By considering the two cases $x_j \leqslant \sup_{i \geqslant j+1} x_i$ and the contrary, it is easy to see that

$$(2) \qquad \delta_j - \delta_{j+1} = \max\{a_j x_j, a_j x_j + (x_j - \sup_{i \geqslant j+1} x_i)\} \qquad (j = 1, 2, \ldots),$$

and of course similarly for (y_i). Since, by (2), $a_m y_m < a_m x_m \leqslant \delta_m - \delta_{m+1}$, it follows with the help of (2) for (y_i) that

$$a_m y_m + (y_m - \sup_{i \geqslant m+1} y_i) = \delta_m - \delta_{m+1} \geqslant a_m x_m + (x_m - \sup_{i \geqslant m+1} x_i),$$

and consequently

$$(3) \qquad \sup_{i \geqslant m+1} x_i \geqslant (a_m + 1)(x_m - y_m) + \sup_{i \geqslant m+1} y_i > \sup_{i \geqslant m+1} y_i.$$

Hence for $m+1 \leqslant i \leqslant n$

$$x_i \leqslant y_i \leqslant \sup_{i \geqslant m+1} y_i < \sup_{i \geqslant m+1} x_i,$$

and therefore

$$(4) \qquad \sup_{i \geqslant n+1} x_i = \sup_{i \geqslant m+1} x_i > \sup_{i \geqslant m+1} y_i \geqslant \sup_{i \geqslant n+1} y_i.$$

On the other hand, in the same way that from $x_m > y_m$ we derived (3), from $x_n < y_n$ we can derive

$$\sup_{i \geqslant n+1} x_i < \sup_{i \geqslant n+1} y_i,$$

which contradicts (4). The proof is complete.

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On the probability measures in Hilbert spaces

by

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Introduction. In a topological semigroup we may consider integrals of the following kind. Let $g = g(t)$, $a \leqslant t \leqslant b$ be a mapping of the interval $[a, b]$ into G and $F(\Delta)$, $\Delta \subset [a, b]$ — a function of interval whose values are transformations of G into itself. If the limit of the sums

$$s(P) = \sum_i F(\Delta_i) g(t_i)$$

(where $P = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ is a partition of $[a, b]$ and $t_i \in \Delta_i$) exists as P runs over a normal sequence of partitions, then this limit will be called an *integral of g with respect to F on the interval* $[a, b]$.

The aim of this paper is to study some properties of integrals of this kind, where G is a semigroup of probability measures in a Hilbert space (with the convolution as a semigroup operation and with the topology generated by the weak convergency of measures), while F is assumed to take the values from the space of linear bounded operators in a Hilbert space (and induces a transformation of the semi-group of distributions into itself). § 1 contains the basic definitions and facts of the theory of probability measures in a Hilbert space. In § 2 we define the convolution integral and prove its fundamental properties; § 3 contains some theorems illustrating the applications of the convolution integral (the continual analogous of classical theorems on the limit distributions of sums of independent random variables); finally in § 4, employing the notion of a convolution integral, we construct the Gaussian stochastic process with special properties.

§ 1

1.1. Let H be a real separable Hilbert space with the scalar product $(\cdot, \cdot)$ and with the norm $\|\cdot\|$. Denote by $\mathfrak{M}$ the set of all probability measures in H (i.e. the set of normed regular measures defined on a σ-field $\mathfrak{B}$ of

Borelian subsets of H). We regard $\mathfrak{M}$ as a topological space with a weak convergence of measures. Namely, a sequence of probability measures μ_n is said to be *weakly convergent* to μ ($\mu_n \to \mu$) if for any continuous function bounded in H we have

$$\lim_{n\to\infty} \int f(h)\mu_n(dh) = \int f(h)\mu(dh) \ (^1). \tag{1}$$

To any pair $\mu, \nu \in \mathfrak{M}$ we assign a probability measure $\mu * \nu$ which is called the *convolution* of μ and ν, given by the formula

$$(\mu * \nu)(z) = \int \mu(z-g)\nu(dg) \quad \text{for every } Z \in \mathfrak{B}. \tag{2}$$

$\mathfrak{M}$ is an Abelian semigroup with respect to convolution (2). The weak convergence in $\mathfrak{M}$ is a metric convergence (with the metric of Levy-Prochorov; [14], p. 188) and with this metric $\mathfrak{M}$ is a complete space, while the convolution is a continuous operation. We denote the convolution of probability measures $\nu_1, \nu_2, \ldots, \nu_n$ by $\prod\limits_{k=1}^{n}{}^{*}\nu_k$. For a linear bounded operator in H and for $\mu \in \mathfrak{M}$ we put, by definition,

$$(A\mu)(Z) = \mu(A^{-1}Z) \quad \text{for any } Z \in \mathfrak{B}, \tag{3}$$

where $A^{-1}Z = \{h: Ah \in Z\}$. It is clear that if μ is the distribution of a random variable ξ, then $A\mu$ is the distribution of the random variable $A\xi$ (²).

1.2. An element $M_\mu \in H$ such that

$$(M_\mu, h) = \int (g, h)\mu(dg) \quad \text{for any } h \in H \tag{4}$$

is called the *mathematical expectation* of the probability measure μ.

For $a > 0$ let $\|\mu\|_a = \int \|g\|^a \mu(dg)$.

If $\|\mu\|_1 < \infty$, then the mathematical expectation M_μ exists. If $\|\mu\|_2 < \infty$, then the dispersion operator D_μ of the probability measure μ is defined by

$$(D_\mu g, h) = \int (u - M_\mu, g)(u - M_\mu, h)\mu(du) \quad \text{for every } h, g \in H. \tag{5}$$

A linear operator in H is called an S-operator (compare [14], p. 193) if it is self-adjoint, non-negative and has a finite trace. Hence it follows that every S-operator is compact. The disperssion operator defined for $\|\mu\|_2 < \infty$ by formula (5) is an S-operator. We write the formulae

$$D_{\mu_1 * \mu_2} = D_{\mu_1} + D_{\mu_2} \tag{6}$$

$$D_{A\mu} = AD_\mu A^*, \tag{7}$$

where A^* denotes the operator adjoint to A.

(¹) $\int \ldots$ means an integral over the whole space H.

(²) By a *random variable* ξ with the values from H we mean a measurable mapping of the probability space $(\Omega, \mathfrak{A}, p)$ into $(H, \mathfrak{B})$.

1.3. The characteristic functional $\hat{\mu}(h)$ of the probability measure $\mu \in \mathfrak{M}$ (the Fourier transformation of the measure μ) is defined by the formula

$$\hat{\mu}(h) = \int e^{i(g,h)}\mu(dg) \ (^3). \tag{8}$$

The characteristic functional (8) defines uniquely the probability measure μ. A functional $\varphi(h)$ is the *characteristic functional* of a probability measure $\mu \in \mathfrak{M}$ if and only if φ is positively defined, $\varphi(\theta) = 1$ and φ is continuous at the point $h = 0$ in the topology generated by neighbourhoods of the form (S-topology)

$$\{h: (Bh, h) < 1\},$$

where B is an arbitrary S-operator. We write the formulae

$$\widehat{(A\mu)}(h) = \hat{\mu}(A^* h), \tag{9}$$

$$\widehat{\mu_1 * \mu_2}(h) = \hat{\mu}_1(h)\hat{\mu}_2(h). \tag{10}$$

1.4. A family of S-operators B_t $(t \in T)$ is called *compact* if the following two conditions are satisfied (⁴):

(i) $\sup\limits_{t\in T} \operatorname{Tr} B_t < \infty$,

(ii) $\lim\limits_{m\to\infty} \sup\limits_{t\in T} \sum\limits_{k=m}^{\infty} (B_t e_k, e_k) = 0$, where $\{e_k\}$ is the basis in H.

By δ_x we denote a measure condensed at a point x, i.e. $\delta(Z) = 1$ if $x \in Z$ and $\delta(Z) = 0$ if $x \notin Z$. A sequence of probability measures μ_n is called *shift-compact* if there exists a sequence $\{x_n\}$ of elements of the space H such that the sequence $\mu_n * \delta_{x_n}$ is compact in $\mathfrak{M}$.

In the sequel we shall often make use of the following theorems:

(A) (Prochorov [14]). A family of probability measures μ_t $(t \in T)$ is compact in $\mathfrak{M}$ if and only if for any $\varepsilon > 0$ there exists a compact family of S-operators $B_t^{(\varepsilon)}$ $(t \in T)$ such that

$$1 - \operatorname{Re}\hat{\mu}_t(h) \leqslant (B_t^{(\varepsilon)}h, h) + \varepsilon \quad \text{for} \quad h \in H \text{ and } t \in T.$$

(B) A family of distributions is compact in $\mathfrak{M}$ if and only if for any $\varepsilon > 0$ there exists a compact set $Z_\varepsilon \subset H$ such that

$$\inf_{t\in T} \mu_t(Z_\varepsilon) \geqslant 1 - \varepsilon.$$

(C) If the sequence $\{\mu_n\}$ is compact in $\mathfrak{M}$ and if $\hat{\mu}_n(h) \to \varphi(h)$ for any $h \in H$, then $\mu_n \to \mu$ and $\varphi(h) = \hat{\mu}(h)$ (see [14]).

(³) see [9], [12], [16].

(⁴) see [14]; Tr B denotes the trace of the operator B.

(D) If for a sequence of distributions $\{\mu_n\}$ such that $\|\mu_n\|_2 < \infty$ $(n = 1, 2, \ldots)$ the sequence of corresponding dispersion operators is compact, then the sequence $\{\mu_n\}$ is compact [14].

(E) If a sequence of probability measures $\mu_n = \alpha_n * \beta_n$ is compact in $\mathfrak{M}$, then the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are shiftcompact ([13], p. 203).

§ 2

2.1. Let a one-parameter family of probability measures $\mu(t) \in \mathfrak{M}$ be defined in an interval $a \leqslant t \leqslant b$; let us define for $\Delta \subset [a, b]$ a function of interval $F(\Delta)$ whose values are bounded linear operators in H. If for every normal sequence of partitions $P_n = (\Delta_1^{(n)}, \ldots, \Delta_{k_n}^{(n)})$ of the segment $[a, b]$ the sequence of measures

$$\prod_{k=1}^{k_n}{}^{*} F(\Delta_k^{(n)})\mu(t_k^{(n)}), \qquad t_k^{(n)} \in \Delta_k^{(n)} \ (n = 1, 2, \ldots) \tag{11}$$

converges to the distribution ν (in the sense of weak convergence in $\mathfrak{M}$), then we shall call ν an *convolution integral* of the family $\mu(t)$ with respect to the family $F(\Delta)$ in the interval $[a, b]$ and write

$$\nu = {}^{*}\!\int_a^b \mu(t) F(dt). \tag{12}$$

(If $\mu(t) \equiv \mu$, then we shall write ${}^{*}\!\int_a^b \mu F(dt)$.)

If for any normal sequence of partitions of the segment $[a, b]$ the distributions

$$\mu_{n,k} = F(\Delta_k^{(n)})\mu(t_k^{(n)}), \qquad k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots, \tag{13}$$

are infinitely small [5], then integral (12) will be called a *regular convolution integral*. It is clear that a value of a regular integral can only be an infinitely divisible distribution.

2.2. A function of interval F is called *continuous* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|F(\Delta)\| < \varepsilon$ provided $|\Delta| < \delta$ ($|\Delta|$ denotes the length of the interval Δ).

If integral (12) exists, the family $\mu(t)$ is compact in $\mathfrak{M}$ and the family $F(\Delta)$ is continuous in $[a, b]$, then integral (12) is regular.

In fact, let $\{P_n\}$ be a normal sequence of partitions of the segment $[a, b]$. Then

$$\max_{1 \leqslant k \leqslant k_n} \|F(\Delta_k^{(n)})\| \to 0 \quad \text{for} \quad n \to \infty.$$

[5] i.e. for any $\varepsilon > 0$ $\sup\limits_{1 \leqslant k \leqslant k_n} \mu_{n,k}\{\|x\| \geqslant \varepsilon\} \to 0$ as $n \to \infty$.

Moreover, by the compactness of the set of measures $\mu(t)$ there exists in H a compact subset Z_ε such that

$$\inf_{a \leqslant t \leqslant b} \mu_t(Z_\varepsilon) \geqslant 1 - \varepsilon$$

(see theorem (B) of § 1). In particular, for any $\varepsilon > 0$ there exists an $r_\varepsilon > 0$ such that

$$\sup_{a \leqslant t \leqslant b} \mu_t\{\|x\| \geqslant r_\varepsilon\} \leqslant \varepsilon.$$

Let $\eta > 0$ be given. We have

$$\sup_{1 \leqslant k \leqslant k_n} \big(F(\Delta_k^{(n)})\mu(t_k^{(n)})\big)\{\|x\| \geqslant \eta\} \leqslant \sup_{1 \leqslant k \leqslant k_n} \mu_{t_k^{(n)}}\{\|F(\Delta_k^{(n)})\| \cdot \|x\| \geqslant \eta\}$$

$$\leqslant \sup_{1 \leqslant k \leqslant k_n} \mu_{t_k^{(n)}}\{\|x\| \sup_{1 \leqslant k \leqslant k_n} \|F(\Delta_k^{(n)})\| \geqslant \eta\} \leqslant \sup_{1 \leqslant k \leqslant k_n} \mu_{t_k^{(n)}}\left\{\|x\| \geqslant \frac{\eta}{\sup\limits_{1 \leqslant k \leqslant k_n} \|F(\Delta_k^{(n)})\|}\right\} \leqslant \varepsilon$$

for n sufficiently large.

2.3. We observe that in the definition of integral (12) the additivity of the set-function F is not required. Convolution integral (12) can be regarded as a generalization of the ordinary Riemann integral of a mapping x of a segment $[a, b]$ into the Hilbert space H. In fact, put $\mu(t) = \delta_{x(t)}$, $F(\Delta) = |\Delta| I$, where I is the identity operator in H. Then convolutions defining the convolution integral take the form

$$\prod_k{}^{*} |\Delta_k^{(n)}|\, \delta_{x(t_k^{(n)})} = \delta_{\sum_k x(t_k^{(n)})|\Delta_k^{(n)}|}$$

and the existence of convolution integral (12) is equivalent to the existence of the Riemann integral, the following equality being valid:

$${}^{*}\!\int_a^b \mu(t) F(dt) = \delta_{\int_a^b x(t)dt}.$$

In what follows we make use of the following lemma:

2.4. Lemma. *If $\{x_n\}$ is a sequence non-compact in H, then there exist subsequences of natural numbers $\{k_n\}$ and $\{l_n\}$ such that the sequence $\{z_n\} = \{x_{k_n} - x_{l_n}\}$ is non-compact.*

Proof. Considering, if necessary, a subsequence of the sequence $\{x_n\}$ we can assume that there exists an $\varepsilon > 0$ such that $\|x_n - x_m\| \geqslant \varepsilon$ for $n \neq m$. Put $k_1 = 1, l_1 = 2$ and consider the sequence $\{x_3 - x_n\}$, $n = 4, 5, \ldots$ Since

$$\|(x_3 - x_n) - (x_3 - x_m)\| \geqslant \varepsilon \quad \text{for} \quad n \neq m,$$

here exists a natural number s such that

$$\|(x_1 - x_2) - (x_3 - x_s)\| \geqslant \frac{\varepsilon}{2}.$$

Put $k_2 = 3, l_2 = s$ and consider the sequence $\{x_s - x_n\}, n > s$. As before, we can find a natural number r such that

$$\|(x_1 - x_2) - (x_s - x_r)\| \geqslant \varepsilon/2 \quad \text{and} \quad \|(x_3 - x_s) - (x_s - x_r)\| \geqslant \varepsilon/2.$$

Put $k_3 = s+1, l_3 = r$. Proceeding in the same way we obtain two sequences of natural numbers $\{k_n\}$ and $\{l_n\}$ such that

$$\|(x_{k_n} - x_{l_n}) - (x_{k_m} - x_{l_m})\| \geqslant \varepsilon/2 \quad \text{for} \quad n \neq m,$$

which completes the proof of the lemma.

2.5. In the sequel we denote by $\overline{\mu}$ the probability measure defined by the formula $\overline{\mu}(Z) = \mu(-Z)$ for $Z \in \mathfrak{B}$. Put also $|\mu|^2 = \overline{\mu} * \mu$. Clearly $\widehat{|\mu|^2}(h) = |\hat{\mu}(h)|^2$.

Integral (12) defined above possesses a number of regular properties which are put together in the following theorem.

2.6. THEOREM. (i) *If there exists a (regular) convolution integral*

$$(*) \qquad {}^*\!\!\int_a^b \mu(t) F(dt)$$

and $[c, d] \subset [a, b]$, then there exists also a (regular) convolution integral

$${}^*\!\!\int_a^d \mu(t) F(dt).$$

(ii) *If there exist integrals*

$$\nu_i = {}^*\!\!\int_a^b \mu_i(t) F(dt) \quad (i = 1, 2)$$

then there exists an integral

$$\nu = {}^*\!\!\int_a^b (\mu_1(t) * \mu_2(t)) F(dt)$$

and the equality $\nu = \nu_1 * \nu_2$ holds.

(iii) *If there exists an integral $(*)$, $a < c < b$, then there exist integrals*

$${}^*\!\!\int_a^c \mu(t) F(dt), \qquad {}^*\!\!\int_c^b \mu(t) F(dt)$$

and the equality

$${}^*\!\!\int_a^b = {}^*\!\!\int_a^c * {}^*\!\!\int_c^b$$

holds.

(iv) *If A is an operator in H,*

$$\nu = {}^*\!\!\int_a^b \mu(t) F(dt),$$

then

$$A\nu = {}^*\!\!\int_a^b \mu(t) (AF)(dt)$$

and if A is commutative with the operators $F(\Delta)$, $\Delta \subset [a, b]$, then also

$$A\nu = {}^*\!\!\int_a^b (A\mu(t)) F(dt).$$

(v) *If the integral $(*)$ exists, then there also exist integrals*

$${}^*\!\!\int_a^b \overline{\mu}(t) F(dt), \qquad {}^*\!\!\int_a^b |\mu|^2(t) F(dt)$$

and the relations

$${}^*\!\!\int_a^b \overline{\mu}(t) F(dt) = \overline{{}^*\!\!\int_a^b \mu(t) F(dt)}, \qquad {}^*\!\!\int_a^b |\mu|^2 F(dt) = \Big|{}^*\!\!\int_a^b \mu F(dt)\Big|^2$$

hold.

(vi) *If*

$$\Big\|{}^*\!\!\int_a^b \mu(t) F(dt)\Big\|_2 < \infty, \qquad a < c < b,$$

then

$$\Big\|{}^*\!\!\int_a^c \mu(t) F(dt)\Big\|_2 < \infty.$$

Proof. Ad (i). Clearly it suffices to prove that if there exists an integral $(*)$, then there exist integrals

$${}^*\!\!\int_a^c \mu(t) F(dt) \quad \text{and} \quad {}^*\!\!\int_c^b \mu(t) F(dt)$$

for any c from the interval $[a, b]$. Let the integral $(*)$ exist. We state that for any normal sequence of partitions $P_n = (\Delta_1^{(n)}, \ldots, \Delta_{k_n}^{(n)})$ of the interval $[a, c]$, the sequence of measures

$$(14) \qquad \nu_n = \prod_{k=1}^{k_n}{}^* F(\Delta_k^{(n)}) \mu(t_k^{(n)}), \qquad t_k^{(n)} \in \Delta_k^{(n)}, \; n = 1, 2, \ldots,$$

is compact in $\mathfrak{M}$. In fact, suppose that (14) is not compact in $\mathfrak{M}$. Put

$$(15) \qquad \mu_n = \prod_{l=1}^{l_n} F(\overline{\Delta}_l^{(n)}) \mu(s_l^{(n)}), \qquad s_l^{(n)} \in \overline{\Delta}_l^{(n)},$$

where $\overline{P}_n = (\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots, \Delta_{l_n}^{(n)})$ is a normal sequence of partitions of the interval $[c, b]$. By the existence of the integral (*) we have

$$\mu_n * \nu_n \to {}^*\!\!\int_a^b \mu(t)F(dt) \quad \text{as} \quad n \to \infty. \tag{16}$$

In particular, the sequence $\{\nu_n * \mu_n\}$ is compact in $\mathfrak{M}$. Hence it follows that the sequences $\{\nu_n\}$ and $\{\mu_n\}$ are shiftcompact (by theorem (E) of § 1). Thus there exist sequences of elements of the space H, $\{x_n\}$ and $\{y_n\}$, such that the sequences of measures $\{\nu_n * \delta_{x_n}\}$ and $\{\mu_n * \delta_{y_n}\}$ are compact. The sequence $\{x_n\}$ is non-compact (for if it were compact, then the sequence of measures $\{\nu_n\} = \{\nu_n * \delta_{x_n} * \delta_{(-x_n)}\}$ would be compact, contrary to our supposition). Put

$$y_n = (-x_n)+z_n \quad (n = 1, 2, \ldots). \tag{17}$$

Since the sequence $\nu_n * \delta_{x_n} * \mu_n * \delta_{(-x_n)} * \delta_{z_n} = \nu_n * \mu_n * \delta_{z_n}$ is compact in $\mathfrak{M}$ and the sequence $\nu_n * \mu_n$ is convergent, then the sequence $\{z_n\}$ is compact in H.

Now choose for the sequence $\{x_n\}$ sequences of natural numbers $\{k_n\}$ and $\{l_n\}$ such as in Lemma 2.4. Thus the sequence $\{x_{k_n} - x_{l_n}\}$ is non-compact. Since the sequences of distributions $\{\nu_n * \delta_{x_n}\}$ and $\{\mu_n * \mu_n * \delta_{z_n - x_n}\}$ are compact, the sequence of probability measures

$$\{\nu_{k_n} * \delta_{x_{k_n}} * \mu_{l_n} * \delta_{z_{l_n}} * \delta_{(-x_{l_n})}\} \tag{18}$$

is also compact in $\mathfrak{M}$. Moreover, the sequence $\{\nu_{k_n} * \mu_{l_n} * \delta_{z_{l_n}}\}$ is compact and the sequence $\{x_{k_n} - x_{l_n}\}$ is not compact. Thus sequence (18) cannot be compact in $\mathfrak{M}$. Thus the supposition that the sequence of probability measures (14) is not compact in $\mathfrak{M}$ has led us to a contradiction. In view of the symmetry of our considerations we have also proved the compactness of the sequence of distributions of form (15). Now suppose that the integral

$$ {}^*\!\!\int_a^c \mu(t)F(dt) \tag{19}$$

does not exist. Then there exist two normal sequences of partitions of the interval $[a, c]$ such that the corresponding sequences of measures of form (14), ν_n' and ν_n'', converge to different limits: $\nu_n' \to \nu'$, $\nu_n'' \to \nu''$, $\nu' \neq \nu''$.

Let $\{\mu_n\}$ be a convergent sequence of distributions of form (15) corresponding to a normal sequence of partitions of the interval $[c, b]$. Let $\mu_n \to \mu$. Thus we have

$$\nu_1 * \mu = \nu_2 * \mu = {}^*\!\!\int_a^b \mu(t)F(dt),$$

which is imposible for $\nu_1 \neq \nu_2$.

Thus we have completed the proof of (i).

Ad (ii). The proof easily follows from the definition of the convolution integral and from the continuity of the convolution.

Ad (iii). The existence of the integrals follows from (i), the other properties are an immediate corollary to the definition of convolution integral.

Ad (iv). This property follows from the formula

$$A(\mu * \nu) = (A\mu) * (A\nu).$$

Ad (v). This property follows from (ii) and the following simple relationships: $\overline{\mu * \nu} = \overline{\mu} * \overline{\nu}$; $\overline{A\mu} = A\overline{\mu} = (-A)\mu$.

Ad (vi) This results from the following lemma (comp. Cramer [1]): *If* $\|\mu\|_2 < \infty$, $\mu = \nu_1 * \nu_2$, *then* $\|\nu_i\|_2 < \infty$ $(i = 1, 2)$.

In fact, the inequality

$$2(\|g+h\|^2 + \|h\|^2) = \|g\|^2 + \|g+2h\|^2 \geqslant \|g\|^2$$

implies the estimation

$$\begin{aligned}\nu_1(K)\int \|g\|^2 \nu_2(dg) &= \int_K \int_H \|g\|^2 \nu_1(dh)\nu_2(dg) \\ &\leqslant 2\int_H \int_H \|g+h\|^2 \nu_1(dh)\nu_2(dg) + 2\int_K \|h\|^2 \nu_1(dh) \\ &= 2\int \|h\|^2 \mu(dh) + 2\int_K \|h\|^2 \nu_1(dh) < \infty,\end{aligned}$$

where K is a sphere in H such that $\nu_1(K) > 0$, whence $\|\nu_2\|_2 < \infty$.

§ 3

3.1. A probability distribution $\mu \in \mathfrak{M}$ is called *normal* if

$$\hat{\mu}(h) = \exp[i(h_0, h) - \tfrac{1}{2}(Dh, h)],$$

where $h_0 \in H$ and D is an S-operator. Then h_0 is the mathematical expectation of the probability measure μ and D its dispersion operator (hence, in particular, it follows that every S-operator is the dispersion operator of a certain probability measure $\mu \in \mathfrak{M}$).

3.2. THEOREM. *Let* $\|\mu\|_2 < \infty$, $M_\mu = \Theta$, A_t, $a \leqslant t \leqslant b$ *be a continuous family of bounded linear operators in* H *such that*

$$\sup_{a \leqslant t \leqslant b} \|A_t\| = k < \infty.$$

Put $\mu(t) = A_t\mu$, $F(\Delta) = \sqrt{|\Delta|}\cdot I$, *where* I *is the unity operator in* H, Δ — *an interval included in* $[a, b]$, $|\Delta|$ — *the length of the interval* Δ. *Then there exists a regular convolution integral*

$$\nu = {}^{*}\!\int_a^b \mu(t)F(dt), \tag{20}$$

where ν *is the normal probability measure given by the characteristic functional*

$$\hat{\nu}(h) = \exp\Big[-\tfrac{1}{2}\int_a^b (A_t D A_t^* h, h)\,dt\Big], \tag{21}$$

where D *is the dispersion operator of the probability measure* μ *and* A^* *denotes the operator adjoint to* A.

Proof. Let $P = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ be a partition of the interval $[a, b]$, $t_k \in \Delta_k$. First of all we will show that the right-hand side of formula (21) is indeed the characteristic functional of a normal distribution, i.e. that the operator B defined by the integral

$$Bh = \int_a^b A_\tau D A_\tau^* h\,d\tau, \qquad h \in H,$$

is an S-operator (the integral exists for every $h \in H$ since the operator family is by assumption continuous). The operator B, as the limit of linear forms of self-adjoint non-negative operators with non-negative coefficients, is self-adjoint and non-negative. We also have $\operatorname{Tr} B < \infty$, which follows from the estimation

$$\operatorname{Tr}\Big(\sum_k |\Delta_k| A_{t_k} D A_{t_k}^*\Big) = \sum_k |\Delta_k| \operatorname{Tr}(A_{t_k} D A_{t_k}^*) \leqslant k^2(b-a)\operatorname{Tr} D.$$

From the same estimation it follows that all sequences of distributions defining the integral given by formula (20) corresponding to normal sequences of partitions of the interval $[a, b]$ are compact. Thus, according to the theorem of Prochorov, we have to prove that the sequence of corresponding characteristic functionals converge to $\hat{\nu}(h)$. We have

$$\hat{\mu}(h) = 1 - \tfrac{1}{2}(Dh, h) + o(\|h\|^2).$$

Thus

$$\log \widehat{\prod_{k=1}^{n} (\sqrt{|\Delta_k|}I)\mu(t_k)}(h) = \sum_{k=1}^{n} \log\big(1 - \tfrac{1}{2}|\Delta_k|(D_{t_k}h, h) + |\Delta_k|\,\|h\|^2 w(\sqrt{|\Delta_k|}h)\big),$$

where $D_t = A_t D A_t^*$ and $w(h) \to 0$ as $h \to 0$. Hence it easily follows that for $\max\limits_{1 \leqslant k \leqslant k_n} |\Delta_k^{(n)}| \to 0$ $(n \to \infty)$

$$\lim_{n\to\infty} \widehat{\prod_{k=1}^{k_n} (\sqrt{|\Delta_k^{(n)}|}I)\mu(t_k^{(n)})}(h) = \exp\Big[-\tfrac{1}{2}\int_a^b (D_\tau h, h)\,d\tau\Big].$$

The regularity of integral (20) follows from criterion 2.2. Indeed, the compactness of the family of measures $\mu(t) = A_t\mu$, $a \leqslant t \leqslant b$, follows from the compactness of the family of operators D_t and of theorem (D) (§ 1).

The theorem just proved has of course its prototype in the classical Lindenberg-Levy theorem.

3.3. A family D_t, $a \leqslant t \leqslant b$, of S-operators is said to be *summable* with respect to a family of operators $F(\Delta)$, $\Delta \subset [a, b]$ to an S-operator D if for every normal sequence of partitions of the interval $[a, b]$: $P_n = (\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots, \Delta_{k_n}^{(n)})$

1° the sequence of operators

$$\sum_{k=1}^{k_n} F(\Delta_k^{(n)}) D_{t_k^{(n)}} F(\Delta_k^{(n)})^*$$

is compact,

2° the limit

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \big(F(\Delta_k^{(n)}) D_{t_k^{(n)}} F(\Delta_k^{(n)})^* h, h\big) = (Dh, h)$$

exists for every $h \in H$ (obviously $t_k^{(n)} \in \Delta_k^{(n)}$)

3.4. Theorem. *Let a family of probability measures* $\mu(t)$ *such that* $\|\mu(t)\|_2 < \infty$, $M\mu(t) = \theta$ *for* $a \leqslant t \leqslant b$ *be given. Let* D_t *denote the dispersion operator of the distribution* $\mu(t)$. *Let the family* D_t, $a \leqslant t \leqslant b$, *be summable with respect to* $F(\Delta)$ *to an* S*-operator* D. *Then the regular integral*

$${}^{*}\!\int_a^b \mu(t)F(dt) = \nu, \tag{22}$$

where

$$\hat{\nu}(h) = \exp[-\tfrac{1}{2}(Dh, h)] \tag{23}$$

exists if and only if for any normal sequence of partitions $\{P_n\}$ *of the interval* $[a, b]$ *and for any* $\varepsilon > 0$ *the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int\limits_{\|F(\Delta_k^{(n)})h\| \geqslant \varepsilon} \|F(\Delta_k^{(n)})h\|^2 \mu_{t_k^{(n)}}(dh) = 0 \tag{24}$$

exists. (Clearly, in formula (24), $t_k^n \in \Delta_k^{(n)}$.)

We omit the proof of this theorem, because it can be obtained by an easy modification of the argument employed by Kandelaky and Sazonov in [8], § 4.

Condition (24) is to be treated as a generalization of the wellknown Lindeberg condition for one-dimensional distributions. Now we shall formulate a theorem which has no analogon in the theory of one-dimensional distributions.

3.5. THEOREM. *Let $\mu(t)$, $a \leqslant t \leqslant b$, be a family of probability distributions in H such that $M\mu(t) = \theta$, $\|\mu(t)\|_2 < \infty$, $D_t \leqslant D$, where D_t is the dispersion operator of $\mu(t)$ and D is a certain S-operator. Let E_t, $a \leqslant t \leqslant b$, be a continuous resolution of the identity in H. Thus there exists a convolution integral*

$$(25)\qquad {}^{*}\!\int_a^b \mu(t)E(dt) = \delta_\theta,$$

where $E(\Delta) = E_t - E_s$ if $\Delta = [s, t]$.

Proof. Let P_n be a normal sequence of partitions of the interval $[a, b]$, $t_k^{(n)} \in \Delta_k^{(n)}$. We will show that

$$(26)\qquad \lim_{n\to\infty} \sum_{k=1}^{k_n} \big(E(\Delta_k^{(n)}) D_{t_k^{(n)}} E(\Delta_k^{(n)})h, h\big) = 0 \quad \text{for any } h \in H.$$

Since $D_t \leqslant D$ for $a \leqslant t \leqslant b$, also $E(\Delta)D_tE(\Delta) \leqslant E(\Delta)DE(\Delta)$. Thus it suffices to show (26) for $D_t \equiv D$. Let $\{e_v\}$ be a basis in H, diagonal with respect to the operator D, i.e. $De_v = \lambda_v e_v$ $(v = 1, 2, \ldots)$. Obviously $\lambda_v \geqslant 0$. Moreover,

$$\sum_{v=1}^{\infty} \lambda_v = \operatorname{Tr} D < \infty.$$

Then we have

$$(27)\qquad \sum_{k=1}^{k_n} \big(E(\Delta_k^{(n)})DE(\Delta_k^{(n)})h, h\big) = \sum_{v=1}^{\infty} \lambda_v \sum_{k=1}^{k_n} \big(E(\Delta_k^{(n)})h, e_v\big)^2.$$

But

$$\sum_{k=1}^{k_n} \big(E(\Delta_k^{(n)})h, e_v\big)^2 \leqslant \sum_{k=1}^{k_n} \big(E(\Delta_k^{(n)})h, h\big)\big(E(\Delta_k^{(n)})e_v, e_v\big) \leqslant \max_{1\leqslant k\leqslant k_n} \big(E(\Delta_k^{(n)})h, h\big).$$

In view of the uniform continuity of the function (E_t, h, h) in the interval $[a, b]$ the last expression converges to zero. Passing in (27) to the limit as $n \to \infty$ we obtain (26). It may easily be verified that the sequence

$$(28)\qquad S_n = \sum_{k=1}^{k_n} E(\Delta_k^{(n)}) D_{t_k^{(n)}} E(\Delta_k^{(n)})$$

of dispersion operators of the distributions

$$(29)\qquad \mu_n = \prod_{k=1}^{k_n}{}^{*} E(\Delta_k^{(n)})\mu(t_k^{(n)})$$

is compact and thus the sequence of measures is compact. So it suffices to show that, for any $h \in H$, $\hat{\mu}_n(h) \to 1$ as $n \to \infty$. This, however, follows from (26) and from the inequality

$$1 - \operatorname{Re}\hat{\mu}_n(h) = \int \big(1 - \cos(g_0 h)\big)\mu_n(dg) \leqslant \tfrac{1}{2}\int (g, h)^2 \mu_n(dg).$$

By an easy modification of the proof of Theorem 3.5 we obtain the following

3.6. THEOREM. *If $\|\mu\|_2 < \infty$, the resolution $\{E_t\}$ of identity in H has a finite number of points of discontinuity (on the left) $t_1, t_2, \ldots, t_n$, then there exists a convolution integral*

$${}^{*}\!\int_a^b \mu E(dt) = \delta_{x_0} * \prod_{k=1}^{n}{}^{*} (E_{t_k} - E_{t_k - 0})\mu,$$

where

$$x_0 = \Big\{I - \sum_{k=1}^{n} (E_{t_k} - E_{t_k-0})\Big\} M_\mu.$$

§ 4

4.1. The notion of convolution integral has a natural interpretation in the theory of stochastic processes with independent increments. Let ξ_t, $a \leqslant t \leqslant b$, be a stochastic process in the Hilbert space (i.e. let the random variables ξ_t take values from the Hilbert space H).

A process ξ_t, $a \leqslant t \leqslant b$, is called a *process with independent increments* if

1° $\xi_a \equiv \theta$,

2° for every system of numbers $t_1, \ldots, t_n$ (increasing successively) from the interval $[a, b]$ the random variables $\xi_{t_2} - \xi_{t_1}$, $\xi_{t_3} - \xi_{t_2}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent.

A process ξ_t is said to be *gaussian* if all variables $\xi_t - \xi_s$ $(s < t)$ have normal distributions. As we know, gaussian processes play a fundamental part in the theory of processes in R_n with independent increments, also because of the fact that if almost all trajectories of the process are continuous then the process is a gaussian process. This theorem holds also in the case of processes with values from the Hilbert space H. Indeed, let ξ_t, $a \leqslant t \leqslant b$, be a process in H with independent increments and with continuous trajectories. Let $e_1, e_2, \ldots$ be a basis in H and let E_j for $j = 1, 2, \ldots$ denote the projection operator on the subspace of H spanned over $e_1, \ldots, e_j$. Then the process $E_j\xi_t$ for a fixed j is a process in R_j with independent increments and with continuous trajectories, and thus a gaussian process. Consider a random variable $\xi_t - \xi_s$ for $s < t$. Let μ be its distribution. Then the variable $E_j(\xi_t - \xi_s)$ have a normal distribution $E_j\mu$. It is easy to prove that the limit $\lim E_j\mu$ exists and is a normal distribution equal to μ, which completes the proof.

4.2. Suppose that in a gaussian process the random variables ξ_t have normal distributions $N(m_t, D_t)$. If the process has continuous trajectories, then for $t \to s$ we have $N(m_t, D_t) \to N(m_s, D_s)$. Hence it follows that for any h, for $t \to s$

(a) $$(D_t h, h) \to (D_s h, h),$$

(b) $$\|m_t - m_s\| \to 0.$$

We shall prove

4.3. Theorem. *If a process ξ_t, $a \leqslant t \leqslant b$, is a gaussian process, the random variables ξ_t have distributions $N(m_t, D_t)$, the mathematical expectations and dispersion operators satisfy conditions* (a) *and* (b), *then the following continuity condition is satisfied*:

(30) $$\lim_{n\to\infty} \sum_{k=1}^{k_n} \operatorname{Prob}(\|\xi_{t_k^{(n)}} - \xi_{t_{k-1}^{(n)}}\| \geqslant \varepsilon) = 0$$

as $\max_{1\leqslant k\leqslant k_n} |t_k^{(n)} - t_{k-1}^{(n)}| \to 0$ $(n \to \infty)$.

Proof. For the simplicity of writing we assume $m_t \equiv 0$, $a \leqslant t \leqslant b$. First we observe that if ξ is a gaussian random variable whose mathematical expectation equals zero and with the dispersion operator D, then

(31) $$M\|\xi\|^4 \leqslant 3(\operatorname{Tr} D)^2.$$

In fact, let $\{e_i\}$ be the basis in H, diagonal with respect to the operator D, $De_i = \lambda_i e_i$. Then the real random variables (ξ, e_i) $(i = 1, 2, \ldots)$ are gaussian random variables and $M(\xi, e_i)^2 = (De_i, e_i)$ and, moreover, $M(\xi, e_i)^4 = 3\lambda_i$ $(i = 1, 2, \ldots)$. Thus we have

$$M\|\xi\|^4 = M\Big(\sum_{i=1}^{\infty} (\xi, e_i)^2\Big)^2 = M\Big[\sum_{i=1}^{\infty} (\xi, e_i)^4 + \sum_{i\neq j} (\xi, e_i)^2(\xi, e_j)^2\Big]$$
$$= 3\sum_{i=1}^{\infty} \lambda_i + \sum_{i\neq j} \lambda_i\lambda_j \leqslant 3(\operatorname{Tr} D)^2.$$

Making use of (31) we obtain the following estimation:

(32) $$\sum_{k=1}^{k_n} P(\|\xi_{t_k^{(n)}} - \xi_{t_{k-1}^{(n)}}\| \geqslant \varepsilon) \leqslant \frac{3}{\varepsilon^4} \sum_{k=1}^{k_n} \operatorname{Tr}(D_{t_k^{(n)}} - D_{t_{k-1}^{(n)}})^2.$$

If $s \leqslant t$, then $D_s \leqslant D_t$. Hence $\operatorname{Tr} D_s \leqslant \operatorname{Tr} D_b$. Thus if $\{e_i\}$ is the basis in H, then the series

$$\operatorname{Tr} D_s = \sum_{i=1}^{\infty} (D_s e_i, e_i)$$

converges uniformly in the whole interval $a \leqslant s \leqslant b$ and it represents a continuous function with respect to condition (a). Thus the last member in equality (32) tends to zero, which completes the proof.

4.4. If the convolution integral

$$\overset{*}{\int_a^b} \mu(t) F(dt)$$

is given, then there exists in H a stochastic process with independent increments ξ_t, $a \leqslant t \leqslant b$, such that for $s < t$ the distribution of the random variable $\xi_t - \xi_s$ is the integral

$$\overset{*}{\int_s^t} \mu(t) F(dt).$$

This follows from a well-known Theorem of Kolmogoroff ([10], III, § 4). Although Kolmogoroff deals in his argument with real random variables, it can be extended, as Spacek [17] has noticed, to random variables with values in metric complete spaces.

4.5. A process ξ_t, $a \leqslant t \leqslant b$, is called a *process with orthogonal increments with respect to the resolution of identity in* H $(E_t, a \leqslant t \leqslant b)$ if for any two numbers $s < t$ from the interval $[a, b]$ we have

$$\operatorname{Prob}\{\xi_t - \xi_s \in (E_t - E_s)H\} = 1.$$

A process ξ_t, $a \leqslant t \leqslant b$, is called an *$A$-orthogonal process with respect to the resolution of the identity in* H if for any two numbers $s < t$ from the interval $[a, b]$

$$\operatorname{Prob}\{\xi_t - \xi_s \in A(E_t - E_s)H\} = 1.$$

4.6. Now we construct in the space $H = L_2(0, 1)$ a gaussian process ξ_t, $0 \leqslant t \leqslant 1$,

(i) with independent increments;

(ii) with A-orthogonal increments with respect to a continuous resolution of identity in H, where A is the selfadjoint and compact operator, $(Ah, h) > 0$ provided $h \neq \theta$ and the sum of squares of the eigenvalues of the operator A is finite;

(iii) satisfying continuity condition (30);

(iv) such that the dispersion operator $D_{s,t}$ of the random variable $\xi_t - \xi_s$ satisfies the condition $(D_{s,t}h, h) > 0$ provided the vector Ah is not orthogonal to the subspace $(E_t - E_s)H$, i.e. if $(E_t - E_s)Ah \neq \theta$.

We observe that there exists no analogous gaussian process with orthogonal increments. More exactly, for the operator A equal to the identity operator there exists no gaussian process satisfying conditions (i), (ii) and (iv). In fact, if the mentioned process existed, then the random variable $\xi_1 = \xi_1 - \xi_0$ would have a normal regular distribution μ (i.e. its dispersion operator D would satisfy the condition $(Dh, h) > 0$ for $h \neq 0$). On the other hand, for an arbitrary partition $P = (\Delta_1, \ldots, \Delta_n)$ of the interval $[0, 1]$ we would have

$$\mu = \prod_{i=1}^{n}{}^{*} \mu_{\Delta_i} = \prod_{i=1}^{n}{}^{*} E(\Delta_i)\mu,$$

where μ_{Δ_i} is the distribution of a random variable $\xi_{\Delta_i} = \xi_{t_i} - \xi_{t_{i-1}}$. Hence it follows that the distribution μ is equal to the convolution integral

$$*\int_0^1 \mu E(dt),$$

which in view of Theorem 3.6 equals δ_{M_μ}. Thus $\mu = \delta_{M_\mu}$, which contradicts the regularity in H of the measure μ.

We proceed to the construction of our process. Let $H = L_2(0, 1)$. By e_v $(v = 1, 2, \ldots)$ we denote the functions of the Haar system in $L_2(0, 1)$ (see e.g. [7]). We introduce the following notation:

$$e_v(x) = \begin{cases} c_v > 0 & \text{in the interval } \Delta_v^+, \\ -c_v & \text{in the interval } \Delta_v^-, \\ 0 & \text{at the remaining points of the interval } (0, 1). \end{cases} \tag{33}$$

The support of the function e_v (i.e. the union of the intervals Δ_v^+ and Δ_v^-) we denote by Δ_v. Let $\{\lambda_v\}$ be a sequence of positive numbers such that

$$\sum_{v=1}^{\infty} \lambda_v c_v^2 < \infty. \tag{34}$$

Define the S-operator D by the formula

$$De_\nu = \lambda_\nu e_\nu \quad (\nu = 1, 2, \ldots). \tag{35}$$

We define the gaussian distribution in H putting

$$\hat{\mu}(h) = \exp[-\tfrac{1}{2}(Dh, h)], \quad h \in H. \tag{36}$$

Let $E(\Delta)$, $\Delta \subset [0, 1]$ be the resolution of the identity in H defined by the formula

$$\big(E(\Delta)f\big)(x) = \chi_\Delta(x)f(x), \tag{37}$$

where χ_Δ is the characteristic function of the interval Δ. Put

$$F(\Delta) = \frac{1}{\sqrt{|\Delta|}} E(\Delta), \tag{38}$$

where $|\Delta|$ denotes the length of the interval Δ. Let $G(\Delta)$ be defined by the formula

$$G(\Delta) = AF(\Delta), \tag{39}$$

where the operator A is given by the formula

$$A\varphi_v = \mu_v \varphi_v, \tag{40}$$

where $\mu_v > 0$, $\sum_{v=1}^{\infty} \mu_v < \infty$ and $\{\varphi_v\}$ is the basis in H. We will prove that the convolution integral

$$*\int_0^1 \mu G(dt) \tag{41}$$

exists. To this end denote by $P_n = (\Delta_1^{(n)}, \ldots, \Delta_{k_n}^{(n)})$ a normal sequence of partitions of the interval $[0, 1]$. To the sequence $\{P_n\}$ corresponds the sequence of gaussian distributions in H

$$\prod_{k=1}^{k_n} G(\Delta_k^{(n)})\mu \tag{42}$$

with the dispersion operators

$$B_n = \sum_{k=1}^{k_n} G(\Delta_k^{(n)}) D G(\Delta_k^{(n)})^*. \tag{43}$$

Put

$$D_n = \sum_{k=1}^{k_n} F(\Delta_k^{(n)}) D F(\Delta_k^{(n)}). \tag{44}$$

Observe that

$$(D_n h, h) = \sum_{v=1}^{\infty} \lambda_v \sum_{k=1}^{k_n} (F(\Delta_k^{(n)})h, e_v)^2. \tag{45}$$

Now we will show that for any continuous function h in the interval $[0, 1]$ and any $v = 1, 2, \ldots$ the limit

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F(\Delta_k^{(n)})h, e_v)^2 = e_v^2 \int_{\Delta_v} (h(x))^2\,dx \tag{46}$$

exists.

In fact, we have

$$\sum_{k=1}^{k_n} (F(\Delta_k^{(n)})h, e_v)^2 = \sum_{k=1}^{k_n} \frac{1}{|\Delta_k^{(n)}|}\Big(\int_{\Delta_k^{(n)}} h(x)\,e_v(x)\,dx\Big)^2 \tag{47}$$

$$= c_v^2 \sum_{k=1}^{k_n} \frac{1}{|\Delta_k^{(n)}|}\Big(\int_{\Delta_k^{(n)}\cap\Delta_v^+} h(x)\,dx - \int_{\Delta_k^{(n)}\cap\Delta_v^-} h(x)\,dx\Big)^2.$$

Excepting at most four intervals all the other intervals of the partition P_n lie either in Δ_v^+ or in Δ_v^- and for such interval we have the equality

$$\Big(\int_{\Delta_k^{(n)}\cap\Delta_v^+} - \int_{\Delta_k^{(n)}\cap\Delta_v^-}\Big)^2 = \Big(\int_{\Delta_k^{(n)}} h(x)\,dx\Big)^2.$$

Since

$$\max_{1\leqslant k\leqslant k_n} |\Delta_k^{(n)}| \to 0$$

and the function h is continuous in $[0, 1]$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F(\Delta_k^{(n)})h, e_v)^2 = c_v^2 \lim \sum_{\Delta_k^{(n)}\subset\Delta_v} \frac{1}{|\Delta_k^{(n)}|}\Big(\int_{\Delta_k^{(n)}} h(x)\,dx\Big)^2$$

$$= c_v^2 \lim_{n\to\infty} \sum_{\Delta_k^{(n)}\subset\Delta_v} \frac{1}{|\Delta_k^{(n)}|}\cdot\big(|\Delta_k^{(n)}|\,h(x_k^{(n)})\big)^2 = c_v^2 \int_{\Delta_v} (h(x))^2\,dx$$

(obviously $x_k^n \in \Delta_k^{(n)}$).

Now we show that formula (46) holds also for an arbitrary function $h \in L_2(0, 1)$. In fact, let $\varepsilon > 0$ be given. Establish a continuous function f

such that $\|f-h\| < \varepsilon$ ($\|\cdot\|$ denotes the norm in $L_2(0, 1)$). Estimate the difference

$$\Big|\sum_{k=1}^{k_n} (F(\Delta_k^{(n)})h, e_v)^2 - c_v^2 \int_{\Delta} (h(x))^2\,dx\Big|$$

$$= \Big|\sum_{k=1}^{k_n} (F(\Delta_k^{(n)})(h-f+f), e_v)^2 - c_v^2 \int_{\Delta} (h(x))^2\,dx\Big|$$

$$= \Big|\sum_{k=1}^{k_n} (F(\Delta_k^{(n)})(h-f), e_v)^2 + \sum_{k=1}^{k_n} (F(\Delta_k^{(n)})f, e_v)^2 +$$

$$+ 2\sum_{k=1}^{k_n} (F(\Delta_k^{(n)})(h-f), e_v)\cdot(F(\Delta_k^{(n)})f, e_v) - c_v^2 \int h^2\,dx\Big|$$

$$= \Big|\sum\nolimits_1 + \sum\nolimits_2 + 2\sum\nolimits_3 - c_v^2 \int_{\Delta_v} h^2(x)\,dx\Big|,$$

$$\Big|\sum\nolimits_1\Big| \leqslant c_v^2 \sum_k \frac{1}{|\Delta_k^{(n)}|}\Big(\int_{\Delta_k^{(n)}} |h-f|\,dx\Big)^2 \leqslant c_v^2 \sum_k \frac{1}{|\Delta_k^{(n)}|}|\Delta_k^{(n)}|^2\,\|h-f\|^2 < c_v^2\varepsilon^2,$$

$$\Big|\sum\nolimits_2 - c_v^2 \int_{\Delta_v} h^2\,dx\Big| \leqslant \varepsilon + c_v^2\Big[\int_{\Delta_v} h^2\,dx - \int_{\Delta_v} f^2\,dx\Big]$$

for n sufficiently large (since $\sum_2 \to c_v^2 \int_{\Delta_v} f^2\,dx$). The expression in square brackets on the right-hand side of the inequality does not exceed

$$\|h-f\|(\varepsilon + 2\|h\|) < \varepsilon(\varepsilon + 2\|h\|);$$

finally

$$\Big|\sum\nolimits_3\Big| \leqslant \sum_k \frac{1}{\sqrt{|\Delta_k^{(n)}|}}\int_{\Delta_k^{(n)}} |h-f|\,dx\cdot\frac{1}{\sqrt{|\Delta_k^{(n)}|}}\int_{\Delta_k^{(n)}} |f|\,dx \leqslant \|h-f\| : \|f\| < \varepsilon(\varepsilon + \|h\|).$$

These estimations easily imply formula (46) for an arbitrary function $h \in L_2(0, 1)$. From (34), (45) and (46) we have

$$\lim (D_n h, h) = \sum_{v=1}^{\infty} \lambda_v c_v^2 \int_{\Delta_v} (h(x))^2\,dx \quad \text{for any } h \in H. \tag{48}$$

Since

$$(B_n h, h) = (D_n A h, A h), \tag{49}$$

we have

$$\lim_{n\to\infty} (B_n h, h) = \sum_{v=1}^{\infty} \lambda_v c_v^2 \int_{\Delta_v} (A h(x))^2\,dx. \tag{50}$$

To end the proof of the existence of convolution integral (41) it suffices to show the compactness of the sequence of operators $\{B_n\}$ and make use of theorems 1.4, (D), (C). The compactness of the sequence of operators B_n follows from the formula

$$\mathrm{Tr}\,B_n \leqslant \sum_{v=1}^{\infty}\lambda_v c_v^2 \sum_{i=1}^{\infty} \big(\Delta\varphi_i(x)\big)^2 dx \leqslant \sum_{v=1}^{\infty}\lambda_v c_v^2 \sum_{i=1}^{\infty}\mu_i^2, \tag{51}$$

where $\{\varphi_i\}$ is the basis in H from formula (40). Thus integral (41) exists. Let ξ_t, $0 \leqslant t \leqslant 1$, be a stochastic process in H with independent increments such that for $0 \leqslant s \leqslant t < 1$ the distribution of the random variables $\xi_t - \xi_s$ is the integral (see 4.4)

$$\overset{*}{\int_{\Delta}} = \overset{*}{\int_{s}^{t}} \mu(t)\,G(dt). \tag{52}$$

From the generalized theorem of Cramer ([1], 4, § 6.3) it follows that integral (52) is a normal distribution and thus the process ξ is a gaussian process. Now we show that our process is a process with Δ-orthogonal increments with respect to the resolution of identity (37), the operator Δ being defined by formula (40). From the construction of integral (40), in particular from formulas (43)-(45), (48)-(50), it follows that the integral is the limit of distributions μ_n given by the formula

$$\hat{\mu}_n(h) = \exp\Big[-\tfrac{1}{2}\sum_{v=1}^{\infty}\lambda_v \sum_{k=1}^{k_n}\big(F(\Delta_k^{(n)}\cap \Delta)\Delta h, e_v\big)^2\Big], \tag{53}$$

where $\Delta = [s, t]$. On the other hand,

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\big(F(\Delta_k^{(n)}\cap\Delta)\Delta h, e_v\big)^2 = c_v^2 \int_{\Delta\cap\Delta_v} \big(\Delta h(x)\big)^2 dx. \tag{54}$$

Thus, by (48)-(50), integral equals the limit

$$\lim_{n\to\infty} \Delta E(\Delta)\,\nu_n, \tag{55}$$

where the distributions ν_n are given by the characteristic functionals

$$\hat{\nu}_n(h) = \exp\big[-\tfrac{1}{2}(D_n h, h)\big] \tag{56}$$

(the operators D_n are defined by formula (44)). From (55) it follows that the distribution of the random variable $\xi_t - \xi_s$ is condensed on the subspace $\Delta E(\Delta)H$, i.e. condition (ii) is satisfied. To prove condition (iii) it suffices to verify condition 4.2 (a). As can easily be seen from the construction of integral (41),

$$\mathrm{Tr}\,D\Big(\overset{*}{\int_0^t}\mu G(dt)\Big) = \sum_{v=1}^{\infty}\lambda_v c_v^2 \sum_{i=1}^{\infty}\int_{\Delta_v\cap[0,t]}\big(\Delta\varphi_i(x)\big)^2 dx = \sum_{v}\lambda_v c_v^2\sum_{i}\mu_i^2 \int_{\Delta_v\cap[0,t]}\varphi_i^2\,dx,$$

whence it follows that the function

$$d(t) = \mathrm{Tr}\,D\Big(\overset{*}{\int_0^t}\mu G(dt)\Big) \ (^6)$$

is continuous on the interval $[0, 1]$, which completes the proof of property (iii) of our process. It remains to prove property (iv), which we obtain from the formula

$$\Big(D\Big(\overset{*}{\int_{\Delta}}\mu G(d\tau)\Big)h, h\Big) = \sum_{v=1}^{\infty}\lambda_v c_v^2 \int_{\Delta_v}\big(E(\Delta)\Delta h(x)\big)^2 dx.$$

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(6) $D(\ldots)$ denotes the dispersion operator of the distribution in brackets.

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