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On ordered topological spaces

by

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1. A topological space X is called *ordered* if there exists a transitive relation $\prec$, called *order* in X, satisfying the following two conditions (see [2], p. 38):

(i) If $x, y \in X$, then one and only one of the relations $x \prec y$, $x = y$, $y \prec x$ holds.

(ii) If $x, y \in X$ and $x \prec y$, then there exist neighbourhoods $U(x)$ of x and $U(y)$ of y such that $x \prec y'$ and $x' \prec y$ whenever $x' \in U(x)$ and $y' \in U(y)$.

Condition (ii) can be, as is easy to observe, replaced by the following one

(ii') If $p \in X$, then the sets $\{x \in X\colon x \prec p\}$ and $\{x \in X\colon p \prec x\}$ are both open in X.

In what follows an ordered space will always mean an ordered topological space.

There are many examples of ordered spaces. Such are, for instance, the diagrams $\{(x, f(x))\colon x \in R\}$, where $f\colon R \to Y$ is any function mapping the real line R into a topological space Y, with the topology inherited from $R \times Y$. A great variety of such diagrams, interesting from the topological point of view, can be found already in the case of $Y = R$, cf. [14]. Another set of examples of ordered spaces is provided by metric separable spaces whose all quasicomponents are single points (cf. [10], II, p. 93; see also the Remark following Theorem 6 of this paper).

Ordered spaces have several interesting properties (for instance, they are all Hausdorff spaces) and they have already been studied to some extent, e.g. in [2]. The aim of the present paper is to conduct this study further.

Thus, in part 2 of this paper we shall show that in ordered spaces quasicomponents coincide with components (Theorem 1) and we shall discover a close affinity between ordered connected spaces on one hand and irreducibly connected spaces on the other hand (Theorem 3), both results to be applied later.

Eilenberg proved [2] that a connected space X is ordered if and only if

(α) the Cartesian square $X \times X$ is no longer connected after removing its diagonal,

and this theorem became his most useful tool in discovering the properties of separable connected ordered spaces. Then he obtained a theorem stating that a separable connected space X has property (α) if and only if

(β) there exists a contraction of X into the real segment $I = [0, 1]$

(by a *contraction* we mean a one-to-one and continuous mapping, cf. [15] or [17]).

In part 3 of this paper we refrain from the assumptions of either separability or connectedness. We find in it a generalization of Eilenberg's latter result, providing in Theorem 4 a necessary and sufficient condition for an ordered space to have property (β), and in this way promoting (β) to one of our main tools.

Theorem 4, which can be viewed as an internal (nexus) characterization of ordered topological spaces having property (β), has in part 4 of this paper a counterpart in Theorem 5, which contains an external (connexus) characterization of all metric separable ordered spaces by means of embeddings into irreducible continua of type λ, thus revealing an affinity between the two. As a corollary we receive some new characterizations of irreducible connected metric separable spaces (Corollary 5).

In part 5 we give an effective construction (i.e., without using the axiom of choice) of punctiform ordered metric separable connected spaces of arbitrary, finite or infinite, dimension (Theorem 6).

The final part—6—is devoted to compactifications of metric separable ordered spaces: it contains an estimation of the deficiency, i.e., of the lowest dimension of the set of points which must be added to a metric separable ordered space in order to obtain a metric compact space (Theorem 7).

2. If X is an ordered space and $\precsim$ an order in it, then one can define in X the so-called *order topology* (more precisely, the $\precsim$ order topology), i.e., the topology which has a subbase consisting of all sets of the form

$$A_p = \{x \in X\colon x \precsim p\} \quad \text{and} \quad B_p = \{x \in X\colon p \precsim x\}, \quad \text{where } p \in X.$$

The two topologies on X do not in general coincide (for if they do, then $\dim X \leqslant 1$, and we shall construct in the sequel ordered spaces X of any dimension $\dim X = n > 1$). All that can be said here follows from (ii'). Namely, the topology in an ordered space X is finer than the topology induced by the order in X (let us recall that topology τ' is *finer* than topology τ if each τ-open set is τ'-open; see [6], p. 38).

In particular, A_p and B_p are separated for each $p \in X$, and if X is connected, then it is not hard to prove that both $A_p \cup (p)$ and $(p) \cup B_p$ are also connected [1].

The *quasicomponent* of a point x in a topological space X is, by definition, equal to the intersection of all closed-open subsets of X containing x. In other words, the quasicomponent of x in X consists of all points y of X such that X is *connected between x and y*, and, as such, it contains the component of x in X (cf. [10], II, p. 92 ff.). As simple examples show, a quasicomponent is generally bigger than a component.

If X is an ordered space and $\precsim$ an order in it, then for any two points c and d of X such that $c \precsim d$ symbol I_c^d will denote the set

$$I_c^d = A_d \cap B_c .$$

Lemma 1. *Let X be an ordered topological space with an order $\precsim$ and let c and d be two points of X such that $c \precsim d$. If any of the four sets I_c^d, $(c) \cup I_c^d$, $I_c^d \cup (d)$, $(c) \cup I_c^d \cup (d)$ is not connected, then X itself is not connected between c and d.*

Proof. Let A be any of the four sets and suppose that A is not connected. Then

$$(1) \qquad A = C \cup D, \quad \text{where } C \neq 0 \neq D \text{ and } C \cap \bar{D} \cup \bar{C} \cap D = 0 .$$

Take points $c_1 \in C$ and $d_1 \in D$, suppose $c_1 \precsim d_1$, and consider the set

$$A_1 = (c_1) \cup I_{c_1}^{d_1} \cup (d_1) .$$

Put $C_1 = C \cap A_1$ and $D_1 = D \cap A_1$.
As easily follows from (1), we have

$$A_1 = C_1 \cup D_1, \quad C_1 \neq 0 \neq D_1, \quad C_1 \cap \bar{D}_1 \cup \bar{C}_1 \cap D_1 = 0 .$$

And since, in view of $c_1 \precsim d_1$, we have also

$$A_{c_1} \cap \bar{B}_{d_1} \cup \bar{A}_{c_1} \cap B_{d_1} = 0 ,$$

the union

$$X = (A_{c_1} \cup C_1) \cup (D_1 \cup B_{d_1}) ,$$

is a decomposition of X into two non-empty and separated subsets the first of which contains c and the second d.

Theorem 1. *If X is an ordered space, then the components of X coincide with the quasicomponents.*

[1] Mrs. D. Zaremba-Szczepkowicz has observed that connected ordered spaces can be characterized by the following property: for any 3 points of X, one of them separates X between the other two.

Proof. To prove that the quasicomponents of X coincide with the components, it suffices to show that each quasicomponent is connected.

Assume, a contrario, that a quasicomponent Q is not connected. This means that there exist points $c \in Q$ and $d \in Q$ such that $c \precsim d$ and that Q is not connected between c and d. Hence Q does not contain any connected subset containing c and d, and so the set $(c) \cup I_c^d \cup (d)$ is not connected between c and d either. Therefore, by Lemma 1, X itself is not connected between c and d, and this is a contradiction of the assumption that both c and d belong to the same quasicomponent of X.

THEOREM 2. *Let X be a connected ordered space and $\precsim$ an order in it. A subset A of X is connected if and only if there exist in X points c and d such that A is equal to one of the following four sets:*

$$(2) \qquad I_c^d, \quad (c) \cup I_c^d, \quad I_c^d \cup (d), \quad (c) \cup I_c^d \cup (d).$$

Proof. By virtue of Lemma 1, any of the four sets in question is connected.

To prove the converse implication, consider a connected subset A. If $c', d' \in A$ and $c' < d'$, then, in view of Lemma 1, $I_{c'}^{d'} \subset A$. Let

$$A' = \bigcup_{\substack{c' < d' \\ c', d' \in A}} I_{c'}^{d'}.$$

Consider now two cuts of X: the first defined by the equalities $X_1 = \{x \in X\colon x \precsim c'\}$ for all $c' \in A$ and $X_2 = X - X_1$, and the second by the equalities $\tilde{X}_1 = \{x \in X\colon x \precsim d'\}$ for some $d' \in A$ and $\tilde{X}_2 = X - \tilde{X}_1$. Each cut determines a point. Let c be the point determined by the first cut, and d—by the second. It should be obvious that A is equal to one of the four sets (2) (and to which of them depends on whether $c \in A$ and $d \in A$).

COROLLARY 1. *Let X be a connected ordered space and $\precsim$ an order in it. If $f\colon X \to I$ is a contraction and W a connected subset of $f(X)$, then $f^{-1}(W)$ is a connected subset of X.*

In fact, the contraction f either preserves an order in X or reverses it (cf. [2], p. 42), and therefore, by Theorem 2, connected subsets of X are in a one-to-one correspondence (yielded by f) with connected subsets of I.

A connected Hausdorff space X is said to be *irreducibly connected between a and b* (shortly, *irreducibly connected*) if X contains a and b but no proper connected subset of X does.

THEOREM 3. *If X is an irreducibly connected space, then it is an ordered connected space with a first and a last element.*

And, conversely, if X is an ordered connected space, then either it is irreducibly connected or becomes such after completing by one or two points. More precisely, if X has both a first and a last element it is irreducibly connected between them, and if X lacks one of them or both, we must add one or, respectively, two new points to make it such.

Proof. If X is irreducibly connected between its two points a and b, then the set $E(a, b) = X - [(a) \cup (b)]$ of all points of X separating a and b possesses a natural order (cf. [16], p. 43). This order makes X an ordered connected space with a and b as the first and the last elements.

To prove the converse consider an order in X. If X contains a first and a last element (in this order) we add nothing. If it lacks a first element, we add a new point a to X which precedes all the elements of X and the base of neighbourhoods of which is formed by the sets of the form $(a) \cup \{x \in X\colon x \precsim y\}$, where $y \in Y$. Thus a becomes the first element of the augmented X. Similarly, if X lacks a last element, we add another new point b which serves as the last element of the augmented X. The augmented X remains a connected ordered space and is irreducibly connected between its first and its last elements, because any point of X lying between them separates X into two parts one of which contains the first element and the other the last.

3. An ordered space (at the moment not necessarily topological) with order $\precsim$ is said to be *of ordinal separability* $\aleph_0$ (Novotný [13]) if there exists a countable subset $D \subset X$ which is dense in the sense of Hausdorff (cf. [4], p. 89), i.e., such that for any two elements $a, b \in X$, if $a \precsim b$, then the set $J_a^b = (a) \cup \{x \in X\colon a < x < b\} \cup (b)$ contains at least two elements of D.

We shall say that an ordered topological space is *ordinally separable* if there exists an order in it with respect to which it is of ordinal separability $\aleph_0$.

If X is a connected ordered space, then topological separability coincides with ordinal separability. In general, however, the two notions are distinct. There are examples of ordered separable spaces which are not ordinally separable (cf. [13], footnote on p. 98), and, conversely, of ordinally separable spaces which are not separable (such is, for instance, the set of irrational numbers with the usual order and discrete topology).

THEOREM 4. *A topological space X has property (β) if and only if it is an ordinally separable space.*

Proof. If a topological space X has property (β), then there exists a one-to-one and continuous mapping $f\colon X \to I$. A transitive order $\precsim$ in X defined by the formula

$$(3) \qquad x \precsim y \quad \text{if and only if} \quad f(x) < f(y)$$

satisfies both (i) and (ii), which means that X is ordered. And since every subset of I ordered according to $<$ is ordinally separable, in view of (3) also X is ordinally separable. Hence the condition is necessary.

To prove that it is also sufficient take any ordinally separable space X. Being ordinally separable, X can be transformed onto a subset of the real segment I in such a way that x precedes y in the order of X if and only if $f(x) < f(y)$, where f is this transformation (cf. [13], Théorème 1). Clearly, f is one-to-one, but in general it is not continuous. However, if we construct a mapping $g\colon I \to I$ such that g is one-to-one on $f(X)$ and $\varphi = gf$ is continuous, then the proof will be completed, because φ will satisfy (β).

With this end in mind consider a component J of $I - \overline{f(X)}$. J is an open interval and three cases are possible: both end-points of $\bar{J}$ belong to $f(X)$, one end-point of $\bar{J}$ belongs to $f(X)$ and the other does not, no end-point of $\bar{J}$ belongs to $f(X)$. Only the second case is important and requiring intervention, and we contract such a $\bar{J}$ to a single point. Doing this with all such J's we receive a continuous mapping g of I onto itself which is one-to-one on $f(X)$ and does not alter the $<$ order in $f(X)$. We shall show that $\varphi = gf$ is a continuous mapping $\varphi\colon X \to I$.

To prove the continuity of φ it suffices to show that if U is an open subset of I, then the counter-image $\varphi^{-1}(U \cap \varphi(X))$ is open in the order topology in X. Without loss of generality we may assume, in view of (ii′), that U is an element of a subbase of I, that is, say, $U = \{a \in I\colon a < b\}$ for some $b \in I$. If $b \in \varphi(X)$ or $b \in \overline{U \cap \varphi(X)}$, the proof is achieved, for then $\varphi^{-1}(U \cap \varphi(X)) = \{x \in X\colon x \precneqq \varphi^{-1}(b)\}$ in the first case and $\varphi^{-1}(U \cap \varphi(X)) = \bigcup_{n=1}^{\infty} \{x \in X\colon x \precneqq x_n\}$, where $x_n \in X$, $\varphi(x_n) < b$ and $\lim_{n\to\infty} \varphi(x_n) = b$, in the second case are both open in the order topology in X. And if $b \in I - [\varphi(X) \cup \overline{U \cap \varphi(X)}]$, then the component J of $I - \overline{\varphi(X)}$ containing b is such that either both end-points of $\bar{J}$ belong to $\varphi(X)$ or neither of them does. Denoting by $b' \in \varphi(X)$ the right-hand end-point of $\bar{J}$ in the first case, we have

$$\varphi^{-1}(U \cap \varphi(X)) = \{x \in X\colon x \precneqq \varphi^{-1}(b')\}\,.$$

In the second case, choosing a sequence $\{x_n\}$ of points of X such that $\varphi(x_n) < b$ for $n = 1, 2, \ldots$ and for each $x \in X$, if $\varphi(x) < b$, then also $\varphi(x) < \varphi(x_n)$ for some $n = 1, 2, \ldots$, we have

$$\varphi^{-1}(U \cap \varphi(X)) = \bigcup_{n=1}^{\infty} \{x \in X\colon x \precneqq x_n\}\,.$$

Thus in both remaining cases $\varphi^{-1}(U \cap \varphi(X))$ is also shown to be an open subset of X in the order topology, and thus the proof is completed.

Corollary 2. *If X is an ordinally separable space, then X contains at most $\aleph_0$ components distinct from single points.*

Indeed, if $f\colon X \to I$ is a contraction asserted by Theorem 4, then each component of X distinct from a single point goes under f onto a certain subinterval of I and the images under f of any two distinct components are disjoint.

Corollary 3. *Let X be an ordinally separable space. If X is connected and, moreover, either locally connected or peripherally compact, then it is homeomorphic to a subset of the real line.*

More precisely, if X has both a first and a last element, then it is homeomorphic to a closed segment, and if X lacks one of them or both then it is homeomorphic to a segment without one or, respectively, two end-points.

The proof follows from Theorem 4 and from a result of V. V. Proizvolov [15] and G. T. Whyburn [17] which states that a contraction into the real line of a connected Hausdorff space which is either locally connected or peripherally compact is a homeomorphism.

4. A continuum (i.e., a compact metric connected space) K is said to be *irreducible between a and b* if K contains a and b but no proper subcontinuum of K contains them. An irreducible continuum K between a and b is of *type λ* if there exists a continuous mapping $\eta\colon K \to I$ such that $\eta(a) = 0$, $\eta(b) = 1$ and that each *section* $\eta^{-1}(t)$, $0 \leqslant t \leqslant 1$, is a non-dense subcontinuum of K ([10], II, § 43).

Theorem 5. *A metric separable space X is ordered if and only if there exist an irreducible continuum K of type λ and a homeomorphism $h\colon X \to K$ such that $h(X)$ meets each section of K at one point at most.*

If, moreover, X is connected, then a continuum K of type λ and a homeomorphism $h\colon X \to K$ can be found such that $h(X)$ is a dense subset of K meeting each internal section $\eta^{-1}(t)$, $0 < t < 1$, at precisely one point and each boundary section $\eta^{-1}(0)$ and $\eta^{-1}(1)$ at one point at most.

Proof. The condition is clearly sufficient and so we shall only prove that it is necessary. Let X be a metric separable ordered space. Being metric separable, it can be topologically embedded into a Euclidean or a Hilbert cube. We may then assume that $X \subset I^n$ for some $n = 1, 2, \ldots, \aleph_0$.

Let

$$f\colon X \to I$$

be a one-to-one and continuous mapping (existing by Theorem 4) of X into I. If X is connected, then so is $f(X)$, and therefore in such a case we may assume that $f(X)$ is dense in I.

As is well known, the correspondence $x \to (f(x), x)$ is a one-to-one and bicontinuous mapping (i.e., a homeomorphism) between X and the diagram $\{(f(x), x)\colon x \in X\} \subset I^{n+1}$ of f. Identifying the set X with the diagram of f we shall assume in the sequel that $X \subset I^{n+1}$ and that the projection of X onto the first axis of $I^{n+1} = I \times I^n$ is one-to-one and that, for X connected, it covers I (with the possible exception of the end-points of I).

For the sake of simplicity, a point of the cube I^{n+1} will be denoted by (t, z), where t is its first coordinate and z a system of n others, and D will mean the set of all dyadic-rational points of I.

For construction purposes we must slightly modify the position of X in I^{n+1}. Let S be a subset of X countable and dense in X. Clearly, $f(S)$ is then a subset of $f(X)$ countable and dense in $f(X)$. Condensing and distending where necessary (but without altering the order in $f(X)$) we may secure for S a position in which $f(S)$ consists of dyadic-rational points only. Therefore we shall assume that

(4) $\quad$ the set $\{(t, x) \in X\colon t \in D\}$ is dense in X.

Having thus modified the position of X in I^{n+1} we can begin the construction.

First we shall subject the cube I^{n+1}, and together with it the set X, to a non-continuous deformation by pulling apart I^{n+1} to a Cartesian product $C \times I^n$, where C is the ternary Cantor set lying in I.

Let

$$s\colon C \to I$$

be „la fonction scalariforme" of Cantor mapping C onto I by glueing the end-points of intervals contiguous to C to dyadic-rational points in I (cf. [10], I, p. 236).

Being continuous, the function s induces an upper semi-continuous decomposition of C, $C = \bigcup_{t \in I} s^{-1}(t)$. In particular, we then have ([10], II, p. 42)

(5) $\quad$ if $t_n \in I$ for $n = 0, 1, \ldots$ and $\lim_{n\to\infty} t_n = t_0$, then $\operatorname{Ls}_{n\to\infty} s^{-1}(t_n) \subset s^{-1}(t)$.

The „pulling apart" of I^{n+1} consists in the conversion of a point (t, z) of I^{n+1} into a subset $(s^{-1}(t), z)$ of $C \times I^n$ consisting of one point or two. Denoting it by ψ we then have

$$\psi\colon I^{n+1} \to C \times I^n,$$

where $\psi(t, z) = (s^{-1}(t), z)$ for each $(t, z) \in I^{n+1}$.

One can say that each cube $t \times I^n$, where $t \in D$, is doubled and the two are pushed apart, and that each cube $t \times I^n$ with $t \in I - D$ is only slightly shifted.

Now we insert a straight-line segment parallel to I between any two pushed apart cubes $t' \times I^n$ and $t'' \times I^n$, where $s(t') = s(t'') = t$ is dyadic-rational, in such way that

(6) $$\psi(X) \subset \bar{H},$$

where H is the union of all those segments.

Namely, if $t \in f(X)$, that is, if there exists a point $(t, x) \in X$, then we join by a segment the two points (t', x) and (t'', x), both belonging to $\psi(X)$. And if $t \in I - f(X)$, then $\psi(X)$ does not meet $s^{-1}(t) \times I^n = t' \times I^n \cup t'' \times I^n$ and we join by a segment an arbitrary pair of points (t', z) and (t'', z), $z \in I^n$.

Adding to $C \times I^n$ the union H of all segments thus defined we come to a continuum $(C \times I^n) \cup H$ containing $\psi(X)$. Moreover, (4) implies (6).

The second step of the construction consists in contracting each segment of H separately to a single point. To do this, consider a decomposition of the cube I^{n+1} into segments of H and single points. Clearly, this decomposition is upper semi-continuous and its hyperspace is I^{n+1}. By virtue of Aleksandrov's theorem (cf. [10], II, p. 42) there exists a continuous mapping

$$g\colon I^{n+1} \to I^{n+1}$$

such that the counter-images $g^{-1}(p)$, $p \in I^{n+1}$, coincide with the elements of our decomposition.

We shall now show that the composition h of $\psi|X$ and $g|\psi(X)$, written simply as $h = g\psi$, is a homeomorphism from X onto $h(X)$.

In fact, h is one-to-one, because $\psi|X$ doubles some points of X into pairs of $\psi(X)$ and $g|\psi(X)$ glues together all such pairs and only such pairs.

To show that h is continuous consider a sequence $\{p_n\}$ of points of X such that $\lim_{n\to\infty} p_n = p_0 \in X$. If $p_i = (t_i, x_i)$ for $i = 0, 1, 2, \ldots$, then $h(p_i) = g(s^{-1}(t_i), x_i)$ and so we have to prove that the sequence $\{g(s^{-1}(t_n), x_n)\}$ is convergent and that

(7) $$\lim_{n\to\infty} g(s^{-1}(t_n), x_n) = g(s^{-1}(t_0), x_0).$$

To this end note first that in view of (5) we have

$$\operatorname{Ls}_{n\to\infty} s^{-1}(t_n) \subset s^{-1}(t_0).$$

Hence the set $\bigcup_{i=0}^{\infty} s^{-1}(t_i)$ is compact and so, by the continuity of g, we infer ([1], p. 23 and [10], I, p. 243) that

$$\operatorname{Ls}_{n\to\infty} g(s^{-1}(t_n), x_n) = g\big(\operatorname{Ls}_{n\to\infty} (s^{-1}(t_n), x_n)\big) = g\big(\operatorname{Ls}_{n\to\infty} s^{-1}(t_n), \operatorname{Ls} x_n\big) \subset g(s^{-1}(t_0), x_0).$$

But $g(s^{-1}(t_0), x_0)$ is a single point and so (7) is proved.

And to show that also $h^{-1}=\psi^{-1}g^{-1}$ is continuous, consider a sequence $\{q_n\}$ of points of $h(X)$ such that $\lim\limits_{n\to\infty} q_n = q_0 \in h(X)$. If $q_i=(u_i, z_i)$ for $i=0,1,2,\ldots$, then $h^{-1}(q_i)=\psi^{-1}g^{-1}(u_i, z_i)=\psi^{-1}(U'_i, z'_i)=\big(s(U'_i), z'_i\big)$, where $(U'_i, z'_i)=g^{-1}(u_i, z_i)$ and U'_i consists of one point or two. But g is continuous. Therefore the assumption $\lim\limits_{n\to\infty}(u_n, z_n)=(u_0, z_0)$ implies that $\operatorname*{Ls}\limits_{n\to\infty} g^{-1}(u_n, z_n)\subset g^{-1}(u_0, z_0)$, i.e., that $\operatorname*{Ls}\limits_{n\to\infty}(U'_n, z'_n)\subset(U'_0, z'_0)$, whence, as before,

$$\psi^{-1}[\operatorname*{Ls}_{n\to\infty}(U'_n, z'_n)]=\operatorname*{Ls}_{n\to\infty}\big(s(U'_n), z'_n\big)\subset\psi^{-1}(U'_0, z'_0)=\big(s(U'_0), z'_0\big)\,.$$

Hence the sequence of points $\{s(U'_n), z'_n\}$ is convergent and its limit is the point $\big(s(U'_0), z'_0\big)$, and so $\lim\limits_{n\to\infty} h^{-1}(q_n)=h^{-1}(q_0)$.

The function h, being one-to-one and bicontinuous, is thus shown to be a homeomorphism.

Finally, turn to the question of finding a continuum K satisfying our theorem. For that purpose note first that the continuum $N=g[(C\times I^n)\cup H]$ has a natural upper semi-continuous decomposition of type λ

$$N=\bigcup_{t\in I} g[s^{-1}(t)\times I^n]\,.$$

Defining the mapping $\eta\colon N\to I$ by the formula

$$\eta(p)=t \quad\text{ whenever }\quad p\in g[s^{-1}(t)\times I^n]\,,$$

we easily verify that each section $\eta^{-1}(t)$ of N meets $h(X)$ at one point at most (if X is connected, then each internal section $\eta^{-1}(t)$, $0<t<1$, meets $h(X)$ at precisely one point). However, N itself is, in general, far from being irreducible and so it cannot serve as K.

Choose two points in N, one in $\eta^{-1}(0)$ and the other in $\eta^{-1}(1)$. If $\eta^{-1}(0)$ meets $h(X)$, we choose the point $\eta^{-1}(0)\cap h(X)$; otherwise the choice is arbitrary. Similarly for $\eta^{-1}(1)$. Now, N contains a subcontinuum $\widetilde{K}$ irreducible between the two chosen points ([10], II, p. 132). The continuum $\widetilde{K}$ contains $g(H)$, because each point of $g(H)$ cuts N into two disjoint and separated sets containing, one each, the two chosen points. Hence, in view of (6), $\widetilde{K}$ must contain $h(X)$.

We shall now show that $\eta|\widetilde{K}$ maps $\widetilde{K}$ onto I in such a way that

(8) each set $(\eta|\widetilde{K})^{-1}(t)$ is a subcontinuum of $\widetilde{K}$.

Indeed, otherwise there would exist a dyadic-rational t_0 for which $(\eta|\widetilde{K})^{-1}(t_0)$ would not be connected. Let P be its component containing the point $g(H)$ belonging to $(\eta|K)^{-1}(t_0)$ and R—any other of its components. Take disjoint open neighbourhoods, U of P and V of R. It should be obvious that U and V may be chosen in such a way that for some t_1, if t lies between t_0 and t_1, then the boundary of U

and the boundary of V are both disjoint with $(\eta|\widetilde{K})^{-1}(t_1)$. Without loss of generality we may assume that $t_0<t_1$. If t_2 is a dyadic-rational such that $t_0<t_2<t_1$, then two cases are possible: either the point $g(H)\cap\eta^{-1}(t_2)$ belongs to U or not. If it does, then the subset of $\widetilde{K}$ consisting of all points $p\in V$ for which $t_0\leqslant\eta(p)\leqslant t_2$, is closed-open in $\widetilde{K}$ and so $\widetilde{K}$ is not connected. And if it does not, then the subset of $\widetilde{K}$ consisting of all points q for which $\eta(q)\leqslant t_0$ and of all points $r\in U$ for which $t_0\leqslant\eta(r)\leqslant t_2$, is closed-open in $\widetilde{K}$ and so $\widetilde{K}$ is not connected either. Hence in both cases we have obtained a contradiction.

Thus the proof of (8) is finished.

Since for each t, $\eta^{-1}(t)$ contains one point from $h(X)$ at most (for X connected and $0<t<1$ one point precisely), then a fortiori so does $(\eta|\widetilde{K})^{-1}(t)=\widetilde{K}\cap\eta^{-1}(t)$. Hence if we could assert in addition that each set $(\eta|\widetilde{K})^{-1}(t)$ is non-dense in $\widetilde{K}$, $\widetilde{K}$ would be the continuum K we want: it is irreducible and contains $h(X)$ in the required manner, and $\eta|\widetilde{K}$ would give the required decomposition of type λ. As a matter of fact, each set $(\eta|\widetilde{K})^{-1}(t)$ is non-dense in $\widetilde{K}$ in the particular case of X being connected with $f(X)$ which is dense in I, because then $\overline{h(X)}=\widetilde{K}$ and the required property easily follows. In general, however, we should replace each set $(\eta|\widetilde{K})^{-1}(t)$ with a non-empty interior (relative to $\widetilde{K}$) by an arbitrary irreducible continuum of type λ and in this way come at last to the required K.

The proof of Theorem 5 is thus completed.

Corollary 4. *Let a and b two points of a metric separable connected space X. The following four properties are then equivalent*:

(a) *X is irreducible connected between a and b*,

(b) *there exists a one-to-one and continuous mapping $f\colon X\to I$ such that $f(a)=0$ and $f(b)=1$*,

(c) *there exist an irreducible continuum K of type λ and a homeomorphism $h\colon X\to K$ such that $h(X)$ meets each section of K at one point only*,

(d) *X is ordered and contains a first and a last element.*

Proof. The implication (a) $\Rightarrow$ (b) is known under the name of the Lennes theorem (cf. [10], II, p. 103), the implication (b) $\Rightarrow$ (c) follows from Theorem 5, and the implication (c) $\Rightarrow$ (a) is trivial. Hence properties (a), (b) and (c) are equivalent. And since, as we have already mentioned, for connected ordered spaces separability coincides with ordinal separability, in view of Theorem 4 (b) is equivalent to (d).

5. Theorem 5 implies also that every metric separable connected space which is ordered can be embedded into a certain irreducible continuum K of type λ in such a way that each section of K contains one point from X at most. On the other hand, as B. Knaster proved [7] a long time ago, an implication contrary in a sense also holds true. Namely, every irreducible continuum K of type λ contains a certain connected set chosen one point by one section of K.

Knaster's theorem gives a method for proving the existence of some peculiar metric separable ordered spaces (for instance, if K has dense sections not reducing to single points, then X must be *punctiform*, i.e., such that its only subcontinua are single points). Unfortunately, this does not permit to obtain specific examples, because the choice of X from K is not effective (cf. [7], p. 278). For that reason it seems appropriate to give an effective (without the axiom of choice) construction as a proof of the following:

THEOREM 6. *For each finite or infinite dimension n the cube I^{n+1} contains a dense, connected and punctiform subset X of dimension n, which is ordered.*

Proof. For a given n we shall construct X by a proper choice of one point p_t from each cube $t\times I^n$, where $t\in I$. For that purpose divide the first segment I into $2^{\aleph_0}$ subsets E_ξ dense in I and pairwise disjoint, $I=\bigcup_\xi E_\xi$ (such decompositions of I do exist, see [9], p. 252), and let $\mathfrak{R}$ be the family of all subcontinua K_ξ of I^{n+1} meeting the faces $0\times I^n$ and $1\times I^n$ of I^{n+1}. Clearly, family $\mathfrak{R}$ is also of cardinality $2^{\aleph_0}$ and so there exists a one-to-one correspondence between the elements E_ξ of the decomposition of I and the continua K_ξ belonging to $\mathfrak{R}$. Now for each $t\in E_\xi$ choose in the set $(t\times I^n)\cap K_\xi$ a point p_t (in an arbitrary way with the help of an effective definition; for instance, a point of, roughly speaking, minimum coordinates). Denoting by X_ξ the set of all points p_t thus chosen, for a given ξ, we define X as the union of all X_ξ, $X=\bigcup_\xi X_\xi$.

All the steps of the construction can be made effective (cf. [7], where there is a similar effective construction of a biconnected set of arbitrary dimension) and so we may proceed to the proof of the properties of X.

The family $\mathfrak{R}$ consists of all subcontinua K_ξ and so, clearly, X must be dense in I^{n+1}.

To prove that X is connected suppose, to the contrary, that X is not and let $X=A\cup B$ be its decomposition into two subsets non-empty and closed in X. There exists then ([9], p. 234, and [8], footnote on p. 13) a continuum $M\subset I^{n+1}-X$ which separates the cube I^{n+1} between A and B. Consider the projection $r(M)$ of M into I. The set $r(M)$ cannot consist of a single point t, because then M would be equal to $t\times I^n$ in contradiction with $p_t=X\cap(t\times I^n)$. Hence $r(M)$ must be a segment $[r_1,r_2]$, where $0\leqslant r_1<r_2\leqslant 1$. Joining by a straight-line segment a point of M with the first coordinate equal to r_1 to the face $0\times I^n$, and, similarly, apoint of M with the first coordinate equal to r_2 to the face $1\times I^n$, we come to a continuum K_ξ belonging to $\mathfrak{R}$. The set E_ξ is dense in I; let $t\in E_\xi\cap[r_1,r_2]$. Hence there exists a point $p_t\in K_\xi\cap X$, but, in view of our definition of K_ξ, $p_t\in M\cap X$—a contradiction.

The proof that X is punctiform is also simple. Indeed, let Q be a subcontinuum of X. If Q is not a single point, then the projection $r(Q)$ is a segment $[q_1,q_2]$, where $0\leqslant q_1<q_2\leqslant 1$. Clearly, Q must then contain each point $p_{t_0}\in X$ with $q_1<t_0<q_2$, because p_{t_0} disconnects X into two separate subsets, $X-(p_{t_0})=\{p_t\colon t<t_0\}\cup\{p_t\colon t>t_0\}$ and Q meets both. Hence $Q=X\cap([q_1,q_2]\times I^n)$. But $X\cap([q_1,q_2]\times I^n$ is dense in $[q_1,q_2]\times I^n$, whence $Q=[q_1,q_2]\times I^n$. This gives a contradiction, because X meets any $t\times I^n$ at one point only. Hence the only subcontinua of X are single points, i.e., X is punctiform.

Clearly, X contains no open subset of I^n. Hence (cf. [5], p. 44) $\dim X\leqslant n$. On the other hand, however, the proof of the connectedness of X works also to the effect that no pair v and w of points of I^{n+1} with $r(v)\neq r(w)$ can be joined by a subcontinuum of $I^{n+1}-X$. Hence, in view of Mazurkiewicz theorem (cf. [10], II, p. 343) we must also have $\dim X\geqslant n$. It follows then that $\dim X=n$.

Finally, the projection $r\colon X\to I$ gives a one-to-one and continuous mapping of X onto I and so X has property (β). This means, in view of Theorem 4, that X is ordered.

Remark. A connected metric separable space X with a point p such that $X-(p)$ contains no non-trivial connected subset is called *pulverable* and its subset $X-(p)$ a *pulverized* space (see [1]). Clearly, every component of a pulverized space is a single point. Sometimes, however, more is true. Namely, there exist (e.g., [8]) pulverized spaces of arbitrarily large dimension whose quasicomponents are all single points. Since the set of the quasicomponents of a metric separable space is ordered (cf. [10], II, p. 93), such pulverized spaces are good examples of metric separable ordered spaces with no non-trivial connected subsets and of arbitrarily large dimension.

6. All spaces under discussion here are metric separable.

By the *deficiency* of a topological space X we mean the least integer n, denoted by $\operatorname{def}X$, for which there exists a compactification $\widetilde{X}$ of X with the property $n=\dim(\widetilde{X}-X)$. The notion is due to J. de Groot [3].

THEOREM 7. *If X is an ordered and non-compact topological space of dimension n, then $n-1\leqslant\operatorname{def}X\leqslant n$.*

If, moreover, all components of X are single points, then $\operatorname{def}X=n$.

Proof. As is well known (cf. [5], p. 64), if a topological space has dimension n, then there exists a compactification $\widetilde{X}$ of X such that $\dim X=n$. A fortiori, $\operatorname{def}X\leqslant\dim(\widetilde{X}-X)\leqslant\dim\widetilde{X}=n$.

On the other hand, Lelek has shown [11] that if $f\colon X\to Y$ is a mapping of a topological space X into a topological space Y such that each counter-image $f^{-1}(y)$, $y \in Y$, is locally compact, then

$$\dim X \leqslant \dim Y + \max\{\dim f, \operatorname{def} X\},$$

where $\dim f$ denotes the greatest of dimensions $\dim f^{-1}(y)$ for $y \in Y$.

Applying this inequality to our case of X being an ordered topological space, f a contraction into the real line (existing by Theorem 4), and Y the real line, we come to the inequality

$$n-1 \leqslant \operatorname{def} X.$$

In fact, since X is not compact by hypothesis, we have $\operatorname{def} X \geqslant 0$, and so

$$\max\{\dim f, \operatorname{def} X\} = \operatorname{def} X,$$

because, clearly, $\dim f = 0$.

The second conclusion of Theorem 7 follows from Theorem 1 and from a result of Mazurkiewicz [12] stating that if X is an n-dimensional topological space whose quasicomponents are all single points, then $\operatorname{def} \boldsymbol{X} = n$.

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