$, $\dot{D} = D \cap M = \bigcup_{j=0}^{k} \{d^j\}$, and each component U of $Q - (D \cup M)$ satisfies $\operatorname{diam} U < 2\varepsilon$.*

Proof. Let H be the union of the straight lines $x_1 = k\cdot\varepsilon/12$, $x_2 = l\cdot\varepsilon/12$, where $k, l = 0, \pm 1, \ldots$ Since Q does not contain any square with a side of diameter $\varepsilon/12$, there exists a homeomorphisms $h\colon E_2 \xrightarrow{\text{onto}} E_2$ such that $\varrho\big(p, h(p)\big) < \varepsilon/6$, $h(M)$ is a polygonal simple closed curve, and $h(Q)$ does not contain any point of the form $(k\cdot\varepsilon/12, l\cdot\varepsilon/12)$. Thus we can assume that M and Q themselves possess those properties, and prove the lemma under the assumption $\operatorname{diam}[\widehat{d^j, d^{j+1}}] < \varepsilon + \frac{\varepsilon}{3} = \frac{4}{3}\varepsilon$ for the conclusion $\operatorname{diam} U^j < 2\varepsilon - \frac{\varepsilon}{3} = \frac{5}{3}\varepsilon$.

Now, $H \cap Q$ is the union of mutually disjoint closed segments H_α with the end-points in M. We can evidently assume that d^j is neither a vertex of M nor a vertex of any H_α for $j = 0, 1, \ldots, k$. Let p_α be the centre of H_α and let $H_0 = \bigcup\limits_\alpha H_\alpha - \{p_\alpha\}$. Let V_β be open polyhedral neighbourhoods in Q of vertices in M such that for $V = \bigcup\limits_\beta V_\beta$ we have $d^j \notin V$ for $j = 0, 1, \ldots, k$, $p_\alpha \notin V$ for all α, and $Q - V$ is still a polyhedral topological disc.

The set $Q_0 = Q - (V \cup H_0)$ is simply connected. Let $q_0 \in \operatorname{Int} Q_0$ and let D^j be the shortest polygonal arc joining q_0 with d^j in Q for $j = 0, 1, \ldots, k$. The set $D = \bigcup\limits_{j=0}^{k} D^j$ is obviously a finite dendrite satisfying $D \subset Q$, $\dot{D} = D \cap M = \bigcup\limits_{j=0}^{k} \{d^j\}$. Let us assume that $D^j \cap D^{j+1} = [\widehat{q_0, p_j}]$ and let us write

$$L_j = [\widehat{d^j, p_j}] \cup [\widehat{p_j, d^{j+1}}] \subset D, \qquad M_j = [\widehat{d^j, d^{j+1}}] \subset M$$
$$(j = 0, 1, \ldots, k;\ \operatorname{mod} k).$$

It is easy to see that the arc L_j passes only through those p_α (except p_j) for which H_α meets M_j. It follows that if U^j denotes the component of $Q - (D \cup M)$ bounded by $L_j \cup M_j$, then we have

$$\operatorname{diam} U^j < \operatorname{diam} M_j + 4 \max_\alpha \operatorname{diam} H_\alpha < \frac{4}{3}\varepsilon + 4\cdot\frac{1}{12}\varepsilon = \frac{5}{3}\varepsilon.$$

Making a sufficiently small modification of D, if necessary, we can assume that $\operatorname{ram} D \leqslant 3$.

5.3. Lemma. *Let h_0 be a homeomorphism of $\operatorname{Fr} I^2$ onto the boundary M of a topological disc Q, let $h_0\big(I\times(0)\big) = B^0$, $h_0\big(I\times(1)\big) = \bigcup\limits_{j=1}^{k} B^j$, where $B^j = [\widehat{b^{j-1}, b^j}]$ for $j = 1, 2, \ldots, k$, and let $D \subset Q$ be a finite dendrite such that $\operatorname{ram} D \leqslant 3$, $\dot{D} = D \cap M = \bigcup\limits_{j=0}^{k} \{d^j\}$, where $d^j \in \dot{B}^j$ for $j = 0, 1, \ldots, k$. Then there exist:*

(i) *a retraction $r\colon P = \operatorname{Int} Q \cup \bigcup\limits_{j=0}^{k} \dot{B}^j \to D$ such that $r(\dot{B}^j) = d^j$ for $j = 0, 1, \ldots, k$ and $r(U) \subset \operatorname{Fr} U$ for each component U of $Q - (D \cup M)$;*

(ii) *a homeomorphic extension $h\colon I^2 \xrightarrow{\text{onto}} Q$ of h_0 such that if $h(s, t) \in P$, then*

$$rh\big((s)\times[0, t]\big) = [\widehat{rh(s, 0), rh(s, t)}] \subset D.$$

Proof. To begin with we shall construct a retraction $q\colon P \to D \cup \bigcup\limits_{j=0}^{k} \dot{B}^j$ such that $q(U) \subset \operatorname{Fr} U$ for each component U of $Q - (D \cup M)$ and a homeomorphic extension $h\colon I^2 \xrightarrow{\text{onto}} Q$ of h_0 such that if $h(s, t) \in P$, then

$$qh\big((s)\times[0, t]\big) = [\widehat{qh(s, 0), qh(s, t)}] \subset D \cup \bigcup_{j=0}^{k} \dot{B}^j.$$

We proceed by induction. Let us suppose that $k = 1$ and let $\delta^j = h_0^{-1}(d^j)$ for $j = 0, 1$; $\Delta = [\widehat{\delta^0, \delta^1}]$. Let Γ^j be the component of $I^2 - (\Delta \cup \operatorname{Fr} I^2)$ which contains $(j)\times I$ on its boundary for $j = 0, 1$. Let

$$\pi\colon \Pi = \operatorname{Int} I^2 \cup \bigcup_{j=0}^{1} \big(\dot{I}\times(j)\big) \to \Delta \cup \bigcup_{j=0}^{1} \big(\dot{I}\times(j)\big)$$

be the retraction which projects Γ^j from the point $(j, \frac{1}{2})$ onto $\Delta \cup [\widehat{\delta^0, (j, 0)}] \cup [\widehat{\delta^1, (j, 1)}]$ for $j = 0, 1$. It is clear that if $(s, t) \in \Pi$, then

$$\pi\big((s)\times[0, t]\big) = [\widehat{\pi(s, 0), \pi(s, t)}] \subset \Delta \cup \big(\dot{I}\times(0)\big) \cup \big(\dot{I}\times(1)\big).$$

Let h be a homeomorphic extension of h_0 such that $h(I^2) = Q$ and $h(\Delta) = D$. Then evidently h and $q = h\pi h^{-1}$ have the required properties.

Let us now suppose that the assertion is true for $k-1 \geqslant 1$; we shall prove it for k. Let $d \in D$ be a point of ramification of D which does not lie between d^0 and any other point of ramification. We can assume without real loss of generality that $d \in L = [\widehat{d^{k-1}, d^k}] \subset D$ (see Fig. 2 for the case $k = 3$). Evidently $b^{k-1} \in S = [\widehat{d^{k-1}, d^k}] \subset M$.

Let $M' = (M \cup L) - \dot{S}$ and let Q' be the topological disc bounded by M'. Let $\Sigma = h_0^{-1}(S)$ and let $h_0'\colon \operatorname{Fr} I^2 \to M'$ be a homeomorphism such that $h_0'|\operatorname{Fr} I^2 - \Sigma = h_0|\operatorname{Fr} I^2 - \Sigma$, $h_0'(\Sigma) = L$, and $h_0'^{-1}(d) = h_0^{-1}(b^{k-1}) = \delta \in \Sigma$. Let $D' = (D - L) \cup \{d\}$, $B'^j = B^j$ and $d'^j = d^j$ for $j = 0, 1, \ldots, k-2$, $B'^{k-1} = [\widehat{b^{k-2}, d^{k-1}}] \cup L \cup [\widehat{d^k, b^k}] \subset M'$, $d'^{k-1} = d$. By the inductive assumption, there exist: a retraction $q'\colon P' = \operatorname{Int} Q \cup \bigcup\limits_{j=0}^{k-1} \dot{B}'^j \to D \cup \bigcup\limits_{j=0}^{k-1} \dot{B}'^j$ such that $q(U') \subset \operatorname{Fr} U'$ for each component U' of $Q' - (D' \cup M')$, and a homeomorphic extension $h'\colon I^2 \xrightarrow{\text{onto}} Q'$ of h_0' such that if $h'(s,t) \in P'$, then

$$q'h'\big((s)\times[0,t]\big) = [\widehat{q'h'(s,0),\, q'h'(s,t)}] \subset D' \cup \bigcup_{j=0}^{k-1} B'^j .$$

Let Λ be the half-circle in I^2 with the diameter Σ. Let Γ be the topological disc bounded by $\Lambda \cup \Sigma$. Let $\vartheta\colon I^2 \xrightarrow{\text{onto}} (I^2-\Gamma)\cup\Lambda$ be a homeomorphism such that for each $p \in I^2$ the points $p, \vartheta(p)$ have the same first coordinate. Let us define the homeomorphism $g = h'\vartheta^{-1}\colon (I^2-\Gamma)\cup \Lambda \xrightarrow{\text{onto}} Q'$.

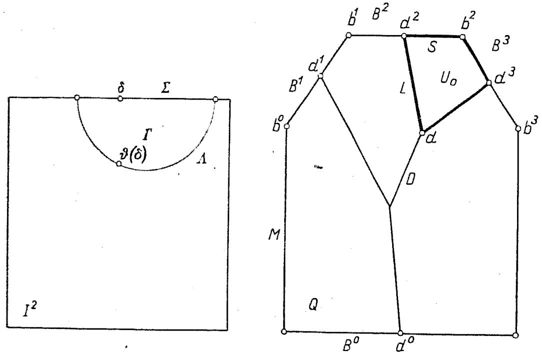

Fig. 2

Let U_0 be the component of $Q-(D\cup M)$ bounded by $L\cup S$, and let $g_0\colon \Gamma \xrightarrow{\text{onto}} U_0\cup L\cup S$ be a homeomorphism such that $g_0|\Lambda = g|\Lambda$, $g_0|\Sigma = h_0|\Sigma$. Then

$$h = \begin{cases} g & \text{on} \quad (I^2-\Gamma)\cup\Lambda ,\\ g_0 & \text{on} \quad \Gamma , \end{cases}$$

is evidently a homeomorphism of I^2 onto Q.

Let $\varrho\colon \Gamma-\{\delta\} \to (\Lambda\cup\Sigma)-\{\delta\}$ be a retraction such that

$$\varrho\big(\overline{[\sigma,\vartheta(\sigma)]}\big) = \begin{cases} \vartheta(\sigma) , & \text{if} \quad \sigma = \delta \\ [\widehat{\sigma,\vartheta(\sigma)}] \subset \Lambda\cup\Sigma-\{\delta\} , & \text{if} \quad \sigma\in\Sigma-\{\delta\} . \end{cases}$$

Let us define a mapping q by the formula

$$q = \begin{cases} q' & \text{on} \quad P' ,\\ g_0\varrho g_0^{-1} & \text{on} \quad \overline{U}_0-\{b^{k-1}\} . \end{cases}$$

Then evidently q is a continuous retraction of P to $D\cup\bigcup_{j=0}^{k} B^j$ and $q(U) \subset \operatorname{Fr} U$ for each component U of $Q-(D\cup M)$.

It remains to prove that if $h(s,t)\in P$, then

$$qh\big((s)\times[0,t]\big) = [\widehat{qh(s,0),\, qh(s,t)}] \subset D\cup\bigcup_{j=0}^{k} B^j .$$

By the inductive assumption it is obvious if $h(s,t)\in P'$. Thus, let us assume that $h(s,t)\in \overline{U}_0-\{b^{k-1}\}$, i.e. that $(s,t)\in\Gamma-\{\delta\}$. Let $\big((s)\times[0,t]\big)\cap\Lambda = (s,t_0)$. Then, by the definition of the retraction ϱ, the arcs $qh\big((s)\times[0,t_0]\big)$ and $qh\big((s)\times[t_0,t]\big)$ have only the point $qh(s,t_0)$ in common, which completes the inductive part of the proof.

To conclude the proof of the lemma, let us consider the retraction $r'\colon D\cup\bigcup_{j=0}^{k} B^j \to D$ such that $r'(B^j) = d^j$ for $j = 0,1,\ldots,k$. Then the retraction $r = r'q$ and the homeomorphism h satisfy conditions (i) and (ii).

5.4. Lemma. *Let us suppose that $Q\subset E_2$ is a topological disc with the polygonal boundary M and that $\varepsilon>0$ is a number such that Q is $\varepsilon/12$-lanky and no side of M is longer than ε. Let $h_0\colon \operatorname{Fr} I^2 \xrightarrow{\text{onto}} M$ be a homeomorphism such that $h_0\big((0)\times I\big)$, $h_0\big(I\times(0)\big) = B^0$, $h_0\big((1)\times I\big)$ are consecutive sides of M and $h_0\big(I\times(1)\big)$ is a broken line with sides B^j $(j = 1,2,\ldots,k)$. Let d^j be the centre of B^j for $j = 0,1,\ldots,k$. Then there exist:*

(i) *a finite dendrite $D\subset Q$ such that*

$$\operatorname{ram} D \leqslant 3 \quad \textit{and} \quad \dot{D} = D\cap M = \bigcup_{j=0}^{k}\{d^j\};$$

(ii) *a local cut $J(d)\subset Q$ of D (at each point $d\in D$) such that $\dot{J}(d)$ is contained in the set of vertices of M and* $\operatorname{diam} J(d) < 4\varepsilon$;

(iii) *a retraction $r\colon P = \operatorname{Int} Q\cup\bigcup_{j=0}^{k} B^j \to D$ such that $r(B^j) = d^j$ for $j = 0,1,\ldots,k$ and $\varrho\big(p, r(p)\big) < 2\varepsilon$ for $p\in P$;*

(iv) *a homeomorphic extension $h\colon I^2 \xrightarrow{\text{onto}} Q$ of h_0 such that if $h(s,t)\in P$, then*

$$rh\big((s)\times[0,t]\big) = [\widehat{rh(s,0),\, rh(s,t)}] \subset D .$$

Proof. By virtue of Lemma 5.2, there exists a finite dendrite D satisfying (i) and such that each component U of $Q-(D\cup M)$ satisfies $\operatorname{diam} U < 2\varepsilon$. Hence we can construct a local cut $J(d)$ satisfying (ii) for each $d\in D$. Now, we can apply Lemma 5.3 and note that 5.3 (i) implies (iii), while 5.3 (ii) is exactly (iv).

5.5. Lemma. *Let $X\subset E_2$ be a continuum which does not separate the plane and let $\delta(\varepsilon)$ be a real function defined for $\varepsilon>0$. Then* (see Fig. 3):

(i) *There exists a sequence of topological discs Q_n with polygonal boundaries B_n, where $n = 0,1,\ldots,$ such that*

(a) $X = \bigcap_{n=0}^{\infty} Q_n$, $Q_{n+1} \subset \operatorname{Int} Q_n$, $Q_n \subset B(X, \delta(\varepsilon_n))$ *for* $n = 0, 1, \ldots$, *where* $\varepsilon_0 = 1$ *and* $\varepsilon_n = \operatorname{dist}(X, B_{n-1})$ *for* $n \geqslant 1$,

(b) *if* $b_n^1, b_n^2, \ldots, b_n^{m_n}, b_n^{m_n+1} = b_n^1$ *are the consecutive vertices of* B_n *and* $B_n^i = [\overline{b_n^i, b_n^{i+1}}]$, *then* $\operatorname{diam} B_n^i < 2^{-n}$ *for* $i = 1, 2, \ldots, m_n$; $n = 0, 1, \ldots$;

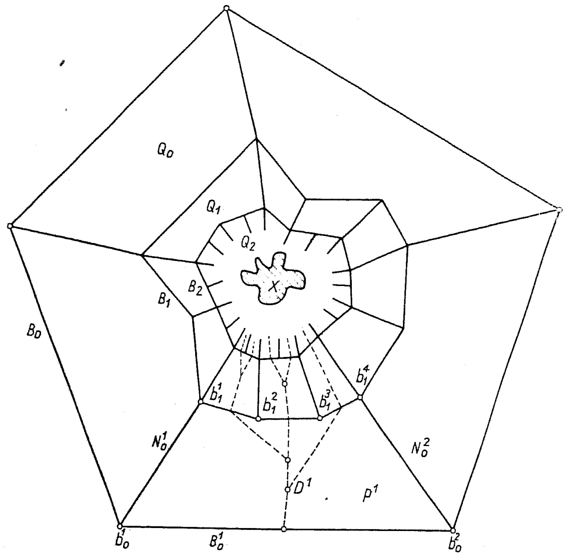

Fig. 3

(ii) *For every* b_n^i *there exists a* $b_{n+1}^{v(i)}$ *such that if* $N_n^i = [\overline{b_n^i, b_n^{v(i)}}]$, *then* $N_n^i \subset \operatorname{Int} Q_n - Q_{n+1}$, $N_n^{i'} \cap N_n^{i''} = 0$ *for* $i' \neq i''$, *and* $\operatorname{diam} N_n^i < 2^{-n}$ *for* $i = 1, 2, \ldots, m_n$; $n = 0, 1, \ldots$;

(iii) *If* $N = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{m_n} N_n^i$ *and* P^ν $(\nu = 1, 2, \ldots, m_0)$ *are components of* $Q_0 - (N \cup X)$, *then for every* $\nu = 1, 2, \ldots, m_0$ *there exists a canal* $D^\nu \subset P^\nu$ *in* X;

(iv) *For every* $\nu = 1, 2, \ldots, m_0$ *there exists a retraction* r^ν: $P^\nu \to D^\nu$ *such that* $\varrho(p, r^\nu(p)) < 2^{-n+2}$ *for* $p \in P^\nu \cap Q_n$ $(n = 0, 1, \ldots)$;

(v) *There exists a homeomorphism* h: $S^1 \times [0, \infty) \xrightarrow{\text{onto}} Q_0 - X$ *such that*

(a) $h^{-1}(B_n) = S^1 \times (n)$, $h^{-1}(N_n^i) = h^{-1}(b_n^i) \times [n, n+1]$ *for* $i = 1, 2, \ldots$ $\ldots, m_n$; $n = 0, 1, \ldots$,

(b) *if* $p = h(s, t) \in P^\nu$, *then* $r^\nu h((s) \times [0, t]) = [\widehat{r^\nu h(s, 0), r^\nu h(s, t)}] \subset D^\nu$ $(\nu = 1, 2, \ldots, m_0)$.

Proof. Let us consider the sequence Q_n satisfying conditions (i)-(iii) of Lemma 5.1. Choosing a suitable subsequence, if necessary, we can assume that $Q_n \subset B(X, \delta(\varepsilon_n))$ for $n = 0, 1, \ldots$, where $\varepsilon_0 = 1$ and $\varepsilon_n = \operatorname{dist}(X, B_{n-1})$ for $n \geqslant 1$. Thus condition (i)(a) of our lemma is satisfied. Moreover, by the same argument, we can assume that $Q_n - X$ is $12 \cdot 2^{-n}$-lanky for $n = 0, 1, \ldots$

Subdividing B_n, if necessary, we can assume that condition (i)(b) is also satisfied.

For every vertex $b_n^i \in B_n$ let $I(b_n^i)$ be the segment described in 5.1 (iii). It evidently meets B_{n+1}; let $c_{n+1}^i \in I(b_n^i) \cap B_{n+1}$ be that point of the intersection which is nearest to b_n^i. We can evidently annex the points c_{n+1}^i to the set of vertices of B_{n+1}. Thus starting from $n = 0$ we proceed by induction and define $b_{n+1}^{v(i)} = c_{n+1}^i$. Condition (ii) of our lemma now follows from 5.1 (iii).

Conditions (iii)-(v) will be verified simultaneously. It is clear that there exists a homeomorphism g: $\bigcup_{n=0}^{\infty} B_n \cup N \xrightarrow{\text{into}} S^1 \times [0, \infty)$ such that $g(B_n) = S^1 \times (n)$ and $g(N_n^i) = g(b_n^i) \times [n, n+1]$ for $i = 1, 2, \ldots, m_n$; $n = 0, 1, \ldots$

Let Q_n^i be the topological disc bounded by the polygonal simple closed curve

$$M_n^i = N_n^i \cup B_n^i \cup N_n^{i+1} \cup \bigcup_{j=1}^{v(i+1)-v(i)} B_{n+1}^{v(i)+j-1} \quad \text{for} \quad i = 1, 2, \ldots, m_n; \; n = 0, 1, \ldots$$

Since we have assumed that $Q_n - X$ is $12 \cdot 2^{-n}$-lanky, we infer that Q_n^i is also $12 \cdot 2^{-n}$-lanky. Obviously, no side of M_n^i is longer than 2^{-n}. Let d_n^i be the centre of B_n^i for $i = 1, 2, \ldots, m_n$; $n = 0, 1, \ldots$

The homeomorphism $h_0 = g^{-1}$ restricted to $g(M_n^i)$ can be considered as a homeomorphism $h_{0,n}^j$ defined on $\operatorname{Fr} I^2$. We can now apply Lemma 5.4 to each of the topological discs Q_n^i and $\varepsilon = 2^{-n}$. Thus for each $i = 1, 2, \ldots, m_n$ and $n = 0, 1, \ldots$ there exist: a finite dendrite D_n^i, a retraction r_n^i, and an extension h_n^i of $h_{0,n}^i$ satisfying conditions (i)-(iv) of that lemma with suitable adaptations.

It is clear from the construction and from 5.4 (i) that each component P^ν of $Q_0 - (N \cup X)$ $(\nu = 1, 2, \ldots, m_0)$ contains exactly one component D^ν of $\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{m_n} D_n^i$ and that D^ν is a tree for $\nu = 1, 2, \ldots, m_0$. By 5.4 (ii), for every $d \in D_n^i$ there exists a local cut $J(d) \subset Q_n^i$ of D_n^i such that

$$\dot{J}(d) \subset \{b_n^i\} \cup \{b_n^{i+1}\} \cup \bigcup_{j=1}^{v(i+1)-v(i)+1} \{b_{n+1}^{v(i)+j-1}\} \quad \text{and} \quad \operatorname{diam} J(d) < 2^{-n+2}.$$

Since each vertex of B_n can be joined in N with X by a simple closed arc of diameter $< 2^{-n+1}$, we easily infer that for each $d \in D^\nu \cap Q_n$ there exists a bridge $L(d)$ from d to X such that $\operatorname{diam} L(d) < 2^{-n+3}$. It follows that D^ν is a canal in X for $\nu = 1, 2, \ldots, m_0$ and condition (iii) is satisfied.

By virtue of 5.4(iii), the retractions r_n^i $(i = 1, 2, \ldots, m_n;\ n = 0, 1, \ldots)$ define for each $\nu = 1, 2, \ldots, m_0$ a retraction $r^\nu\colon P^\nu \to D^\nu$ satisfying condition (iv).

Finally, the homeomorphisms h_n^i define a homeomorphism $h\colon S^1 \times [0, \infty) \xrightarrow{\text{onto}} Q_0 - X$. Condition (v)(a) follows directly from the properties of h_0. Condition (v)(b) follows from 5.4(iv).

6. Auxiliary Theorem. *If $X \subset E_2$ is a continuum which does not separate the plane and $f\colon X \to E_2$ is a continuous mapping such that $f(\operatorname{Fr} X) \subset X$ and $f(x) \neq x$ for every $x \in X$, then there exists a simple canal D in X such that $f(\operatorname{Lim} D) \subset \operatorname{Lim} D$.*

Proof. By the compactness of X, there exists an $e > 0$ such that

(1) $$\varrho\big(x, f(x)\big) > 4e \quad \text{for} \quad x \in X.$$

We can evidently assume that

(2) $$e > 1.$$

There exist an open set $U \subset X$ and an extension $f^*\colon U \to E_2$ of the mapping f such that

(3) $$\varrho\big(p, f^*(p)\big) > 3e \quad \text{for} \quad p \in U.$$

Since $f(\operatorname{Fr} X) \subset X$, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

(4) if $p \in U - X$ and $\operatorname{dist}(p, \operatorname{Fr} X) < \delta(\varepsilon)$, then $\operatorname{dist}\big(f^*(p), X\big) < \varepsilon$.

Moreover, replacing U by a smaller set if necessary, we can assume that

(5) $$\operatorname{dist}(E_2 - U, X) = \delta(1).$$

Making use of Lemma 5.5, we infer that conditions (i)-(v) of that lemma are satisfied. Let us adopt the notation of that lemma and let us define Q_t and B_t by the formulae

$$Q_t = h\big(S^1 \times [t, \infty)\big), \quad B_t = h\big(S^1 \times (t)\big) \quad \text{for} \quad 0 \leqslant t < \infty.$$

Conditions (4) and (5) imply that we can assume

(6) $$f^*\colon Q_1 \to E_2, \quad f^*(Q_{n+1} - X) \subset \operatorname{Int} Q_n \quad \text{for} \quad n = 0, 1, \ldots$$

It can also be assumed that

(7) $$\text{if } p = h(s, t) \in N, \text{ then } f^*(p) \notin h\big((s) \times [0, \infty)\big).$$

Indeed, let $N = N' \cup N''$, where N' is the union of those N_n^i for which b_n^i is of the form $b_n^{v(i)}$, and N'' is the union of the remaining N_n^i.

If $p = h(s, t) \in N'$ and $f^*(p) \in h\big((s) \times [0, \infty)\big)$, then, by (6), p and $f^*(p)$ would lie in the same component of N, which is impossible because of (3), (2), and 5.5(ii). Let V_n^i be disjoint neighbourhoods of the segments $N_n^i \subset N''$ such that $\operatorname{diam} V_n^i < 2^{-n}$, and let V be the union of those V_n^i. Let f^{**} be a continuous mapping satisfying (6) and such that $f^{**}|Q_1 - V = f^*|Q_1 - V$, $f^{**}(N_n^i) = f^*(b_{n+1}^{v(i)})$ for each $N_n^i \subset N''$, and $\varrho\big(f^{**}(p), f^*(p)\big) < a_n = \max_i \operatorname{diam} f^*(V_n^i)$ for $p \in Q_n - Q_{n+1}$. Since $a_n \to 0$ as $n \to \infty$, f^{**} is another extension of f satisfying (6). Moreover, if $p = h(s, t) \in N_n^i \subset N''$ and $f^{**}(p) = f^*(b_{n+1}^{v(i)}) \in h\big((s) \times [0, \infty)\big)$, then $b_{n+1}^{v(i)} = h(s, n+1) \in N'$ and $f^*(b_{n+1}^{v(i)}) \in h\big((s) \times [0, \infty)\big)$, which is impossible by the preceding argument. Thus f^{**} satisfies (7) and we shall simply write f^* instead of f^{**}. However, condition (3) must now be replaced by

(8) $$\varrho\big(p, f^*(p)\big) > 2e \quad \text{for} \quad p \in Q_1.$$

For every $t \in [0, \infty)$ let $\pi_t\colon S^1 \times [0, \infty) \to S^1 \times [t, \infty)$ be the retraction defined as follows:

(9) $$\pi_t(s, \tau) = \begin{cases} (s, \tau), & \text{if} \quad \tau \geqslant t; \\ (s, t), & \text{if} \quad \tau \leqslant t. \end{cases}$$

Moreover, let $\varphi_t\colon Q_0 \to Q_t$ be defined by the formula:

(10) $$\varphi_t(p) = \begin{cases} p, & \text{if} \quad p \in X; \\ h\pi_t h^{-1}(p), & \text{if} \quad p \in Q_0 - X. \end{cases}$$

Finally, let $f_t\colon Q_1 \to E_2$ be defined by the formula:

(11) $$f_t(p) = \begin{cases} f^*(p), & \text{if} \quad p \in X; \\ \varphi_t f^*(p), & \text{if} \quad p \in Q_1 - X. \end{cases}$$

It is clear that the mappings π_t and φ_t are continuous for each $t \in [0, \infty)$. Let us note that, in fact, f_t is continuous as a function of (t, p).

Let us define K_t by the formula

(12) $$K_t = \{p \in Q_1\colon f_t(p) = p\} \quad \text{for} \quad t \in [0, \infty), \quad K = \bigcup_{t \in [0, \infty)} K_t.$$

The definition implies, by (11), that

(13) $$K \subset Q_1 - X.$$

Let us note that if $p = h(s, \tau) \in Q_0 - X$, then

(14) $p \in K_t$ if and only if $\tau = t$ and $f^*(p) = h(s, \tau')$, where $\tau' \in [0, \tau)$.

Indeed, if $p \in K_t$, then by (12), (11), and (13), we have $p = f_t(p) = \varphi_t f^*(p)$. If $f^*(p) \in X$, then by (10), $p = \varphi_t f^*(p) = f^*(p) \in X$, contrary to $p \in K_t \subset K \subset Q_1 - X$. Since $f^*(p) \in Q_0$ for $p \in K_t \subset K \subset Q_1 - X$, this implies that $f^*(p) \in Q_0 - X$; let $f^*(p) = h(s', \tau')$. Then, by (10), $p = \varphi_t f^*(p)$

$= h\pi_t h^{-1}h(s', \tau') = h\pi_t(s', \tau')$. If $\tau' \geqslant t$, then (9) would imply $p = h\pi_t(s', \tau') = h(s', \tau') = f^*(p)$, contrary to (8). Thus $\tau' < t$ and, by (9), we get $h(s, \tau) = p = h\pi_t(s', \tau') = h(s', t)$, whence $s = s'$ and $\tau = t$; thus $f^*(p) = h(s, \tau')$, where $\tau' \in [0, \tau)$.

Conversely, if $f^*(p) = h(s, \tau')$, where $0 \leqslant \tau' < \tau = t$, then by (11), (10), and (9), we get $f_t(p) = \varphi_t f^*(p) = h\varphi_t h^{-1}h(s, \tau') = h\pi_t(s, \tau') = h(s, t) = p$; thus $p \in K_t$.

Condition (14) implies, by (13), that

(15) $$K_t \subset B_t \quad \text{for} \quad t \in [1, \infty).$$

The definition (12) of K_t implies now that K_t is closed in B_t for every $t \in [1, \infty)$, whence

(16) $$K \text{ is closed in } Q_1 - X.$$

Moreover, (7) and (14) imply

(17) $$K \cap N = 0.$$

Let $F_t(p) = \overrightarrow{p, f_t(p)}$ for $p \in Q_1$ and $t \in [0, \infty)$; thus $F_t(p) = 0$ if and only if $p \in K_t$. For each simple closed curve $M \subset Q_1 - K_t$ let $\vartheta_t(M)$ denote the characteristic of the vector field F_t on M. Since homotopic vector fields have the same characteristics, we infer that

(18) $$\text{if } M \subset Q_1 - \bigcup_{t \in [t_0, t_1]} K_t, \text{ then } \vartheta_{t_0}(M) = \vartheta_{t_1}(M).$$

Let us note that

(19) $$\vartheta_t(B_\tau) = \begin{cases} 0, & \text{if} \quad t < \tau, \\ \neq 0, & \text{if} \quad t > \tau, \end{cases} \quad \text{for every } t \in [0, \infty), \tau \in [1, \infty).$$

Indeed, in both cases $\vartheta_t(B_\tau)$ is correctly defined, because (15) yields $B_\tau \subset Q_1 - K_t$. Moreover, if $t < \tau$, then by (15), $K_t \cap Q_\tau = 0$; hence F_t is a non-vanishing vector field on Q_τ. This evidently implies that $\vartheta_t(B_\tau) = 0$. If however $t > \tau$, then (9), (10), and (11) give $f_t(B_\tau) \subset Q_t \subset Q_\tau$, whence evidently $\vartheta_t(B_\tau) \neq 0$.

Let $b_0^\nu = h(s_0^\nu, 0)$ for $\nu = 1, 2, \ldots, m_0$. For every $\nu = 1, 2, \ldots, m_0$ and $1 \leqslant \tau' < \tau'' < \infty$ let $Q^\nu_{\tau',\tau''} = h(\widehat{[s_0^\nu, s_0^{\nu+1}]} \times [\tau', \tau''])$ and let $M^\nu_{\tau',\tau''} = \operatorname{Fr} Q^\nu_{\tau',\tau''}$. Since $M^\nu_{\tau',\tau''} \subset B_{\tau'} \cup B_{\tau''} \cup N$, we infer from (15) and (17) that

(20) $$M^\nu_{\tau',\tau''} \subset Q_1 - K_t \quad \text{for} \quad \tau' \neq t \neq \tau''.$$

Moreover, if $t \notin [\tau', \tau'']$ then by the same argument $K_t \cap Q^\nu_{\tau',\tau''} = 0$, whence

(21) $$\vartheta_t(M^\nu_{\tau',\tau''}) = 0 \quad \text{for} \quad t \notin [\tau', \tau''].$$

If $1 \neq t \neq 2$, then by (15) and (20), the numbers $\vartheta_t(B_1)$, $\vartheta_t(B_2)$, and $\vartheta_t(M^\nu_{1,2})$ are correctly defined for each $\nu = 1, 2, \ldots, m_0$ and we have $\vartheta_t(B_1) = \vartheta_t(B_2) + \sum_{\nu=1}^{m_0} \vartheta_t(M^\nu_{1,2})$. Moreover, if $1 < t < 2$, then by (19), $\vartheta_t(B_1) \neq 0 = \vartheta_t(B_2)$, whence there exists a ν_0 such that $\vartheta_t(M^{\nu_0}_{1,2}) \neq 0$. By virtue of (20) and (18), we can assume that ν_0 is the same for all $t \in (1, 2)$. Hence

(22) $$\vartheta_t(M^{\nu_0}_{1,2}) \neq 0 \quad \text{for} \quad t \in (1, 2).$$

We shall show that if $\tau > 2$, then also

(23) $$\vartheta_t(M^{\nu_0}_{1,\tau}) \neq 0 \quad \text{for} \quad t \in (1, \tau).$$

Indeed, if $1 < t_0 < 2$, then by (20) the numbers $\vartheta_{t_0}(M^{\nu_0}_{1,\tau})$, $\vartheta_{t_0}(M^{\nu_0}_{1,2})$, and $\vartheta_{t_0}(M^{\nu_0}_{2,\tau})$ are correctly defined and $\vartheta_{t_0}(M^{\nu_0}_{1,\tau}) = \vartheta_{t_0}(M^{\nu_0}_{1,2}) + \vartheta_{t_0}(M^{\nu_0}_{2,\tau})$. By (22) and (21), we have $\vartheta_{t_0}(M^{\nu_0}_{1,2}) \neq 0 = \vartheta_{t_0}(M^{\nu_0}_{2,\tau})$, whence $\vartheta_{t_0}(M^{\nu_0}_{1,\tau}) \neq 0$. On the other hand, if $1 < t_1 < \tau$, then by (20) and (18), we infer that $\vartheta_{t_1}(M^{\nu_0}_{1,\tau}) \neq 0$, which is exactly (23).

Let P be the component of $Q_0 - (N \cup X)$ which contains $b_0^{\nu_0}$, $b_0^{\nu_0+1}$ on its boundary, and let $L' = h((s_0^{\nu_0} \times [0, \infty))$, $L'' = h((s_0^{\nu_0+1}) \times [0, \infty))$. We shall show that

(24) $$P \cap K \text{ cuts } P \cap Q_1 \text{ between } L' \cap Q_1 \text{ and } L'' \cap Q_1.$$

Indeed, otherwise there would exist a simple closed arc $L = \widehat{[l', l'']}$ such that $L \subset (P \cap Q_1) - K$, $l' \in L' \cap Q_1$, $l'' \in L'' \cap Q_1$. By (16), we can assume that $L \cap B_1 = 0$. Let us consider τ sufficiently large to ensure $L \subset Q_1 - Q_\tau$. Thus $L \cup M^{\nu_0}_{1,\tau}$ is the union of two simple closed curves M' and M'' with the common arc L. Let us assume that $M' \cap B_1 \neq 0$. Making use of (15), we infer that the numbers $\vartheta_1(M'')$, $\vartheta_\tau(M')$ are correctly defined and, since F_1 is not vanishing inside M'' nor F_τ is vanishing inside M', we have $\vartheta_1(M'') = 0 = \vartheta_\tau(M')$. Moreover, since $(L \cup L' \cup L'') \cap K = 0$, we can apply (18) and infer that $\vartheta_t(M'') = 0 = \vartheta_t(M')$ for every $t \in (1, \tau)$. Hence $\vartheta_t(M^{\nu_0}_{1,\tau}) = \vartheta_t(M') + \vartheta_t(M'') = 0$ for $t \in (1, \tau)$, contrary to (23), and this contradiction proves (24).

Making use of Theorem 52.III.1 from [8] (p. 335) we infer that there exists a connected set $K_0 \subset P \cap K$ such that

(25) $$K_0 \text{ cuts } P \cap Q_1 \text{ between } L' \cap Q_1 \text{ and } L'' \cap Q_1.$$

Let $D = D^{\nu_0}$ and $r = r^{\nu_0}$ be the canal and the retraction described in 5.5 (iii), (iv). The set $H = r(K_0)$ is a connected subset of D. Moreover

(26) $$\bar{H} \cap \operatorname{Lim} D \neq 0.$$

Indeed, (25) yields $K_0 \cap Q_n \neq 0$ for $n = 1, 2, \ldots$ Let $p_n \in K_0 \cap Q_n$ for $n = 1, 2, \ldots$; choosing a subsequence if necessary, we can assume that $p_n \to p_0$. Making use of 5.5 (iv), we infer that $r(p_n) \to p_0$, whence $p_0 \in \operatorname{Lim} D \cap \bar{H}$.

Lemmas 3.1 and 4.3 imply that there exists a simple canal $\widetilde{D} \subset H$ in X. We shall prove that

$$f(\operatorname{Lim}\widetilde{D}) \subset \operatorname{Lim}\widetilde{D}. \tag{27}$$

Let d_0 be the origin of D and $\tilde{d}_0$ the origin of $\widetilde{D}$. We can obviously assume that the arc $[\widetilde{d_0, \tilde{d}_0}] \subset D$ is disjoint with $\widetilde{D}$ beyond the origin $\tilde{d}_0$. Let n_0 be a natural number such that $[\widetilde{d_0, \tilde{d}_0}] \cap Q_{n_0} = 0$. Let us suppose that $x \in \operatorname{Lim}\widetilde{D}$. Then there exists a sequence $d_n \in \widetilde{D}$ such that $d_n \to x$. We can assume that $d_n \in Q_n$ for $n = 1, 2, \ldots$ Since $d_n \in \widetilde{D} \subset H$, we have $d_n = r(k_n)$, where $k_n \in K_0$ for $n = 1, 2, \ldots$ Condition 5.5 (iv) ensures that $\varrho(d_n, k_n) < 2^{-n+2}$, whence also $k_n \to x$. Thus $f^*(k_n) \to f(x)$. We infer from (12) and (14) that if $k_n = h(s_n, t_n)$ for $n = 1, 2, \ldots$, then $f^*(k_n) = h(s_n, t_n')$, where $t_n' \in [0, t_n)$. Thus $rf^*(k_n)$ is defined, and since $x \in \operatorname{Fr} X$, we have $f(x) \in X$. Thus $\operatorname{dist}(f^*(k_n), X) \to 0$ and by 5.5 (iv),

$$rf^*(k_n) \to f(x). \tag{28}$$

We shall show that

$$rf^*(k_n) \in \widetilde{D} \text{ for } n \geqslant n_0+1. \tag{29}$$

Indeed, by 5.5 (v), $D_n = rh((s_n) \times [0, t_n)) = [\widetilde{rh(s_n, 0), rh(s_n, t_n)}] = [\widetilde{d_0, d_n}]$. If $n > n_0$, then $D_n = [\widetilde{d_0, \tilde{d}_0}] \cup [\widetilde{\tilde{d}_0, d_n}]$. Since $rf^*(k_n) = rh(s_n, t_n')$, where $t_n' \in [0, t_n)$, we have $rf^*(k_n) \in D_n$. But if $n > n_0+1$, then evidently $rf^*(k_n) \in Q_{n_0}$, whence $rf^*(k_n) \in [\widetilde{\tilde{d}_0, d_n}] \subset \widetilde{D}$.

From (28) and (29) we infer that $f(x) \in \overline{\widetilde{D}}$. But since $x \in \operatorname{Lim}\widetilde{D} \subset \operatorname{Fr} X$, we have $f(x) \in X$, i.e. $f(x) \notin \widetilde{D}$. Thus $f(x) \in \operatorname{Lim}\widetilde{D}$ and we have proved (27). This completes the proof of the theorem.

7. Main theorem. In this section we give the proof of the theorem announced in the introduction. We begin with a lemma whose proof is a standard application of Brouwer's reduction theorem.

7.1. LEMMA. *If $0 \neq X = \hat{X} \subset E_2$ is a continuum and $f\colon X \to E_2$ is a mapping such that $f(\operatorname{Fr} X) \subset X$, then there exists Z_0 irreducible with respect to the following properties:*

(i) *Z is a non-empty subcontinuum of X;*

(ii) $Z = \hat{Z}$;

(iii) $\operatorname{Fr} Z \subset \operatorname{Fr} X$;

(iv) $f(\operatorname{Fr} Z) \subset Z$.

Moreover, if there exists a Z satisfying (i)-(iii) *and also*

(iv′) $f(\operatorname{Fr} Z) \subset \operatorname{Fr} Z$,

then there exists a Z_0 irreducible with respect to properties (i)-(iii), (iv)′.

Proof. Since X satisfies conditions (i)-(iv) and there exists a Z satisfying (i)-(iii), (iv)′, by Brouwer's reduction theorem ([8], p. 27), it is sufficient to prove that if Z_n satisfies (i)-(iv) (resp. (i)-(iii), (iv)′) and $Z_{n+1} \subset Z_n$ for $n = 1, 2, \ldots$, then $Z = \bigcap_{n=1}^{\infty} Z_n$ satisfies also conditions (i)-(iv) (resp. (i)-(iii), (iv)′).

(i) Z is a non-empty subcontinuum of X as the intersection of a decreasing sequence of non-empty continua.

(ii) $Z = \hat{Z}$, for $E_2 - Z = \bigcup_{n=1}^{\infty} (E_2 - Z_n)$ is connected as the union of connected sets with a non-empty intersection.

(iii) Since E_2 is locally connected, we have

$$\operatorname{Fr}\Big(\bigcup_{n=1}^{\infty} (E_2 - Z_n)\Big) \subset \overline{\bigcup_{n=1}^{\infty} \operatorname{Fr}(E_2 - Z_n)}$$

(comp. [8], p. 168), whence $\operatorname{Fr} Z \subset \overline{\bigcup_{n=1}^{\infty} \operatorname{Fr} Z_n}$. Since $\operatorname{Fr} Z_n \subset \operatorname{Fr} X$ for $n = 1, 2, \ldots$, we have $\bigcup_{n=1}^{\infty} \operatorname{Fr} Z_n \subset \operatorname{Fr} X$, whence

$$\overline{\bigcup_{n=1}^{\infty} \operatorname{Fr} Z_n} \subset \operatorname{Fr} X \quad \text{and} \quad \operatorname{Fr} Z \subset \operatorname{Fr} X.$$

(iv), (iv)′. Let us note that if $A = \bar{A} \subset X$ and $\operatorname{Fr} A \subset \operatorname{Fr} X$, then $\operatorname{Fr} A = A \cap \operatorname{Fr} X$. Hence $\operatorname{Fr} Z_n = Z_n \cap \operatorname{Fr} X$ for $n = 1, 2, \ldots$ and $\operatorname{Fr} Z = Z \cap \operatorname{Fr} X$. Thus

$$f(\operatorname{Fr} Z) = f\Big(\bigcap_{n=1}^{\infty} Z_n \cap \operatorname{Fr} X\Big) \subset \bigcap_{n=1}^{\infty} f(Z_n \cap \operatorname{Fr} X) = \bigcap_{n=1}^{\infty} f(\operatorname{Fr} Z_n).$$

Now, if $f(\operatorname{Fr} Z_n) \subset Z_n$ for $n = 1, 2, \ldots$, then $f(\operatorname{Fr} Z) \subset Z$. But if $f(\operatorname{Fr} Z_n) \subset \operatorname{Fr} Z_n$ for $n = 1, 2, \ldots$, then

$$f(\operatorname{Fr} Z) \subset \bigcap_{n=1}^{\infty} \operatorname{Fr} Z_n = \bigcap_{n=1}^{\infty} (Z_n \cap \operatorname{Fr} X) = Z \cap \operatorname{Fr} X = \operatorname{Fr} Z.$$

7.2. Proof of Theorem 1.1. Let $Z_0 \subset X$ be a set irreducible with respect to properties (i)-(iv) of Lemma 7.1. By the Auxiliary Theorem 6, there exists a simple canal D in Z_0 such that $f(\operatorname{Lim} D) \subset \operatorname{Lim} D$. By Lemmas 3.2 and 4.2, $\operatorname{Lim} D$ is a non-empty subcontinuum of $\operatorname{Fr} Z_0$. By Lemmas 2.1, 2.2, and 2.3, we infer that $Z_1 = \widehat{\operatorname{Lim} D}$ is a non-empty subcontinuum of Z_0 such that $Z_1 = \hat{Z}_1$ and $\operatorname{Fr} Z_1 = \operatorname{Lim} D$. Hence

$$\operatorname{Fr} Z_1 = \operatorname{Lim} D \subset \operatorname{Fr} Z_0 \subset \operatorname{Fr} X \quad \text{and} \quad f(\operatorname{Fr} Z_1) = f(\operatorname{Lim} D) \subset \operatorname{Lim} D = \operatorname{Fr} Z_1.$$

Thus Z_1 satisfies (i)-(iii), (iv)′ and there exists a set irreducible with respec to those properties. We can assume without any loss of generality that Z_0 is the irreducible set.

Since Z_1 satisfies conditions (i)-(iii), (iv)′ and $Z_1 \subset Z_0$, we have $Z_1 = Z_0$. Hence $\operatorname{Fr}\hat{Z}_0 = \operatorname{Fr} Z_0 = \operatorname{Fr} Z_1 = \operatorname{Lim} D$, and since by Lemma 4.1, D is a canal in $\operatorname{Fr} Z_0$, we infer from Lemmas 3.2 and 4.4 that $X_0 = \operatorname{Fr} Z_0$ is an indecomposable continuum.

We have, moreover,

$$X_0 = \operatorname{Fr} Z_0 \subset \operatorname{Fr} X \quad \text{and} \quad f(X_0) = f(\operatorname{Fr} Z_0) \subset \operatorname{Fr} Z_0 = X_0 .$$

To prove that $f(X_0) = X_0$ let us notice that $f(X_0)$ is a non-empty subcontinuum of Z_0. By Lemmas 2.1, 2.2, and 2.3, we infer that $X_1 = \widehat{f(X_0)}$ is a non-empty subcontinuum of Z_0 such that $X_1 = \hat{X}_1$ and $\operatorname{Fr} X_1 = f(X_0)$. Hence

$$\operatorname{Fr} X_1 = f(X_0) \subset X_0 = \operatorname{Fr} Z_0 \subset \operatorname{Fr} X \text{ and } f(\operatorname{Fr} X_1) = ff(X_0) \subset f(X_0) = \operatorname{Fr} X_1 .$$

Since $X_1 \subset Z_0$ and Z_0 is irreducible with respect to properties (i)-(iii), (iv)′, we have $X_1 = Z_0$, i.e.,

$$f(X_0) = \operatorname{Fr} X_1 = \operatorname{Fr} Z_0 = X_0 .$$

8. Corollaries and problems. From Theorem 1.1 we get the following

8.1. Corollary. *If X is a plane continuum which does not separate the plane and every indecomposable subcontinuum of $\operatorname{Fr} X$ has the fixed point property for continuous mappings, then X has the fixed point property for continuous mappings.*

Hence, in particular, we get the following

8.2. Corollary. *If X is a plane continuum which does not separate the plane and $\operatorname{Fr} X$ is hereditarily decomposable, then X has the fixed point property for continuous mappings.*

8.3. Corollary. *Let X be a plane continuum which does not separate the plane and let $\operatorname{Int} X$ have a finite number of components. If $\operatorname{Fr} X = \bigcup_{i=1}^{\infty} X_i$, where X_i for $i = 1, 2, \ldots$ is a continuum which has hereditarily (with respect to subcontinua) the fixed point property for continuous mappings, then X has the fixed point property for continuous mappings.*

Indeed, otherwise there would exist a fixed point free mapping $f\colon X \to X$ and an indecomposable continuum $X_0 \subset \operatorname{Fr} X$ such that $f(X_0) = X_0$. If there exists an $i \geqslant 1$ such that $X_0 \subset X_i$, then by the assumption, there exists a fixed point in X_0. Let us therefore assume that the sets $A_i = X_0 \cap X_i$ are proper subsets of X_0 for $i = 1, 2, \ldots$ Evidently $X_0 = \bigcup_{i=1}^{\infty} A_i$. Moreover, since $\operatorname{Int} X$ has a finite number of components, Janiszewski's Theorem ([8], p. 355) easily implies that each of the compact sets A_i has a finite number of components. Thus X_0 has been represented in the form of the union of a countable number of continua, which is impossible in view of [8], p. 150.

In particular, the Corollary can be applied to X_i's which are snake-like continua [5].

8.4. Corollary. *Let X be a plane continuum which does not separate the plane and let $\operatorname{Int} X$ have a finite number of components. If any two points of $\operatorname{Fr} X$ can be joined in $\operatorname{Fr} X$ by a hereditarily decomposable subcontinuum, then X has the fixed point property for continuous mappings.*

Indeed, otherwise there would exist an indecomposable continuum $X_0 \subset \operatorname{Fr} X$. Let $x', x'' \in X_0$ be two points belonging to distinct composants of X_0. By the assumption, there exists a hereditarily decomposable continuum $X_1 \subset \operatorname{Fr} X$ such that $x', x'' \in X_1$. Consequently, we infer that $X_0 \cap X_1$ is not connected and, by Janiszewski's Theorem, $X_0 \cup X_1$ separates the plane. Since X_0 contains uncountably many composants ([8], p. 150) and $\operatorname{Int} X$ has finitely many components, this easily leads to a contradiction.

In particular, the Corollary can be applied to $\operatorname{Fr} X$ which is arcwise connected.

The eample of S. Kinoshita [7] shows that there exists an acyclic 2-dimensional arcwise connected continuum without the fixed point property. However, the space of Kinoshita is not planar and this fact seems to be essential. The following hypothesis would be true:

8.5. Hypothesis. *If X is a plane continuum which does not separate the plane and is arcwise connected, then X has the fixed point property for continuous mappings,*

provided the following problem had a positive answer.

8.6. Problem. *Let X be a plane continuum which does not separate the plane and is arcwise connected. Is $\operatorname{Fr} X$ necessarily hereditarily decomposable?*

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Post algebras as semantic bases of some many-valued logics

by

V. G. Kirin (Zagreb)

1. Introduction. The subject of the present paper are the many-valued functional calculi of the first order of Turquette–Rosser (without equality) and the role of Post algebras therein which is fully analogous in many respects to that played by Boolean algebras in the classical two-valued case.

The main results are obtained by means of a certain formalism of Gentzen and the diagram of formulas due to H. Rasiowa and R. Sikorski introduced by them in [6].

The author desires to express his profound indebtendness to Professor H. Rasiowa for the problem itself and for the kindly advice as well.

2. Post algebras. A distributive lattice with zero (e_0) and unit (e_{n-1}), which contains an n-element chain $e_0, e_1, \ldots, e_{n-2}, e_{n-1}$ (with $n \geqslant 2$) and is such that for each x there are n y's with the property that

$$x = (e_0 \vee y_0) \wedge (e_1 \vee y_1) \wedge \ldots \wedge (e_{n-1} \vee y_{n-1})$$

if

$$y_r \vee y_s = e_{n-1} \quad \text{for} \quad r \neq s \quad \text{and} \quad y_0 \wedge \ldots \wedge y_{n-1} = e_0 ,$$

is called a *Post algebra of order* n (¹). Together with the additional conditions: for every x

$$x \vee e_{i-1} = e_i \quad \text{implies} \quad x = e_i \quad (i = 1, \ldots, n-1)$$

and

$$x \wedge e_1 = e_0 \quad \text{implies} \quad x = e_0 ,$$

Post algebras are fully characterized. (Cf. [1] and, for another axiomatization, [7].)

Since the uniqueness of those y's has been proved in [1], we write henceforth

$$y_i = k_i(x) \quad \text{for} \quad i = 0, \ldots, n-1 .$$

(¹) One obtains Boolean algebras, for $n = 2$, by putting $y_0 = x$ and $y_1 = \neg x$ (the complement of x).