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Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications

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1. INTRODUCTION

The main purpose of the present paper is to give a new presentation as well as new applications of the results contained in Grothendieck's paper [17]. In this remarkable paper Grothendieck outlined the theory of tensor products of Banach spaces. The climax of this paper was a theorem called by Grothendieck "the fundamental theorem of the metric theory of tensor products". This theorem is equivalent to the following assertion:

Let $\{a_{i,j}\}_{i,j=1}^n$ be a finite matrix of real numbers such that

$$\Big|\sum_{i,j=1}^{n} a_{i,j} t_i s_j\Big| \leqslant 1$$

whenever $|t_i| \leqslant 1$, $|s_j| \leqslant 1$. Then for every set of unit vectors $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ in a Hilbert space

$$\Big|\sum_{i,j} a_{i,j}(x_i, y_j)\Big| \leqslant K,$$

where K is an absolute constant and $(\cdot,\cdot)$ denotes the inner product in the Hilbert space.

This inequality, as well as many of its applications, are meaningful and interesting also outside the framework of tensor product theory. Though the theory of tensor products constructed in Grothendieck's paper has its intrisic beauty we feel that the results of Grothendieck and their corollaries can be more clearly presented without the use of tensor products. The paper of Grothendieck is quite hard to read [1] and its results are not generally known even to experts in Banach space theory. In fact, by using these results some problems which were posed by various authors in the last decade can be easily solved. All these considerations persuaded us to write this paper in its present form. We do not use here the notion of tensor products.

[1] An elegant exposition of the introductory part of [17] can be found in [56].

In Section 2 we present a proof of the inequality mentioned above and of its immediate consequences. The proof we present is just a reformation of the argument of Grothendieck. The proof is elementary and no knowledge of functional analysis is needed for its understanding.

Section 3 is devoted to functional analytic preliminaries. In particular, we introduce in it the class of $\mathscr{L}_p$-spaces, $1 \leqslant p \leqslant \infty$. These are Banach spaces whose finite-dimensional subspaces are the "same" as those of an $L_p(\mu)$ space for some measure μ. These spaces are introduced since most of the results proved in the present paper depend not on the whole Banach space but rather on the structure of its finite-dimensional subspaces. We present also the notion of p-absolutely summing operators ($1 \leqslant p < \infty$) which is due to Pietsch [51] (cf. Saphar [52], [58] for $p = 2$) and which for $p = 1$ goes back to Grothendieck. The applications of the inequality of Section 2 to the theory of Banach spaces are made through the use of this notion of p absolutely summing operators. This is done in Section 4. We prove there that every operator from an $\mathscr{L}_1$-space to a Hilbert space is 1-absolutely summing and that this property characterizes, in a certain sense, $\mathscr{L}_1$ and Hilbert spaces respectively. As a corollary it follows that the inequality of Section 2 (which was stated above) characterizes Banach spaces which are isomorphic to Hilbert spaces. It also is shown in Section 4 that every operator from an $\mathscr{L}_\infty$ space to an $\mathscr{L}_p$ space, $1 \leqslant p \leqslant 2$, is 2-absolutely summing.

The results of Section 4 are used in Section 5 for obtaining factorization theorems for certain classes of operators. The main result here is that every linear operator T from an $\mathscr{L}_p$-space X into an $\mathscr{L}_r$-space Y where $p > 2 > r$ can be represented as $T = T_1 T_0$, where T_0 is a linear operator from X into a suitable Hilbert space H and T_1 is a linear operator from H into Y.

Section 6 is devoted to various applications of the preceding results. One application is the following: In the spaces l_1 and c_0 all normalized unconditional bases are equivalent to the usual unit basis. The space l_1 (resp. c_0) is the only complemented subspace of an $\mathscr{L}_1$ (resp. $\mathscr{L}_\infty$) space which has an unconditional basis. A qualitative version of this result gives a new connection between the projection and symmetry constants of a finite-dimensional space X and its distance from the space l_∞^n (with $n = \dim X$).

The results in Sections 4 and 5 concerning operators defined on $\mathscr{L}_p$-spaces provide a tool for proving that certain subspaces of $\mathscr{L}_p$-spaces are not complemented subspaces. We show in Section 6 how to use this tool in order to give a new proof to the result of D. J. Newman that the Hardy space H_1 is not a complemented subspace of $L_1(\mu)$ (where μ is the Haar measure on the circle).

Another application which is presented in Section 6 is Grothendieck's characterization of a Hilbert space as the only Banach space which is

isomorphic to a subspace of an $\mathscr{L}_1$-space and to a quotient space of an $\mathscr{L}_\infty$-space. We also present in this section several characterizations, due essentially to Grothendieck, of Hilbert-Schmidt and trace-class operators in a Hilbert space.

Section 7 is devoted to a study of subspaces of $L_p(\mu)$-spaces. This study clarifies somewhat the relation between general $\mathscr{L}_p$-spaces and $L_p(\mu)$-spaces. We show in particular that every $\mathscr{L}_p$-space, $1 < p < \infty$, is isomorphic to a complemented subspace of an $L_p(\mu)$-space for a suitable measure μ. Examples, given in Section 8, show that this is no longer true if $p = 1$ or ∞ and that unless $p = 2$ the class of $\mathscr{L}_p$-spaces properly includes the class of spaces isomorphic to $L_p(\mu)$-spaces. In Section 7 it is also shown that by combining known results it is now possible to give a complete solution to the problem of the linear dimension of $L_p(\mu)$-spaces (cf. Banach [2]).

The last section contains, besides the examples mentioned above, some open problems and various additional remarks and results. The main result in this section is the proof of the existence of a "universal" non-weakly compact operator.

Notation and terminology are given in Section 3. Let us only mention here that unless stated otherwise we consider only spaces over the reals though all the results and proofs carry over to the complex case.

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2. THE BASIC INEQUALITY

In this section we present the inequalities which form the basis of most of the proofs in the following sections. These inequalities are of interest in themselves and may be of use also to mathematicians who are not working in Banach space theory.

Let $S = S^n = \{x \in E^n; \|x\| = 1\}$ denote the $(n-1)$-dimensional sphere in the n-dimensional real Euclidean space $E = E^n$. Let m be the rotation invariant Borel measure on S normalized so that $m(S) = 1$. Let

$$(x, y) = \sum_{i=1}^{n} x^i y^i$$

denote the usual inner product of the vectors $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ in E^n. For real t let $\operatorname{sign} t = t/|t|$ if $t \neq 0$ and $\operatorname{sign} 0 = 0$.

LEMMA 2.1. *Let $x, y \in S^n$; then*

$$\int_{S^n} \operatorname{sign}(x, u) \operatorname{sign}(y, u)\, dm(u) = 1 - \frac{2}{\pi}\theta(x, y), \tag{2.1}$$

where $\theta = \theta(x, y)$ is the unique number satisfying $\cos\theta = (x, y)$ and $0 \leqslant \theta \leqslant \pi$ (i.e. θ is the angle between x and y).

Proof. We choose the basis in E^n in such a way that $x = (1, 0, \ldots, 0)$ and $y = (\cos\theta, \sin\theta, 0, 0, \ldots, 0)$. Let g be a bounded measurable function on S^n. Using polar coordinates $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ we express the integral $\int_{S^n} g(u)\,dm(u)$ by the $(n-1)$-dimensional Lebesgue integral. We have the relation

$$\int_{S^n} g(u)\,dm(u) = |S^n|^{-1} \int_{I^{n-1}} g\big(u(\varphi)\big) J(\varphi)\,d(\varphi),$$

where

$$u(\varphi) = \big(u^1(\varphi), u^2(\varphi), \ldots, u^n(\varphi)\big), \qquad u^1(\varphi) = \prod_{i=1}^{n-1} \sin\varphi_i,$$

$$u^k(\varphi) = \cos\varphi_{k-1} \prod_{i=k}^{n-1} \sin\varphi_i \quad \text{for} \quad k = 2, 3, \ldots, n-1,$$

$$u^n(\varphi) = \cos\varphi_{n-1},$$

$$I^{n-1} = \{\varphi\colon 0 \leqslant \varphi_1 < 2\pi;\ 0 \leqslant \varphi_i \leqslant \pi \text{ for } i = 2, 3, \ldots, n-1\},$$

$$J(\varphi) = \prod_{i=2}^{n-1} (\sin\varphi_i)^{i-1},$$

$$|S^n| = \int_{I^{n-1}} J(\varphi)\,d(\varphi) = 2\pi \prod_{i=2}^{n-1} \int_0^\pi (\sin\varphi_i)^{i-1}\,d\varphi_i.$$

Let $h(u) = (x, u)(y, u) = u^1(u^1\cos\theta + u^2\sin\theta)$. Then

$$h\big(u(\varphi)\big) = \Big[\prod_{i=2}^{n-1} \sin\varphi_i\Big]^2 \sin\varphi_1(\sin\varphi_1\cos\theta + \cos\varphi_1\sin\theta).$$

Hence, for $g(u) = \operatorname{sign}[h(u)]$, we get

$$g\big(u(\varphi)\big) = \operatorname{sign}[\sin\varphi_1 \sin(\varphi_1 + \theta)] = f(\varphi_1, \theta).$$

Clearly, $f(\varphi_1, \theta)$ is equal to $+1$ on the intervals $(0; \pi-\theta)$ and $(\pi; 2\pi-\theta)$, and is equal to -1 on the intervals $(\pi-\theta; \pi)$ and $(2\pi-\theta; 2\pi)$. Thus

$$\int_{S^n} g(u)\,dm(u) = |S^n|^{-1} \int_{I^{n-1}} f(\varphi_1, \theta) J(\varphi)\,d(\varphi)$$

$$= |S^n|^{-1} \int_0^{2\pi} f(\varphi_1, \theta)\,d\varphi_1 \prod_{i=2}^{n-1} \int_0^\pi (\sin\varphi_i)^{i-1}\,d\varphi_i$$

$$= (2\pi)^{-1} \int_0^{2\pi} f(\varphi_1, \theta)\,d\varphi_1 = 1 - 2\theta/\pi.$$

This completes the proof.

We are now ready for the proof of the main result:

THEOREM 2.1. *Let $\{a_{i,j}\}_{i,j=1,2,\ldots,N}$ be a real-valued matrix and let M be a positive number such that*

$$\Big|\sum_{i,j=1}^N a_{i,j} t_i s_j\Big| \leqslant M \tag{2.4}$$

for every real $\{t_i\}_{i=1}^N$ and $\{s_j\}_{j=1}^N$ satisfying $|t_i| \leqslant 1$ and $|s_j| \leqslant 1$. Then for arbitrary vectors $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ in a real inner product space H

$$\Big|\sum_{i,j=1}^N a_{i,j}(x_i, y_j)\Big| \leqslant K_G M \sup_i \|x_i\| \sup \|y_j\|, \tag{2.5}$$

where K_G is the Grothendieck universal constant ($K_G \leqslant \sinh\pi/2 = (e^{\pi/2} - e^{-\pi/2})/2$).

Proof. Let us first make some observations.

1° If a matrix $\{a_{i,j}\}$ satisfies (2.4), then for arbitrary real numbers c_i' and c_j'' $(i, j = 1, 2, \ldots, N)$ the matrix $\{a_{i,j}'\}$ with $a_{i,j}' = c_i' a_{i,j} c_j''$ for $i, j = 1, \ldots, N$ satisfies (2.4) with the constant

$$M' = M \sup_i |c_i'| \sup_j |c_j''|.$$

2° Since every $2N$ vectors in H belong to some $2N$-dimensional linear subspace of H which is isometric to E^{2N}, we may assume without loss of generality that $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ belong to E^{2N}. From observation 1° and a standard homogeneity argument it follows that we may assume also that $\|x_i\| = \|y_j\| = 1$ for every i and j.

For an arbitrary $u \in S^{2N}$ we define $t_i(u) = \operatorname{sign}(u, x_i)$ and $s_j(u) = \operatorname{sign}(u, y_j)$, $i, j = 1, \ldots, N$. By (2.4)

$$-M \leqslant \sum_{i,j=1}^N a_{i,j} t_i(u) s_j(u) \leqslant M \quad \text{for} \quad u \in S^{2N}.$$

Hence by integrating over S^{2N} with respect to the normalized rotation invariant measure we get, by formula (2.1),

$$-\frac{\pi}{2} M \leqslant \sum_{i,j=1}^N a_{i,j}\Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big) \leqslant \frac{\pi}{2} M.$$

Let us put $a_{i,j}^{(1)} = a_{i,j}\big(\pi/2 - \theta(x_i, y_j)\big)$ for $i, j = 1, 2, \ldots, N$. It follows easily from observation 1° that the matrix $(a_{i,j}^{(1)})$ satisfies (2.4) if we replace M by $\pi M/2$. Hence, by repeating the averaging argument we get

$$-\Big(\frac{\pi}{2}\Big)^2 M \leqslant \sum_{i,j=1}^N a_{i,j}^{(1)}\Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big) = \sum_{i,j=1}^N a_{i,j}\Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big)^2 \leqslant \Big(\frac{\pi}{2}\Big)^2 M.$$

In this manner we obtain inductively

$$(2.6)\qquad -\left(\frac{\pi}{2}\right)^n M \leqslant \sum_{i,j=1}^{N} a_{i,j}\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)^n \leqslant \left(\frac{\pi}{2}\right)^n M, \qquad n=1,2,\ldots$$

Since

$$(x_i, y_j) = \cos\theta(x_i, y_j) = \sin\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)^{2n+1}/(2n+1)!,$$

inequality (2.6) implies that

$$\Big|\sum_{i,j=1}^{N} a_{i,j}(x_i, y_j)\Big| \leqslant M\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^{2n+1}\frac{1}{(2n+1)!} = M\sinh\frac{\pi}{2}$$

and this concludes the proof of the theorem.

Corollary 1. *Let $\{a_{i,j}\}$ be a real-valued matrix for which* (2.4) *holds. Then for arbitrary vectors $\{x_i\}_{i=1}^N$ in an inner product space H*

$$(2.7)\qquad \sum_{j=1}^{N}\Big\|\sum_{i=1}^{N} a_{i,j}x_i\Big\| \leqslant K_G M \sup_i \|x_i\|.$$

Proof. Choose for $j=1,\ldots,N$ vectors $y_j\in H$ such that $\|y_j\|=1$ and

$$\Big(\sum_{i=1}^{N} a_{i,j}x_i, y_j\Big) = \Big\|\sum_{i=1}^{N} a_{i,j}x_i\Big\|.$$

By using these x_i and y_j in (2.5) we get (2.7).

Corollary 2. *Let $\{a_{i,j}\}_{i,j=1,2,\ldots}$ be an infinite real matrix and let M be a positive constant such that*

$$(2.8)\qquad \Big|\sum_{i,j=1}^{N} a_{i,j}t_i s_j\Big| \leqslant M \quad \textit{for} \quad |t_i|\leqslant 1,\ |s_j|\leqslant 1, \quad i,j,N=1,2,\ldots$$

Then for an arbitrary real matrix $\{x_{k,i}\}$ such that for some $C>0$

$$(2.9)\qquad \Big(\sum_k x_{k,i}^2\Big)^{1/2} \leqslant C \quad \textit{for} \quad i=1,2,\ldots$$

the following inequalities hold:

$$(2.10)\qquad \sum_j\Big(\sum_k\Big(\sum_i x_{k,i}a_{i,j}\Big)^2\Big)^{1/2} \leqslant K_G CM,$$

"general Littlewood inequality", and

$$(2.11)\qquad \Big(\sum_k\Big(\sum_j\Big|\sum_i x_{k,i}a_{i,j}\Big|\Big)^2\Big)^{1/2} \leqslant K_G CM,$$

"general Orlicz inequality."

Proof. Observe first that (2.8) implies that

$$\sum_i |a_{i,j}| \leqslant M \qquad (j=1,2,\ldots).$$

Since, by (2.9), $|x_{i,k}|\leqslant C$ for every i and k, the series $\sum_i x_{k,i}a_{i,j}$ is absolutely convergent for $k,j=1,2,\ldots$ Therefore, since the sums over k and j in (2.10) and (2.11) have non-negative terms, it is enough to restrict our attention to the case where $\{x_{k,i}\}$ is a matrix with an arbitrary but finite number of elements different from zero (we pass to the general case by a standard limit procedure). Hence in the sequel we shall assume that each of the sums appearing in (2.9), (2.10) or (2.11) has exactly N terms.

Let $x_i=(x_{k,i})$ denote the i-th column of the matrix $\{x_{k,i}\}$ $(i=1,\ldots,N)$. We consider the x_i as vectors in the N-dimensional Euclidean space E^N. Then (2.9) means that $\|x_i\|\leqslant C$ for every i, and thus (2.10) is just a reformulation of (2.7).

Inequality (2.11) is an immediate consequence of (2.10). In fact, let

$$b_{j,k} = \Big|\sum_i x_{k,i}a_{i,j}\Big|.$$

By the triangle inequality for the vectors $b_j=(b_{j,k})$, $j=1,\ldots,N$, in E^N

$$\Big(\sum_k\Big(\sum_j b_{j,k}\Big)^2\Big)^{1/2} \leqslant \sum_j\Big(\sum_k b_{j,k}^2\Big)^{1/2},$$

i.e. the expression in the left-hand side of (2.11) is not larger than the expression in the left-hand side of (2.10).

Remark. If $x_{k,i}=\delta_i^k(=1$ for $i=k$ and $=0$ otherwise), (2.10) reduces to the inequality

$$\sum_j\Big(\sum_i a_{i,j}^2\Big)^{1/2} \leqslant K_G M.$$

This inequality (with a better constant, $\sqrt{3}$ instead of K_G) is due to Littlewood [38] (see also [50], p. 39, and [49]). For the same choice of $x_{k,i}$ formula (2.11) reduces to the inequality

$$\Big(\sum_i\Big(\sum_j|a_{i,j}|\Big)^2\Big)^{1/2} \leqslant K_G M.$$

This inequality was obtained by Orlicz in [42]. As in the proof of Theorem 2.1, the inequalities of Littlewood and Orlicz were obtained from (2.8) by using an averaging procedure. It would be of some interest to know the best possible value for K_G as well as the best constant in the inequalities of Littlewood and Orlicz (i.e. inequalities (2.10) and (2.11) with $x_{k,i} = \delta_i^k$). Grothendieck proves in [17] that $K_G \geqslant \pi/2$.

Let us finally note that if we consider also complex-valued matrices $\{a_{i,j}\}$ for which (2.4) holds, then (2.5) will be valid (in complex or real Hilbert spaces) if K_G is replaced by $2K_G$. In order to see this we have only to take the real and imaginary parts of the matrix $\{a_{i,j}\}$ and to use inequality (2.7) which is equivalent to (2.5).

3. NOTATIONS AND PRELIMINARIES

We begin with some notation. Let X and Y be Banach spaces. We denote by $B(X, Y)$ the space of all the operators from X into Y with the usual operator norm

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\|.$$

By "operator" we always mean a linear and bounded operator. The distance $d(X, Y)$ between the Banach spaces X and Y is defined as $\inf \|T\| \, \|T^{-1}\|$, the infimum is taken over all invertible T in $B(X, Y)$. If no such T exists, i.e., if X and Y are not isomorphic, $d(X, Y)$ is taken as ∞. (Remark. Clearly d is not a metric but we find it more convenient to use d instead of $\log d$ which is a metric. Thus two spaces X and Y are "near" if $d(X, Y)$ is close to 1.)

If X is a subspace of a Banach space Y, we say that X is *complemented in* Y if there is a bounded linear projection from Y onto X. A Banach space X is said to be a *$\mathscr{P}$-space* if it is complemented in every Banach space Y containing it as a subspace. A Banach space X is said to be a *$\mathscr{P}_\lambda$-space*, $1 \leqslant \lambda < \infty$, if for every $Y \supset X$ there is a projection of norm $\leqslant \lambda$ from Y onto X.

A series $\sum x_i$ of elements in a Banach space X is said to be *unconditionally convergent* if the series $\sum x_{\sigma(i)}$ converges for every permutation σ of the integers. The series $\sum x_i$ is said to *converge absolutely* if $\sum_i \|x_i\| < \infty$. A set $\{x_i\}_{i=1}^\infty$ is called a *basis* of the space X if for every $x \in X$ there is a unique sequence of reals $\{a_i\}_{i=1}^\infty$ such that $x = \sum_i a_i x_i$. If this series converges unconditionally for every $x \in X$, then $\{x_i\}_{i=1}^\infty$ is said to be an *unconditional basis* of X. More generally, a set $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements of a Banach space X is called an *unconditional basis* of X if for every $x \in X$ there is a unique set of scalars a_γ, $\gamma \in \Gamma$, such that $x = \sum_\gamma a_\gamma x_\gamma$ and this series *converges unconditionally* (in particular, for every x at most a countable number of the a_γ are different from 0).

Most of the results in the coming sections are concerned with $L_p(K, \Sigma, \mu)$ spaces, $1 \leqslant p \leqslant \infty$, i.e., the spaces of measurable functions f on some measure space (K, Σ, μ) for which (if $p < \infty$)

$$\int_K |f(x)|^p \, d\mu(x) < \infty$$

and with norm

$$\|f\| = \Big(\int |f(x)|^p \, d\mu(x)\Big)^{1/p}$$

(if $p = \infty$, the space consists of those measurable f for which $\|f\|$ = essential supremum $|f(x)| < \infty$). We shall often omit the measure space K and the σ-field Σ from the notation and speak simply of an $L_p(\mu)$-space. If (K, Σ, μ) is the unit interval with the Lebesgue measure we shall denote $L_p(K, \Sigma, \mu)$ also by $L_p(0, 1)$. A special kind of an $L_p(\mu)$-space is the space $l_p(\Gamma)$ = the space of all real-valued functions f on the abstract set Γ for which

$$\|f\| = \begin{cases} \Big(\sum_\gamma |f(\gamma)|^p\Big)^{1/p} < \infty & \text{if} \quad p < \infty, \\ \sup_\gamma |f(\gamma)| < \infty & \text{if} \quad p = \infty. \end{cases}$$

If Γ is a countable infinite set, we denote $l_p(\Gamma)$ also by l_p and if Γ consists of a finite number, n say, of elements, we shall denote $l_p(\Gamma)$ also by l_p^n. The subspace of $l_\infty(\Gamma)$ consisting of those f for which $\{\gamma;\ |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$ is denoted by $c_0(\Gamma)$ (or c_0 if Γ is countably infinite).

In the context of the present paper it is more natural to consider a larger class of Banach spaces than the class of $L_p(\mu)$-spaces.

Definition 3.1. A Banach space X is called an $\mathscr{L}_{p,\lambda}$-space, $1 \leqslant p \leqslant \infty$, $1 \leqslant \lambda < \infty$, provided that for every finite-dimensional subspace B of X there is a finite-dimensional subspace E of X containing B such that $d(E, l_p^n) \leqslant \lambda$ (where $n = \dim E$).

A Banach space X is called an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant \infty$, if it is an $\mathscr{L}_{p,\gamma}$-space for some $\lambda \geqslant 1$.

Related notions have been considered recently by various authors, cf. e.g. [35], [19] and [39].

By using subspaces which are generated by the characteristic functions of sets in a decomposition of the measure space into a finite number of subsets, it easily follows and it is well known that every $L_p(\mu)$-space is an $\mathscr{L}_{p,\lambda}$-space for every $\lambda > 1$. By using partitions of unity, it follows also easily that every $C(K)$-space (= the space of continuous functions on a compact Hausdorff space K) is an $\mathscr{L}_{\infty,\lambda}$-space for every $\lambda > 1$. More generally, every Banach space whose dual is isometric to an $L_1(\mu)$-space (e.g. every M space in the sense of Kakutani [29]) is an $\mathscr{L}_{\infty,\lambda}$-space for every $\lambda > 1$ (see [32]).

As we shall see in Section 7 every $\mathscr{L}_p$-space is isomorphic to a subspace of an $L_p(\mu)$-space for a suitable measure μ. In particular, since every subspace of a Hilbert space is again a Hilbert space, the class of $\mathscr{L}_2$-spaces coincides with the class of spaces isomorphic to a Hilbert space. In Section 8 we shall give examples of L_p-spaces which are not isomorphic to $L_p(\mu)$-spaces for $1 \leqslant p < \infty$, $p \neq 2$. For $p = \infty$ it is clear that not every L_∞-space is isomorphic to an $L_\infty(\mu)$-space (observe for example that there are no infinite-dimensional separable $L_\infty(\mu)$-spaces) but it is an open problem whether every L_∞-space is isomorphic to a $C(K)$-space for a suitable compact Hausdorff K (see [35], chapter III, for a discussion of this problem).

We shall often use the following notion which was introduced and studied by Pietsch [51] (cf also Grothendieck [15], p. 160, for $p = 1$ and Saphar [52] for $p = 2$).

Definition 3.2. Let X and Y be Banach spaces, let $T \in B(X, Y)$ and let $1 \leqslant p < \infty$. Put

$$a_p(T) = \inf\Big\{C;\ \Big(\sum_{i=1}^n \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x^*\|\leqslant 1}\Big(\sum_{i=1}^n |x^*(x_i)|^p\Big)^{1/p},\quad x_i \in X,\ i = 1, 2, \ldots, n,\ n = 1, 2, \ldots\Big\}.$$

If $a_p(T) < \infty$, then T is said to be *p-absolutely summing*.

We shall say "absolutely-summing" instead of "1-absolutely summing." The source of this terminology is the easily checked fact that an operator T is absolutely summing if and only if the series $\sum_i Tx_i$ converges absolutely whenever the series $\sum_i x_i$ is unconditionally convergent. (Observe that for every $\{x_i\}_{i=1}^n$

$$\sup\Big\{\sum_{i=1}^n |x^*(x_i)|;\ \|x^*\| = 1\Big\} = \max\Big\{\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|;\ \varepsilon_i = \pm 1,\ i = 1, \ldots, n\Big\}\Big).$$

It is not hard to see that if $p_1 < p_2$, then $a_{p_1}(T) \geqslant a_{p_2}(T)$. This is for example an immediate consequence of the proposition below. Hence every p_1-absolutely summing operator is also p_2-absolutely summing.

The following basic result is due to Pietsch [51]. (In Pietsch's original formulation the measure μ is concentrated on the unit ball of X^*.)

PROPOSITION 3.1. *Let $1 \leqslant p < \infty$ and let $T \in B(X, Y)$. If T is p-absolutely summing, then there is a probability measure ($=$ regular positive Borel measure with total mass 1) μ on the compact space $K^* =$ the w^* closure of the set of all extreme points of the unit ball of X^*, such that*

$$\|Tx\| \leqslant a_p(T)\Big(\int_{K^*} |x^*(x)|^p\,d\mu(x^*)\Big)^{1/p},\qquad x \in X. \tag{3.1}$$

Conversely, if for some $T \in B(X, Y)$ there is a probability measure μ on K^* such that (3.1) holds with $a_p(T)$ replaced by some constant $C < \infty$, then T is p-absolutely summing and $a_p(T) \leqslant C$.

A detailed proof of this result can be found in [51], Theorem 2. For self-containedness of our paper we indicate briefly the proof of the first part of the proposition (the second part is trivial).

Proof. Let $T \in B(X, Y)$ and let $a_p(T) < \infty$. Put

$$W = \Big\{g \in C(K^*);\ g = [a_p(T)]^p \sum_{i=1}^n |f_{x_i}|^p \text{ with } \sum_{i=1}^n \|Tx_i\|^p = 1\Big\},$$

where $f_x(x^*) = x^*(x)$ for $x^* \in K^*$ and $x \in X$.

It immediately follows from the definitions of W and $a_p(T)$ that W is a convex subset of $C(K^*)$ which is disjoint from the set

$$N = \{f \in C(K^*);\ \sup_{x^* \in K^*} f(x^*) < 1\}.$$

We use the fact that

$$\sup_{\|x^*\|=1} \sum_{i=1}^n |x^*(x_i)|^p = \sup_{x^* \in K^*} \sum_{i=1}^n |x^*(x_i)|^p$$

for arbitrary $\{x_i\}_{i=1}^n$ in X. Since N is an open convex set, it follows from the separation theorem and the Riesz representation theorem that there is a measure μ_0 on K^* such that $\int f d\mu_0 < 1$ for $f \in N$ and $\int g d\mu_0 \geqslant 1$ for $g \in W$. Since N contains the cone of negative functions in $C(K^*)$ as well as the open unit ball of this space, it follows that $\mu_0 = a\mu$, where μ is a probability measure and $0 < a \leqslant 1$. For any $x \in X$ with $Tx \neq 0$ the function $g = (|a_p(T) f_x| / \|Tx\|)^p$ belongs to W and hence $\int g d\mu \geqslant \int g d\mu_0 \geqslant 1$, or

$$\|Tx\|^p \leqslant (a_p(T))^p \int_{K^*} |x^*(x)|^p\,d\mu(x^*),$$

and this concludes the proof.

COROLLARY 1. *Let $T \in B(X, Y)$ be a 2-absolutely summing operator. Then there is a probability measure μ on K^* ($=$ the w^* closure of the extreme points of the unit ball in X^*) and an operator $S: L_2(\mu) \to Y$ such that*

(i) $\|S\| = a_2(T)$;

(ii) $T = SJI$, *where $I: X \to C(K^*)$ is the canonical isometry $x \to x(x^*)$ and $J: C(K^*) \to L_2(\mu)$ is the (formal) identity map $f \to f$.*

Proof. Proposition 3.1 asserts the existence of an operator $\tilde{S}$ from the closure of JIX in $L_2(\mu)$ into Y such that $\|\tilde{S}\| = a_2(T)$ and $T = \tilde{S}JI$. Since in the Hilbert space $L_2(\mu)$ there is a projection of norm one onto the closure of JIX, we can extend $\tilde{S}$ in a norm-preserving manner to an operator $S: L_2(\mu) \to Y$. This operator S has the desired properties.

4. ABSOLUTELY SUMMING OPERATORS BETWEEN $\mathscr{L}_p$-SPACES

The first theorem we prove in this section is a reformation of [17], Corollary 1, p. 59.

THEOREM 4.1. *Let X be an $\mathscr{L}_1$-space and let H be a Hilbert space. Then every $T \in B(X, H)$ is absolutely summing.*

Proof. Let λ be such that X is an $\mathscr{L}_{1,\lambda}$-space, let $\{x_i\}_{i=1}^n \subset X$ be such that

$$\sum_{i=1}^{n} |x^*(x_i)| \leqslant \|x^*\|$$

for every $x^* \in X^*$. By Definition 3.1 there is a finite-dimensional subspace $E \subset X$ containing $\{x_i\}_{i=1}^n$ and an operator $S: l_1^m \to E$ $(m = \dim E)$ with $\|S\| = 1$ and $\|S^{-1}\| \leqslant \lambda$. Put $y_i = S^{-1}x_i$, $i = 1, \ldots, n$, and let $a_{i,j}$ be the j-th coordinate of y_j with respect to the usual basis $\{e_j\}_{j=1}^m$ of l_1^m (i.e., $y_i = \sum_{j=1}^{m} a_{i,j}e_j$, $i = 1, \ldots, n$). Let t_i and s_j $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ be real numbers of absolute value $\leqslant 1$ and let y^* be the element in $l_\infty^m = (l_1^m)^*$ whose j-th coordinate is s_j. Then

$$\Big|\sum_{i,j} a_{i,j}t_i s_j\Big| \leqslant \sum_i |t_i| \Big|\sum_j a_{i,j}s_j\Big| \leqslant \sum_i \Big|\sum_j a_{i,j}s_j\Big|$$
$$= \sum_i |y^*(y_i)| = \sum_i |y^*(S^{-1}x_i)| = \sum_i |(S^{-1})^* y^*(x_i)| \leqslant \|(S^{-1})^* y^*\| \leqslant \lambda.$$

Now let

$$u_i = Tx_i = TSy_i = \sum_{j=1}^{m} a_{i,j}TSe_j, \quad i = 1, \ldots, n.$$

Then

$$\sum_i \|u_i\| = \sum_i \Big\|\sum_j a_{i,j}TSe_j\Big\|$$

and by Corollary 1 to Theorem 2.1

$$\sum_{i=1}^{n} \|Tx_i\| \leqslant K_G \lambda \sup_j \|TSe_j\| \leqslant K_G\lambda\|TS\| \leqslant K_G\lambda\|T\|.$$

Thus $a_1(T) \leqslant K_G\lambda\|T\| < \infty$ and this concludes the proof.

It is conceivable that Theorem 4.1 actually characterizes $\mathscr{L}_1$ and Hilbert spaces respectively. By this we mean that the following result may be true. Let X and Y be infinite-dimensional Banach spaces such that every $T \in B(X, Y)$ is absolutely summing. Then X is an $\mathscr{L}_1$-space and Y is isomorphic to a Hilbert space. We shall prove now a partial result in this direction.

THEOREM 4.2. *Let X and Y be infinite-dimensional Banach spaces such that X has an unconditional basis and such that every $T \in B(X, Y)$ is absolutely summing. Then X is isomorphic to $l_1(\Gamma)$ and Y is isomorphic to a Hilbert space.*

Proof. We remark first that by our assumptions there is a constant K such that $a_1(T) \leqslant K\|T\|$ for every $T \in B(X, Y)$. (Use the fact that by Baire's category theorem there is an M such that the subset $\{T;\ a_1(T) \leqslant M\}$ of $B(X, Y)$ has a non-empty interior.)

Let $\{x_i\}_{i=1}^\infty$ be a normalized (i.e. $\|x_i\| = 1$) unconditional basis in X and let n be an integer. (We assume that X is separable, but the same proof works also if x is non-separable and has an unconditional basis $\{x_\gamma\}_{\gamma\in\Gamma}$). By the main lemma of the paper of Dvoretzky-Rogers [12] (cf. also [8], p. 61-63) there are $\{y_i\}_{i=1}^n$ in Y with $\|y_i\| = 1$ for every i and such that

$$\Big\|\sum_{i=1}^{n} \lambda_i y_i\Big\| \leqslant 2\Big(\sum_{i=1}^{n} \lambda_i^2\Big)^{1/2}$$

for every choice of $\{\lambda_i\}_{i=1}^n$. Let $\{\mu_i\}_{i=1}^n$ be positive numbers such that $\sum_{i=1}^{n} \mu_i^2 = 1$, and define $T: X \to Y$ by

$$Tx = \sum_{i=1}^{n} \mu_i a_i y_i \quad \text{if} \quad x = \sum_{i=1}^{\infty} a_i x_i.$$

Let ϱ be a constant such that

$$\Big\|\sum_i \varepsilon_i a_i x_i\Big\| \leqslant \varrho \Big\|\sum_i a_i x_i\Big\|$$

whenever $\varepsilon_i = \pm 1$ and $\sum a_i x_i$ converges. Then clearly

$$|a_i| \leqslant \varrho \Big\|\sum_i a_i x_i\Big\|, \quad i = 1, 2, \ldots,$$

and hence

$$\|Tx\| \leqslant 2\Big(\sum_{i=1}^{n} (a_i\mu_i)^2\Big)^{1/2} \leqslant 2\varrho\|x\|.$$

Consequently, $a_1(T) \leqslant 2\varrho K$. Since

$$\Big\|\sum_{i=1}^{n} \varepsilon_i a_i x_i\Big\| \leqslant \varrho\|x\| \quad \text{for every } x = \sum_{i=1}^{\infty} a_i x_i \in X$$

and every choice of $\varepsilon_i = \pm 1$, we get by the definition of $a_1(T)$ that

$$\sum_{i=1}^{n} |a_i|\mu_i = \sum_{i=1}^{n} \|Ta_ix_i\| \leqslant a_1(T)\varrho\|x\| \leqslant 2\varrho^2 K\|x\|. \tag{4.1}$$

Since (4.1) is valid whenever $\sum_{i=1}^{n}\mu^2 = 1$, we get by Landau's theorem

$$\Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \leqslant 2\varrho^2 K\|x\| \quad \text{if} \quad x = \sum_{i=1}^{\infty} a_ix_i. \tag{4.2}$$

Define now the operator $S\colon X \to Y$ by $Sx = \sum_{i=1}^{n} a_iy_i$ if $x = \sum_{i=1}^{n} a_ix_i$. Then

$$\|Sx\| \leqslant 2\Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \leqslant 4\varrho^2 K\|x\|,$$

and hence $a_1(S) \leqslant 4\varrho^2K^2$. Consequently,

$$\sum_{i=1}^{n}|a_i| = \sum_{i=1}^{n}\|Sa_ix_i\| \leqslant a_1(S)\sup_{\varepsilon_i=\pm 1}\Big\|\sum_{i=1}^{n}\varepsilon_ia_ix_i\Big\| \leqslant a_1(S)\varrho\|x\| \leqslant 4\varrho^3K^2\|x\|.$$

Therefore for every $x = \sum_{i=1}^{n} a_ix_i \in X$ we get

$$\|x\| \leqslant \sum_{i=1}^{\infty}|a_i| \leqslant 4\varrho^3K^2\|x\|$$

and this proves that X is isomorphic to l_1.

Now let Y_0 be a separable subspace of Y. Since every separable Banach space is a quotient space of l_1 [3] there is an operator T_0 from X onto Y_0. By our assumption T_0 is absolutely summing and hence also 2-absolutely summing. By Corollary 1 to Proposition 3.1 there is a Hilbert space H and operators $T_1\colon X \to H$, $T_2\colon H \to Y_0$ such that $T_0 = T_2T_1$. Since T_0 is onto Y_0, T_2 must also be a quotient map and hence Y_0, being isomorphic to a quotient of a Hilbert space, must itself be isomorphic to a Hilbert space. Hence every separable subspace of Y is isomorphic to a Hilbert space. This implies (cf. [34], Lemma 3, or section 7 below) that Y itself is isomorphic to a Hilbert space.

Remark. The proof above did not only show that X is isomorphic to l_1 but that the given unconditional basis in X is equivalent to usual basis of l_1. Thus by combining Theorem 4.1 with the proof of Theorem 4.2 we get that all normalized unconditional bases in l_1 are equivalent. We shall return to this result in a more detailed way in Section 6. We shall state here only the following consequence of the proofs of Theorems 4.1 and 4.2 which shows that also Theorem 2.1 can be used to characterize spaces isomorphic to Hilbert spaces.

COROLLARY 1. *Let Y be a Banach space for which there is a constant K such that the following is true: Let $\{a_{i,j}\}_{i,j=1,\dots,N}$ be any finite real-valued matrix for which*

$$\Big|\sum_{i,j} a_{i,j}t_is_j\Big| \leqslant 1$$

whenever $|t_i| \leqslant 1$ and $|s_j| \leqslant 1$, $i,j = 1,\dots,N$. Then for every choice of $\{y_i\}_{i=1}^N \subset Y$ and $\{y_j^*\}_{j=1}^N \subset Y^*$

$$\Big|\sum_{i,j} a_{i,j}y_j^*(y_i)\Big| \leqslant K\sup_i\|y_i\|\sup_j\|y_j^*\|.$$

Then Y is isomorphic to a Hilbert space.

Proof. By the same proof as that of Theorem 4.1 every operator in $B(l_1, Y)$ is absolutely summing. Hence, by the second part of the proof of Theorem 4.2, Y is isomorphic to a Hilbert space.

THEOREM 4.3. *Let X be an $\mathscr{L}_\infty$-space and let Y be an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant 2$. Then every $T \in B(X, Y)$ is 2-absolutely summing.*

Proof. Let λ and ϱ be such that X is an $\mathscr{L}_{\infty,\lambda}$-space and Y an $\mathscr{L}_{p,\varrho}$-space, and let $\{x_i\}_{i=1}^N \subset X$ be such that

$$\sum_{i=1}^{n}|x^*(x_i)|^2 \leqslant \|x^*\|^2$$

for every $x^* \in X^*$. By our assumption on X there is an integer m and an invertible operator S from l_∞^m into X such that $Sl_\infty^m \supset \{x_i\}_{i=1}^N$, $\|S\| = 1$ and $\|S^{-1}\| \leqslant \lambda$. Put $z_i = S^{-1}x_i \in l_\infty^m$ $(i = 1,\dots,N)$. By our assumption on Y there is a finite-dimensional subspace E of Y containing TSl_∞^m and an invertible operator $U\colon E \to l_p^h$ $(h = \dim E)$ with $\|U\| = 1$ and $\|U^{-1}\| \leqslant \varrho$. Thus we have an operator $T_0 = UTS\colon l_\infty^m \to l_p^h$ and elements $\{z_i\}_{i=1}^n \subset l_\infty^m$ such that for every $z^* \in l_1^m$

$$\begin{aligned}\sum_i \big(z^*(z_i)\big)^2 &= \sum_i \big(z^*(S^{-1}x_i)\big)^2 \\ &= \sum_i \big((S^{-1})^*z^*(x_i)\big)^2 \leqslant \|(S^{-1})^*z^*\|^2 \leqslant \lambda^2\|z^*\|^2.\end{aligned} \tag{4.3}$$

Our aim is to show that $\sum_i\|T_0z_i\|^2$ is bounded by a constant depending only on λ and $\|T_0\|$. Let $\{e_j\}_{j=1}^m$ and $\{f_k\}_{k=1}^h$ be the usual bases in l_∞^m and l_p^h respectively and let $a_{i,j}$ be defined by

$$T_0e_j = \sum_{k=1}^{h} a_{j,k}f_k, \quad j = 1,\dots,m.$$

For every $u^* = (a_1, a_2, \ldots, a_h) \epsilon (l_p^h)^*$ with $\|u^*\| = 1$ and every real $\{t_j\}_{j=1}^m$ and $\{s_k\}_{k=1}^h$ of absolute value $\leqslant 1$ we have

$$(4.4)\qquad \Big|\sum_{j,k} a_{j,k}a_k t_j s_k\Big| = \Big|u_s^*\Big(\sum_j T_0 t_j e_j\Big) \leqslant \|u_s^*\|\,\|T_0\|\,\Big\|\sum_j t_j e_j\Big\| \leqslant \|T_0\|,$$

where by u_s^* we denote the vector $(s_1 a_1, s_2 a_2, \ldots; s_h a_h)$ in $(l_p^h)^*$. Let $z_{i,j}$ denote the j-th coordinate of z_i, i.e.

$$z_i = \sum_{j=1}^m z_{i,j} e_j.$$

By (4.3) we get

$$(4.5)\qquad \sum_{i=1}^n z_{i,j}^2 \leqslant \lambda^2, \quad j = 1, 2, \ldots, m$$

(take $z^* =$ the j-th unit vector in l_1^m, in (4.3)). By (4.4) and (4.5) we get from the generalized Littlewood inequality (2.10)

$$\sum_k a_k\Big(\sum_i\Big(\sum_j z_{i,j}a_{j,k}\Big)^2\Big)^{1/2} \leqslant \lambda K_G\|T_0\|.$$

Since this holds whenever $\Sigma a_k^q = 1$ $(1/p+1/q = 1)$ if $p > 1$, and whenever $\max_k |a_k| = 1$ if $p = 1$, we get by Landau's theorem

$$(4.6)\qquad \Big(\sum_k\Big(\sum_i\Big(\sum_j z_{i,j}a_{j,k}\Big)^2\Big)^{p/2}\Big)^{1/p} \leqslant \lambda K_G\|T_0\|.$$

Put

$$b_{k,i} = \Big|\sum_j z_{i,j}a_{j,k}\Big|^p.$$

By using the triangle inequality in $l_{2/p}$ (recall that $p \leqslant 2$), i.e.

$$\Big(\sum_i\Big(\sum_k b_{k,i}\Big)^{2/p}\Big)^{p/2} \leqslant \sum_k\Big(\sum_i (b_{k,i})^{2/p}\Big)^{p/2},$$

we get from (4.6)

$$(4.7)\qquad \Big(\sum_i\Big(\sum_k\Big|\sum_j z_{i,j}a_{j,k}\Big|^p\Big)^{2/p}\Big)^{1/2} \leqslant \lambda K_G\|T_0\|.$$

Now

$$T_0 z_i = \sum_j z_{i,j}Te_j = \sum_k\Big(\sum_j z_{i,j}a_{j,k}\Big)f_k$$

and hence

$$\|T_0 z_i\| = \Big(\sum_k\Big|\sum_j z_{i,j}a_{j,k}\Big|^p\Big)^{1/p}.$$

Thus we may rewrite inequality (4.7) as

$$\sum_i \|T_0 z_i\|^2 \leqslant \lambda^2 K_G^2\|T_0\|^2 \leqslant \lambda^2 K_G^2\|T\|^2.$$

Consequently

$$\sum_i \|Tx_i\|^2 = \sum_i \|U^{-1}T_0 z_i\|^2 \leqslant \varrho^2\lambda^2 K_G^2\|T\|^2,$$

or $a_2(T) \leqslant \varrho\lambda K_G\|T\| < \infty$ and this concludes the proof.

Remark. For a version of Theorem 4.3 which is valid for $p > 2$ see Proposition 8.2.

Corollary 1. *Let X be a Banach space whose second dual is a $\mathscr{P}$-space and let Y be isomorphic to a subspace of an $L_1(\mu)$-space for some measure μ. Then every $T \epsilon B(X, Y)$ is 2-absolutely summing.*

Proof. Clearly, $a_2(T) \leqslant a_2(T^{**})$ for every operator T since T^{**} is an extension of T (if X is canonically embedded in X^{**}). Therefore it is enough to prove that T^{**} is 2-absolutely summing. Let Z be a $C(K)$-space containing X^{**} isometrically (take e.g. as K the unit ball of X^{***} in its w^*-topology). The space Z, like any $C(K)$-space, is an $\mathscr{L}_\infty$-space. Since X^{**} is a $\mathscr{P}$-space, there is a bounded linear projection, say P, from Z onto X^{**}. The operator $T^{**}P$ maps the $\mathscr{L}_\infty$-space Z into an $\mathscr{L}_1$-space. (We use the fact, due to Kakutani (cf. [28], [29] or [8], p. 100), that the second dual of an $L_1(\mu)$-space is again an $L_1(\mu')$-space for some μ'.). By Theorem 4.3 the operator $T^{**}P$ is 2-absolutely summing. Since T^{**} is the restriction of $T^{**}P$ to X^{**}, we get

$$a_2(T) \leqslant a_2(T^{**}) \leqslant a_2(T^{**}P) < \infty,$$

and this concludes the proof.

Remark. Theorem 4.3 is actually a special case of Corollary 1. This assertion follows from the following two facts.

(i) Every $\mathscr{L}_p$-space, $1 \leqslant p \leqslant 2$, is isomorphic to a subspace of an $L_1(\mu)$-space for some measure μ (see Section 7).

(ii) If X is an $\mathscr{L}_\infty$-space, then X^{**} is a $\mathscr{P}$-space (see [35], Theorems 2.1 and 3.3).

We state now explicitly a special case of Corollary 1:

Corollary 2. *Let X be a Banach space whose dual is an $L_1(\mu)$-space (in particular X may be an abstract M-space in the sense of Kakutani [29]). Let Y be an $L_p(\nu)$-space for some $1 \leqslant p \leqslant 2$ and some measure ν. Then every $T \epsilon B(X, Y)$ is 2-absolutely summing.*

5. HILBERTIAN OPERATORS

Let X and Y be Banach spaces and let $T \in B(X, Y)$. We say that T can be *factored through a Banach space* Z if there exist bounded linear operators $T_0: X \to Z$ and $T_1: Z \to Y$ such that $T = T_1 T_0$. An operator T is called *Hilbertian* if it can be factored through a Hilbert space H.

PROPOSITION 5.1. *Let X and Y be Banach spaces and let $T \in B(X, Y)$. Then the following assertions are equivalent:*

(1) *T is Hilbertian.*

(2) *T^* is Hilbertian.*

(3) *There is a Banach space Z and a Hilbertian operator $S: Z \to Y$ such that $SZ \supset TX$.*

(4) *There is a Banach space Z and a Hilbertian operator $S: X \to Z$ such that $\|Tx\| \leqslant \|Sx\|$ for $x \in X$.*

Proof. The implication $(1) \Rightarrow (2)$ is an immediate consequence of the fact that the dual of an Hilbert space is again a Hilbert space. Conversely, if (2) holds, then T^{**} is Hilbertian and hence T, which is the restriction of T^{**} to X, is also Hilbertian. Hence (1) and (2) are equivalent.

The implications $(1) \Rightarrow (3)$ and $(1) \Rightarrow (4)$ are obvious. Assume now that (3) holds. By the definition of a Hilbertian operator we may assume without loss of generality that Z is a Hilbert space. By considering the orthogonal complement of the kernel of S we may also assume that S is one-to-one. Define now a map $S_0: X \to Z$ by putting $S_0 x = S^{-1}Tx$, $x \in X$. By our assumptions S_0 is a well defined linear map. The fact that S_0 is bounded follows from the closed graph theorem. Indeed, if $\|x_n - x\| \to 0$ and $\|S_0 x_n - h\| \to 0$, then $\|Tx_n - Tx\| \to 0$ and $\|Tx_n - Sh\| = \|SS_0 x_n - Sh\| \to 0$, and thus $Tx = Sh$ or $h = S_0 x$. Hence $T = SS_0$ is a Hilbertian operator and $(3) \Rightarrow (1)$.

Finally, assume that (4) holds. Again, we may assume without loss of generality that Z is a Hilbert space. For every $h \in SX$ define $S_1 h = Tx$, where x is any element in $S^{-1}h$. Since the kernel of T contains the kernel of S, S_1 is a well defined linear map. For every $x \in X$, $\|Tx\| = \|S_1 Sx\| \leqslant \|Sx\|$ and hence $\|S_1 h\| \leqslant \|h\|$. We can extend therefore S_1 to a bounded linear operator $\tilde{S}_1$ from Z into Y. Since $T = \tilde{S}_1 S$, we proved that $(4) \Rightarrow (1)$.

From what we have proved in the preceding sections we easily get the Grothendieck factorization theorem ([17], Corollaire 2, p. 61):

THEOREM 5.1. *Let X be an $\mathscr{L}_\infty$-space and let Y be an $\mathscr{L}_1$-space. Then every $T \in B(X, Y)$ is Hilbertian.*

Proof. By Theorem 4.3, every $T \in B(X, Y)$ is 2-absolutely summing. Hence the result follows by using Corollary 1 to Proposition 3.1.

The proofs of Theorem 4.3 and Proposition 3.1 show that the following more precise version of Theorem 5.1 holds:

THEOREM 5.1′. *Let X be an $\mathscr{L}_{\infty,\lambda}$-space and let Y be an $\mathscr{L}_{1,\varrho}$-space. Let K^* be the w^*-closure of the extreme points of the unit ball of X^*. Then for every $T \in B(X, Y)$ there is a probability measure μ on K^* and an operator $S: L_2(\mu) \to Y$ with $\|S\| \leqslant K_G \|T\| \lambda \varrho$ such that $T = SJI$, where I denotes the canonical isometry $I: X \to C(K^*)$ and $J: C(K^*) \to L_2(\mu)$ is the formal identity map.*

For X a $C(K)$-space, Theorem 5.1′ gets the following simpler form:

COROLLARY 1. *Let $X = C(K)$ and let Y be an $\mathscr{L}_{1,\varrho}$-space and let $T \in B(X, Y)$. Then there is a probability measure μ on K and an operator $S: L_2(\mu) \to Y$ with $\|S\| \leqslant K_G \varrho \|T\|$ such that $T = SJ$, where $J: C(K) \to L_2(\mu)$ is the formal identity map.*

Proof. If $X = C(K)$, then, as well known ([9], p. 441), K^* can be identified canonically with K. Also, since a $C(K)$-space is an $\mathscr{L}_{\infty,1+\varepsilon}$-space for every $\varepsilon > 0$, we get by the proof of Theorem 4.3 that $a_2(T) \leqslant K_G(1+\varepsilon)\varrho\|T\|$ for every $\varepsilon > 0$ and hence $a_2(T) \leqslant K_G \varrho \|T\|$. Hence we can apply the corollary to Proposition 3.1 to get the desired result.

Another variant of Theorem 5.1 is

THEOREM 5.1″. *Let X be a Banach space such that X^{**} is a $\mathscr{P}$-space and let Y be a subspace of an $\mathscr{L}_1$-space. Then every $T \in B(X, Y)$ is Hilbertian.*

Proof. Use Corollary 1 to Theorem 4.3 and Corollary 1 to Proposition 3.1.

In the final result of this section we shall use some results which will be proved only in Section 7.

THEOREM 5.2. *Let X be an $\mathscr{L}_p$-space with $2 \leqslant p \leqslant \infty$ and let Y be an $\mathscr{L}_r$-space with $1 \leqslant r \leqslant 2$. Then every $T \in B(X, Y)$ is Hilbertian.*

Proof. The space X is isomorphic to a quotient space of an $\mathscr{L}_\infty$-space. This is clear if $p = \infty$ and for $p < \infty$ this follows from the results of Section 7. Indeed, by Theorem 7.1, X is isomorphic to a complemented subspace of an $L_p(\mu)$-space. Since $p \geqslant 2$, it follows (cf. Theorem 7.2 and its corollaries) that $L_p(\mu)^*$ is isometric to a subspace of $L_1(\nu)$ for some measure ν. Passing to the duals we get that $X = X^{**}$ is a quotient space of $L_1(\nu)^*$ which is an $\mathscr{L}_\infty$-space.

Now let $U: Z \to X$ be a quotient map, where Z is a suitable $\mathscr{L}_\infty$-space and let $T \in B(X, Y)$. By Theorem 4.3 the operator $TU: Z \to Y$ is 2-absolutely summing and hence (by the corollary to Proposition 3.1) Hilbertian. By $(3) \Rightarrow (1)$ of Proposition 5.1 it follows that T is Hilbertian and this concludes the proof.

In the proof of Theorem 5.2 we used two results from Section 7, namely Theorems 7.1 and 7.2. The use of Theorem 7.1 can be avoided if we use instead the following proposition which shows that for an operator $T \in B(X, Y)$ the property of being a Hilbertian operator is actually a local property, i.e. depends only on the action of T on the set of finite-dimensional subspaces of X.

PROPOSITION 5.2. *Let X and Y be Banach spaces and let $T \in B(X, Y)$. Then the following assertions are equivalent*:

(1) *T is Hilbertian.*

(2) *There is a constant C such that for every finite-dimensional subspace E of X there are operators $T_{0,E}: E \to l_2$ and $T_{1,E}: l_2 \to Y$ such that $T_{1,E}T_{0,E}$ is the restriction of T to E and $\|T_{0,E}\|\,\|T_{1,E}\| \leqslant C$.*

(3) *There is a constant C such that for every finite real-valued matrix $\{a_{i,j}\}_{i,j=1,\ldots,N}$ and elements $\{x_i\}_{i=1}^N \subset X$, $\{y_j^*\}_{j=1}^N \subset Y^*$,*

$$\Big|\sum_{i,j} a_{i,j} y_j^*(Tx_i)\Big| \leqslant CM \sup_i \|x_i\| \sup_j \|y_j^*\|,$$

where

$$M = \sup_{|s_i|\leqslant 1, |t_j|\leqslant 1} \Big|\sum_{i,j} a_{i,j} s_i t_j\Big|.$$

Proof. Clearly, (1) $\Rightarrow$ (2). We shall prove that (1) $\Leftrightarrow$ (3) and since (3) has, like (2), a local character, it will follow that all three assertions are equivalent.

(1) $\Rightarrow$ (3). Assume that $T = T_1 \cdot T_0$, where $T_0: X \to H$, $T_1: H \to Y$ and H is a Hilbert space. Then

$$\sum_{i,j} a_{i,j} y_j^*(Tx_i) = \sum_{i,j} a_{i,j}(T_0 x_i, T_1^* y_j^*)$$

and Theorem 2.1 implies that (3) holds.

(3) $\Rightarrow$ (1). Assume that (3) holds and that U is an operator from $l_1(\Gamma)$ into X. Then, by the proof of Theorem 4.1, the operator $TU: l_1(\Gamma) \to Y$ is absolutely summing and therefore Hilbertian. By taking as U a quotient map, it follows from the implication (3) $\Rightarrow$ (1) of Proposition 5.1 that T is Hilbertian.

Remark. The proof of Proposition 5.2 shows that if (2) holds, then $T = T_1 \cdot T_0$ with $T_0: X \to H$, $T_1: H \to Y$ and $\|T_1\|\,\|T_0\| \leqslant K_G C$.

After this paper has been submitted for publication, the first named author obtained the following strengthening of Proposition 5.2:

Let X and Y be Banach spaces such that there is a projection P from Y^{**} onto Y. Let $1 \leqslant p \leqslant \infty$ and let $T \in B(X, Y)$. Assume that there is a $C > 0$ such that for every finite-dimensional subspace $E \subset X$ there are operators $T_{0,E}: E \to l_p$ and $T_{1,E}: l_p \to Y$ such that $T_{1,E}T_{0,E}$ is the restriction of T to E and $\|T_{0,E}\|\,\|T_{1,E}\| \leqslant C$. Then there is a measure μ and operators $T_0: X \to L_p(\mu)$ and $T_1: L_p(\mu) \to Y$ such that $T = T_1 T_0$ and $\|T\| \leqslant C\|P\|$.

6. APPLICATIONS

Our first application is concerned with the notion of equivalent bases. A basis $\{x_i\}_{i=1}^\infty$ in a Banach space X is said to be *equivalent* to a basis $\{y_i\}_{i=1}^\infty$ in a Banach space Y if the series $\sum_{i=1}^\infty a_i x_i$ converges if and only if the series $\sum_{i=1}^\infty a_i y_i$ converges (and hence, by the closed graph theorem, the mapping $T: X \to Y$ defined by $T(\sum_i a_i x_i) = \sum_i a_i y_i$ is an isomorphism).

THEOREM 6.1. *Let X be a complemented subspace of an $\mathscr{L}_1$-space (resp. $\mathscr{L}_\infty$-space) Y and let $\{x_i\}_{i=1}^\infty$ be a normalized (i.e. $\|x_i\| = 1$ for every i) unconditional basis in X. Then the basis $\{x_i\}_{i=1}^\infty$ is equivalent to the unit vector basis in l_1 (resp. c_0).*

Proof. Let Y be an $\mathscr{L}_{1,\lambda}$ (resp. $\mathscr{L}_{\infty,\lambda}$) space and let P be a projection from Y onto X. Let ϱ be such that

$$\Big\|\sum_{i=1}^\infty a_i x_i\Big\| \leqslant \varrho \Big\|\sum_{i=1}^\infty \varepsilon_i a_i x_i\Big\|$$

whenever $\sum a_i x_i$ converges and $\varepsilon_i = \pm 1$.

We consider first the case when Y is an $\mathscr{L}_{1,\lambda}$ space. Let $\{u_i\}_{i=1}^\infty$ be any sequence of vectors in Y such that $\sum_{i=1}^\infty u_i$ converges unconditionally. Consider the operator $S: c_0 \to Y$ defined by

$$S(a_1, a_2, \ldots) = \sum_{i=1}^\infty a_i u_i.$$

By Theorem 4.3 we get

$$\Big(\sum_i \|u_i\|^2\Big)^{1/2} \leqslant \lambda K_G \sup_{\varepsilon_i=\pm 1} \Big\|\sum_i \varepsilon_i u_i\Big\| \tag{6.1}$$

because for every y^* in Y^* with $\|y^*\| = 1$,

$$\sup_{\varepsilon_i=\pm 1} \Big\|\sum_i \varepsilon_i u_i\Big\| \geqslant \Big(\sum_i |y^*(u_i)|^2\Big)^{1/2}.$$

(Inequality (6.1) is in fact the theorem of Orlicz [42]. Using his argument one can replace in (6.1) K_G by $\sqrt{3}$.)

Let T be the operator from Y into l_2 defined by

$$Ty = (a_1, a_2, \ldots) \quad \text{if} \quad Py = \sum_{i=1}^\infty a_i x_i.$$

By (6.1)

$$\|Ty\| = \Big(\sum_i a_i^2\Big)^{1/2} \leqslant \lambda K_G \sup_{\varepsilon_i} \Big\| \sum \varepsilon_i a_i x_i \Big\| \leqslant \varrho\lambda K_G \|Py\| \leqslant \varrho\lambda K_G \|P\|\,\|y\|.$$

Hence, $\|T\| \leqslant \varrho\lambda K_G \|P\|$ and, by Theorem 4.1, $a_1(T) \leqslant \varrho\lambda^2 K_G^2 \|P\|$. Thus for every $x = \sum_i a_i x_i \in X$

$$\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} \|Ta_i x_i\| \leqslant a_1(T) \sup_{\varepsilon_i} \Big\| \sum_{i=1}^{\infty} \varepsilon_i a_i x_i \Big\| \leqslant \varrho^2\lambda^2 K_G^2 \|P\|\,\|x\|. \tag{6.2}$$

Since, clearly, $\|\sum_i a_i x_i\| \leqslant \sum |a_i|$, (6.2) implies the equivalence of the basis $\{x_i\}_{i=1}^{\infty}$ with the unit basis in l_1.

Assume now that Y is an $\mathscr{L}_{\infty,\lambda}$-space. In order to show that $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vector basis in c_0 it is enough to show that there is a constant M (independent of n and $\{a_i\}_{i=1}^{n}$) such that

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\| \leqslant M \max_i |a_i|.$$

Fix an n and let B_n be the subspace of Y spanned by $\{x_i\}_{i=1}^{n}$. Let Q_n be the projection from X onto B_n defined by

$$Q_n x = \sum_{i=1}^{n} a_i x_i \quad \text{if} \quad x = \sum_{i=1}^{\infty} a_i x_i.$$

Let E_n be a finite-dimensional subspace of Y containing B_n such that $d(E_n, l_\infty^m) \leqslant \lambda$, where $m = \dim E_n$. The restriction P_n of $Q_n P$ to E_n is a projection from E_n onto B_n with $\|P_n\| \leqslant \|Q_n\|\,\|P\| \leqslant \varrho\|P\|$. Let $\{x_j^*\}_{j=1}^{n} \in B_n^*$ be defined by $x_j^*(x_i) = \delta_i^j$. Clearly

$$\Big\| \sum_{i=1}^{n} \varepsilon_i \beta_i x_i^* \| \leqslant \varrho \Big\| \sum_{i=1}^{n} \beta_i x_i^* \Big\|$$

for every choice of real $\{\beta_i\}$ and signs $\{\varepsilon_i\}$. Since $d(E_n^*, l_1^m) \leqslant \lambda$ and $\|x^*\| \leqslant \|P_n^* x^*\| \leqslant \varrho\|P\|\,\|x^*\|$ for every $x^* \in B_n^*$, it follows from the first part of the proof that

$$C \sum_{i=1}^{n} |\beta_i| \leqslant \Big\| \sum_{i=1}^{n} \beta_i x_i^* \Big\|,$$

where the positive constant C depends only on λ, ϱ and $\|P\|$ (but not on n and $\{\beta_i\}_{i=1}^{n}$). Thus

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\| = \sup_{\|\Sigma\beta_i x_i^*\| \leqslant 1} \Big| \sum_{i=1}^{n} a_i \beta_i \Big| \leqslant \sup_{C\Sigma|\beta_i| \leqslant 1} \Big| \sum_{i=1}^{n} a_i \beta_i \Big| \leqslant C^{-1} \max |a_i|,$$

and this concludes the proof.

Remark. The theorem holds, with the same proof also in the non-separable situation. Thus if X (in the statement of Theorem 6.1) has a normalized unconditional basis $\{x_\gamma\}_{\gamma\in\Gamma}$, then this basis is equivalent to the unit basis of $l_1(\Gamma)$ (resp. $c_0(\Gamma)$).

COROLLARY 1. *All normalized unconditional bases in $l_1(\Gamma)$ (resp. $c_0(\Gamma)$) are equivalent to the unit vector basis in $l_1(\Gamma)$ (resp. $c_0(\Gamma)$).*

Corollary 1 solves a problem raised in [44] (cf. also [48]). In [44] it is shown that in l_p, $1 < p < \infty$, $p \neq 2$, there is a normalized unconditional basis which is not equivalent to the unit vector basis. (For $p = 2$, i.e., for the Hilbert space, it is well known that all normalized unconditional bases are equivalent; see [4] and [13]).

COROLLARY 2. *Every complemented subspace of an $\mathscr{L}_1$-space (resp. $\mathscr{L}_\infty$-space), and in particular every $\mathscr{L}_1$-space (resp. $\mathscr{L}_\infty$-space), with an unconditional basis is isomorphic to $l_1(\Gamma)$ (resp. $c_0(\Gamma)$) for a suitable set Γ.*

Since $L_1(0,1)$ is not isomorphic to $l_1(\Gamma)$, Corollary 2 implies in particular that there is no unconditional basis in $L_1(0,1)$ (cf. [44], [54] and [45] for a slightly stronger result). If K is an infinite compact metric space, then by a result of [5] $C(K)$ is isomorphic to c_0 if and only if K is homeomorphic to the space $[a]$ of all ordinal number $\leqslant a$ (with the order topology) for some ordinal a with $\omega \leqslant a < \omega^\omega$, where ω denotes the first infinite ordinal number. Hence as a special case of Corollary 2 we get.

COROLLARY 3. *Let K be a compact metric space; then $C(K)$ has an unconditional basis if and only if K is homeomorphic to the space $[a]$ for some ordinal $a < \omega^\omega$. In particular, the spaces $C(0,1)$ and $C([\omega^\omega])$ have no unconditional bases.*

Corollary 3 was obtained by the second named author in 1958 in his Ph. D. thesis but the proof of it has not been published. The case of $C(0,1)$ is due to Karlin [30] (cf. also [8], p. 77).

COROLLARY 4. *Let X be a separable infinite dimensional Banach space with an unconditional basis. Then X is complemented in every separable space containing it if and only if X is isomorphic to c_0.*

Proof. If X is isomorphic to c_0, then X is complemented in every separable Banach space containing it by a result of Sobczyk [55] (cf. also [44], p. 217). Conversely, every separable Banach space is isometric to a subspace of the $\mathscr{L}_\infty$-space $C(0,1)$ (see [21]) and hence the desired result follows from Corollary 2.

COROLLARY 5. *Let X be a Banach space with an unconditional basis. Then X is isomorphic to $c_0(\Gamma)$ for a suitable set Γ if and only if X^{**} is a $\mathscr{P}$-space.*

Proof. If X is isomorphic to $c_0(\Gamma)$, then X^{**} is isomorphic to $l_\infty(\Gamma)$ which is a $\mathscr{P}_1$-space ([8], p. 94). Conversely, let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a normalized

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unconditional basis of X and assume that X^{**} is a $\mathscr{P}_\lambda$-space. Let Γ_0 be any finite subset of Γ and let B be the subspace of X spanned by $\{x_\gamma\}_{\gamma\in\Gamma_0}$. By the definition of an unconditional basis there is a projection of norm $\leqslant\varrho$ from X onto B (where ϱ does not depend on Γ_0). Hence, by [35], Corollary 3, p. 16, B is a $\mathscr{P}_{\lambda\varrho}$-space. By embedding B in $C(0,1)$ and the proof of Theorem 6.1 we get that there is a constant M (independent of Γ_0) such that for all a_γ, $\gamma\in\Gamma_0$,

$$M^{-1}\sup_{\gamma\in\Gamma_0}|a_\gamma|\leqslant\Big\|\sum_{\gamma\in\Gamma_0}a_\gamma x_\gamma\Big\|\leqslant M\sup_{\gamma\in\Gamma_0}|a_\gamma|.$$

The set $\{x_\gamma\}_{\gamma\in\Gamma}$ is therefore equivalent to the unit basis of $c_0(\Gamma)$ and, in particular, X is isomorphic to $c_0(\Gamma)$.

In order to state more quantitative versions of Theorem 6.1 let us make the following definitions. If X is a Banach space, the *projection constant* $p(X)$ of X is defined as $\inf\{\lambda;\ X \text{ is a } \mathscr{P}_\lambda\text{-space}\}$ ($p(X)=\infty$ if X is not a $\mathscr{P}$-space). The *symmetry constant* $s(X)$ of X is defined by $\inf\{\varrho;$ there is an unconditional basis $\{x_\gamma\}_{\gamma\in\Gamma}$ in X such that

$$\Big\|\sum_\gamma\varepsilon_\gamma a_\gamma x_\gamma\Big\|\leqslant\varrho\Big\|\sum a_\gamma x_\gamma\Big\|$$

whenever $\varepsilon_\gamma=\pm1$ and $\sum a_\gamma x_\gamma$ converges$\}$.

Again, we put $s(X)=\infty$ if X has no unconditional basis. Some equations relating $p(X)$, $s(X)$ and the distance of X from various spaces were obtained recently (for finite-dimensional spaces X) by Gurariĭ, Kadec and Macaev [20] (they called $s(X)$ the *coordinate asymmetry* of X).

Corollary 6. *Let X be a subspace of an $\mathscr{L}_{1,\lambda}$-space Y and assume that there is a projection P from Y onto X. Then*

$$d\big(X,l_1(\Gamma)\big)\leqslant\lambda^2K_G^2\|P\|s^2(X),$$

where Γ is a set whose cardinality is the density character of X.

Proof. Use the first part of the proof of Theorem 6.1.

Corollary 7. *Let X be a finite-dimensional Banach space* ($\dim X=n$, *say*). *Then*

$$d(X,l_\infty^n)\leqslant K_G^2p^2(X)s^2(X).$$

Proof. Let $I\colon X\to C(0,1)$ be an isometry, let $\varepsilon>0$ and let P be a projection of norm $\leqslant p(X)+\varepsilon$ from $C(0,1)$ onto IX. Then P^*I^* is a projection of the $\mathscr{L}_{1,1+\varepsilon}$-space $C(0,1)^*$ onto P^*X^*. Since $s(X)=s(X^*)$, we get, by Corollary 6,

$$d(P^*X^*,l_1^n)\leqslant K_G^2(1+\varepsilon)^2\big(p(X)+\varepsilon\big)\cdot s^2(X).$$

Hence

$$d(X^*,l_1^n)\,d(X^*,PX^*)\,d(PX^*,l_1^n)\leqslant K_G^2(1+\varepsilon)^2\big(p(X)+\varepsilon\big)^2s^2(X).$$

To complete the proof observe that

$$d(X,l_\infty^n)=d(X^*,l_1^n)$$

and let ε tend to zero.

Remark. The infinite-dimensional version of Corollary 7 is useless since there is no infinite-dimensional $\mathscr{P}$-space with an unconditional basis. (Use e.g. the fact that a $\mathscr{P}$-space has no infinite-dimensional separable complemented subspaces (cf. [14] or [44], p. 222). Hence for every infinite-dimensional Banach space $p(X)\cdot s(X)=\infty$.

The preceding result concerning unconditional bases can be easily generalized to Schauder decompositions of Banach spaces. This notion was introduced by Grünblum [18] and studied by McArthur and his students. Let X be a Banach space and let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a set of closed subspaces of X. The set $\{X_\gamma\}_{\gamma\in\Gamma}$ is said to be an *unconditional Schauder decomposition* of X if every $x\in X$ has a unique representation of the form $x=\sum_\gamma x_\gamma$ with $x_\gamma\in X_\gamma$, $\gamma\in\Gamma$, and if this series converges unconditionally for every $x\in X$. Exactly as in the case of an unconditional basis it follows from the definition that there is a constant ϱ such that

$$\Big\|\sum_\gamma\varepsilon_\gamma x_\gamma\Big\|\leqslant\varrho\Big\|\sum_\gamma x_\gamma\Big\|$$

whenever $\varepsilon_\gamma=\pm1$, $x_\gamma\in X_\gamma$ and $\sum x_\gamma$ converges. If $\{X_\gamma\}_{\gamma\in\Gamma}$ is a set of Banach spaces, then by $(\Sigma\oplus X_\gamma)_1$ (resp. $(\Sigma\oplus X_\gamma)_0$) we denote the direct sum of these spaces in the l_1 (resp. c_0) sense. With this notation we have

Corollary 8. *Let X be an $\mathscr{L}_1$-space* (*resp. $\mathscr{L}_\infty$*)*-space and let $\{X_\gamma\}_{\gamma\in\Gamma}$ be an unconditional Schauder decomposition of X. Then X is isomorphic to $(\Sigma\oplus X_\gamma)_1$* (*resp. $(\Sigma\oplus X_\gamma)_0$*).

Proof. The proof is very similar to the proof of Theorem 6.1. We shall sketch only the proof in the $\mathscr{L}_1$-case. For every $\gamma\in\Gamma$ let $x_\gamma^*\in X_\gamma^*$ be a functional with norm 1. For every $x=\sum_\gamma x_\gamma$ in X we have

$$\Big(\sum_\gamma|x_\gamma^*(x_\gamma)|^2\Big)^{1/2}\leqslant\Big(\sum_\gamma\|x_\gamma\|^2\Big)^{1/2}\leqslant M_1\|x\|,$$

where M_1 is a constant which depends only on the constant ϱ of the decomposition and the λ for which X is an $\mathscr{L}_{1,\lambda}$-space. Hence, the operator $T\colon X\to l_2(\Gamma)$ defined by $T\big(\sum_\gamma x_\gamma\big)(\gamma)=x_\gamma^*(x_\gamma)$ is of norm $\leqslant M_1$. By Theorem 4.1

$$\sum_{\gamma\in\Gamma}|x_\gamma^*(x_\gamma)|=\sum_{\gamma\in\Gamma}\|Tx_\gamma\|\leqslant M_2\|x\|,\qquad x=\sum_{\gamma\in\Gamma}x_\gamma,$$

where again M_2 is a constant depending only on λ and ϱ. Since, in particular, M_2 does not depend on the choice of the x_γ^*, $\gamma\in\Gamma$, we get

$$\sum_\gamma \|x_\gamma\| = \sup\Big\{\sum_\gamma |x_\gamma^*(x_\gamma)|;\ x_\gamma^*\in X_\gamma^*,\ \|x_\gamma^*\|=1\Big\} \leqslant M_2\|x\|,$$

and this concludes the proof.

In our next application of the results of the preceding sections we shall consider the complex Banach space $L_1(\mu)$, where μ is the Haar measure on the circle $\{z;\ |z|=1\}$. Let H_1 be the closure of the polynomials $\sum_{k=0}^n a_k z^k$ in L_1 (cf. [23]). We prove first

Proposition 6.1. *There is an operator T in $B(H_1, l_2)$ which is not absolutely summing.*

Proof. For $f\in H_1$ with

$$f(z) = \sum_{k=0}^\infty a_k z^k \quad \text{for} \quad |z|<1$$

we put

$$Tf = \left\{\frac{a_k}{\sqrt{k}}\right\}_{k=1}^\infty .$$

By a theorem of Hardy ([23], p. 70),

$$\sum_{k=1}^\infty |a_k|/k \leqslant \pi\|f\|.$$

Since

$$a_k = \frac{1}{2\pi i}\int \frac{f(z)}{z^k}\,dz$$

we get $|a_k|\leqslant\|f\|$, and hence

$$\sum_{k=1}^\infty |a_k|^2/k \leqslant \pi\|f\|^2.$$

Therefore $T\in B(H_1, l_2)$ and $\|T\|\leqslant\sqrt{\pi}$. Let

$$g(z) = \sum_{k=1}^\infty \frac{1}{\sqrt{k}\,ln(k+1)} z^k.$$

Since

$$\sum_k \big(\sqrt{k}\,ln(k+1)\big)^{-2} < \infty,$$

the series

$$\sum_k \big(\sqrt{k}\,ln(k+1)\big)^{-1} z^k$$

converges unconditionally in H_2 and therefore in H_1. However,

$$\sum_{k=1}^\infty \big\|T\big(\sqrt{k}\,ln(k+1)\big)^{-1}z^k\big\| = \sum_{k=1}^\infty \big(k\,ln(k+1)\big)^{-1} = \infty$$

and therefore T is not absolutely summing.

From Theorem 4.1 and Proposition 6.1 we immediately get the following result of D. J. Newman (cf. [23], p. 154):

Corollary 1. *Every isomorphic image of H_1 in an arbitrary $\mathscr{L}_1$-space X is uncomplemented.*

Proof. The operator T from H_1 to l_2 of Proposition 6.1 does not have (by Theorem 4.1) an extension to an operator from X to l_2.

We pass now to applications centered around properties of Hilbert spaces. The first is Grothendieck's characterization of Hilbert spaces ([17], Proposition 5, p. 66).

Theorem 6.2. *A Banach space X is isomorphic to a Hilbert space if and only if it is isomorphic to a subspace of an $\mathscr{L}_1$-space and to a quotient space of an $\mathscr{L}_\infty$-space.*

Proof. If X is a Hilbert space, then X is isomorphic to a subspace of an $L_1(\mu)$-space. This fact is well known, at least for a separable Hilbert space, since the subspace of $L_1(0,1)$ spanned by the Rademacher functions is isomorphic to l_2 (see [27] for details and further references). In Section 7 (Corollary 1 to Proposition 7.5) we shall present a proof of this fact in the general case. It follows that $X = X^*$ is a quotient space of the $\mathscr{L}_\infty$-space $L_1^*(\mu)$ and this proves one part of the theorem.

We pass to the converse. Let Y be an $\mathscr{L}_1$-space containing X and let T be an operator from an $\mathscr{L}_\infty$-space Z onto X. The operator T considered as an operator from Z into Y is by Theorem 5.1 a Hilbertian operator. Hence there is an operator from a Hilbert space onto X. Thus X is isomorphic to a quotient space of a Hilbert space and it is therefore itself isomorphic to a Hilbert space.

Corollary 1. *Let X be a Banach space such that X and X^* are both isomorphic to subspaces of $\mathscr{L}_1$-spaces. Then X is isomorphic to a Hilbert space.*

Proof. By Proposition 7.1 every $\mathscr{L}_1$-space is isomorphic to a subspace of an $L_1(\mu)$-space. Since the dual of $L_1(\mu)$ is an $\mathscr{L}_\infty$-space, it follows from Theorem 6.2 that X^*, and hence X, is isomorphic to a Hilbert space.

Corollary. 2 *Let X be a Banach space such that X and and X^* are both quotient spaces of $\mathscr{L}_\infty$-spaces. Then X is isomorphic to a Hilbert space.*

Proof. Use Theorem 6.2 and the fact that the dual of an $\mathscr{L}_\infty$-space is isomorphic to a subspace of an $L_1(\mu)$-space (cf. Proposition 7.4)

Remark. Theorem 6.2 and its Corollaries remain valid if we replace everywhere $\mathscr{L}_1$-spaces by $\mathscr{L}_r$-spaces and $\mathscr{L}_\infty$-spaces by $\mathscr{L}_p$-spaces, where $1 \leqslant r \leqslant 2$ and $2 \leqslant p \leqslant \infty$. The general case reduces to the case of $\mathscr{L}_1$-spaces and $\mathscr{L}_\infty$-spaces via results of Section 7 in the same way as in the proof of Theorem 5.2.

The next theorem gives several characterizations of Hilbert-Schmidt operators. For the basic facts concerning these operators the reader may consult the books [10] or [53].

THEOREM 6.3. *Let H_1 and H_2 be Hilbert spaces and let $T \in B(H_1, H_2)$. Then the following assertions are equivalent:*

(1) *T is a Hilbert-Schmidt operator.*

(2) *T has the lifting property, i.e. for every Banach space Y and every epimorphism $U: Y \to H_2$ there is an operator $S: H_1 \to Y$ such that $T = US$.*

(3) *T admits a factorization through $l_1(\Gamma)$.*

(4) *T admits a factorization through some $\mathscr{L}_1$-space.*

(5) *T is absolutely summing.*

(6) *T admits a factorization through $c_0(\Gamma)$.*

(7) *T admits a factorization through an $\mathscr{L}_\infty$-space.*

(8) *T is 2-absolutely summing.*

(9) *T has the extension property, i.e. for every Banach space X and every isomorphism $S: H_1 \to X$ there is an operator $U: X \to H_2$ such that $T = US$.*

Proof. (1) $\Rightarrow$ (2). Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis in H_1. Since T is a Hilbert-Schmidt operator, we have $\sum_\gamma \|Te_\gamma\|^2 < \infty$. By the open mapping theorem there exist $\{y_\gamma\}_{\gamma\in\Gamma}$ in Y such that $\sum_\gamma \|y_\gamma\|^2 < \infty$ and $Uy_\gamma = Te_\gamma$, $\gamma \in \Gamma$. Let $S: H_1 \to Y$ be defined by $Sx = \sum(x, e_\gamma) y_\gamma$, $x \in X$. It is easily checked that S has the desired properties

(2) $\Rightarrow$ (3). This is a consequence of the fact [3] that every Banach space is a quotient space of $l_1(\Gamma)$ for a suitable Γ.

(3) $\Rightarrow$ (4). This implication is obvious.

(4) $\Rightarrow$ (5). This implication is a consequence of Theorem 4.1.

(5) $\Rightarrow$ (1). Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis in H_1. Then the series $\sum_\gamma a_\gamma e_\gamma$ is unconditionally convergent whenever $\sum_\gamma |a_\gamma|^2 < \infty$. Hence, since T is absolutely summing, $\sum_\gamma |a_\gamma|\, \|Te_\gamma\|$ converges whenever $\sum_\gamma |a_\gamma|^2 < \infty$ and therefore $\sum_\gamma \|Te_\gamma\|^2 < \infty$.

(1) $\Rightarrow$ (6). Since T is a Hilbert-Schmidt operator, it is compact. Hence there is an orthonormal basis $\{e_\gamma\}_{\gamma\in\Gamma}$ in H_1 such that $(Te_{\gamma_1}, Te_{\gamma_2}) = 0$ for $\gamma_1 \neq \gamma_2$ (cf. e.g. [53], Section 14). Define $S: H_1 \to c_0(\Gamma)$ by $Sx(\gamma) = (x, e_\gamma)$, $x \in H_1$, $\gamma \in \Gamma$. Define $U: c_0(\Gamma) \to H_2$ by $U(\{a_\gamma\}_{\gamma\in\Gamma}) = \sum_\gamma a_\gamma Te_\gamma$ for $\{a_\gamma\}_{\gamma\in\Gamma}$ in $c_0(\Gamma)$. Since the vectors $\{Te_\gamma\}_{\gamma\in\Gamma}$ are mutually orthogonal and $\sum_\gamma \|Te_\gamma\|^2 < \infty$, U is a well defined operator. Clearly $US = T$.

(6) $\Rightarrow$ (7). This implication is obvious.

(7) $\Rightarrow$ (8). This implication is a consequence of Theorem 4.3.

(8) $\Rightarrow$ (9). It follows from the corollary to Proposition 3.1 that there is a probability measure μ on the unit cell of H_1 and operators $S_1: H_1 \to L_\infty(\mu)$ and $U_1: L_\infty(\mu) \to H_2$ such that $T = U_1 S_1$. Since $L_\infty(\mu)$ is a $\mathscr{P}_1$-space, there is an operator $\tilde{U}: X \to L_\infty(\mu)$ such that $\tilde{U}S = S_1$. The operator $U = U_1\tilde{U}$ has the desired properties.

(9) $\Rightarrow$ (4). Let S be an isomorphic embedding of H_1 into an $\mathscr{L}_1$-space X. Then, by (9), there is an operator $U: X \to H_2$ such that $T = US$ and thus we get a factorization of T through an $\mathscr{L}_1$-space.

Remark. Almost all the implications in Theorem 6.3 are contained in Grothendieck's paper [17] (see in particular Theorem 6 on p. 55). Pietsch [50] and [51] proved the equivalence of (1), (5) and (8). These equivalences imply that T is a Hilbert-Schmidt operator if and only if T is p-absolutely summing for some $p \leqslant 2$. The same is true for $p > 2$ (cf. [47]).

An operator $T: X \to Y$ is called *nuclear* if it can be represented in the form

$$Tx = \sum_{i=1}^{\infty} y_i^*(x) y_i$$

with $\{y_i\}_{i=1}^\infty \subset Y$, $\{y_i^*\}_{i=1}^\infty \subset Y^*$ and $\Sigma\|y_i^*\|\,\|y_i\| < \infty$ (cf. [50] for a discussion of the properties of these operators which were introduced by Grothendieck [15]).

COROLLARY 1. *Let X_i $(i = 1, 2, 3)$ be Banach spaces and let $T_1: X_1 \to X_2$, $T_2: X_2 \to X_3$ be both 2-absolutely summing operators. Then T_2T_1 is nuclear.*

By the Corollary to Proposition 3.1 there are compact Hausdorff spaces K_1 and K_2 and probability measures μ_1 and μ_2 on K_1 and K_2 respectively such that $T_i = S_iJ_iI_i$ $(i = 1, 2)$, where $I_i: X_i \to C(K_i)$ are isometries, $J_i: C(K_i) \to L_2(\mu_i)$ are the formal identity maps and $S_i: L_2(\mu_i) \to X_{i+1}$ are suitable bounded operators. We have thus the following situation:

$$X_1 \xrightarrow{I_1} C(K_1) \xrightarrow{J_1} L_2(\mu_1) \xrightarrow{S_1} X_2 \xrightarrow{I_2} C(K_2) \xrightarrow{J_2} L_2(\mu_2) \xrightarrow{S_2} X_3.$$

By (7) $\Rightarrow$ (1) of Theorem 6.3 the operator $J_2I_2S_1$ is a Hilbert-Schmidt operator. Hence, by [50], Satz. 2, p. 56, $J_2I_2S_1J_1$ is nuclear and consequently $T_2T_1 = S_2(J_2I_2S_1J_1)I_1$ is also nuclear.

Remark. This corollary is in [17] (Corollaire, p. 34), see also Pietsch [51], Theorem 6.

Corollary 2. *Let H_1 and H_2 be Hilbert spaces and let $T \in B(H_1, H_2)$. Then the following assertions are equivalent:*

(1) *T is nuclear.*

(2) *T admits a factorization of the form $T: H_1 \to c_0(\Gamma) \to l_1(\Gamma) \to H_2$.*

(3) *T admits a factorization of the form $T: H_1 \to X \to Y \to H_2$, where X is an $\mathscr{L}_\infty$-space and Y is an $\mathscr{L}_1$-space.*

Proof. The corollary follows immediately from Theorem 4.3, Theorem 6.3 and the well known fact that an operator $T \in B(H_1, H_2)$ is nuclear if and only if it is a product of two Hilbert-Schmidt operators (cf. e.g. [53], Section III, 1).

7. SUBSPACES OF $L_p(\mu)$-SPACES

We show first that for a Banach space X the property of being isomorphic to a subspace of an $L_p(\mu)$-space is a local property, i.e., depends only on the finite-dimensional subspaces of X.

Proposition 7.1. *Let X be a Banach space, let $1 \leqslant p \leqslant \infty$ and let $\lambda \geqslant 1$. Assume that for every finite-dimensional subspace E of X there is a subspace $\tilde{E}$ of l_p such that $d(E, \tilde{E}) \leqslant \lambda$. Then there is a measure μ and a subspace Y of $L_p(\mu)$ such that $d(X, Y) \leqslant \lambda$.*

Proof. Since every Banach space is isometric to a subspace of an $L_\infty(\mu)$-space (e.g. an $l_\infty(\Gamma)$ for a suitable Γ), the proposition is trivial if $p = \infty$. We assume from now on that $p < \infty$.

Let U^* be the unit ball $\{x^*; \|x^*\| \leqslant 1\}$ of X and let $B(U^*)$ be the space of real-valued bounded (not necessarily continuous) functions on U^*. For $x \in X$ let $f_x \in B(U^*)$ be defined by $f_x(x^*) = x^*(x)$, $x^* \in U^*$. Let E be a subspace of X with $\dim E = n < \infty$ and let $T_E: E \to l_p$ be such that

$$\lambda^{-1}\|x\| \leqslant \|T_E x\| \leqslant \|x\|$$

for every $x \in E$. Since $T_E E$ is a finite-dimensional subspace of l_p, there exists an integer m such that

$$\|P_m T_E x\| \geqslant \left(1 - \frac{1}{n}\right)\|T_E x\|$$

for every $x \in E$ (P_m denotes the projection of l_p onto its subspace generated by the first m basis vectors). Thus $\tilde{T}_E = P_m T_E$ is an operator from E into l_p^m such that

$$\lambda^{-1}\left(1 - \frac{1}{n}\right)\|x\| \leqslant \|\tilde{T}_E x\| \leqslant \|x\|, \qquad x \in E.$$

Let $\{\xi_i\}_{i=1}^m$ be the usual basis of $l_q^m = (l_p^m)^*$ $(p^{-1} + q^{-1} = 1)$. Put

$$\varphi_E f = \sum_{i=1}^{m} f(\tilde{T}_E^* \xi_i), \qquad f \in B(U^*). \tag{7.1}$$

The functional φ_E is clearly linear and positive (i.e. $f \geqslant 0 \Rightarrow \varphi_E f \geqslant 0$). For every $x \in E$

$$\varphi_E |f_x|^p = \sum_{i=1}^{m} |\tilde{T}_E^* \xi_i(x)|^p = \sum_{i=1}^{m} |\xi_i(\tilde{T}_E x)|^p = \|\tilde{T}_E x\|^p, \tag{7.2}$$

and hence

$$\frac{1}{\lambda}\left(1 - \frac{1}{n}\right)\|x\| \leqslant (\varphi_E |f_x|^p)^{1/p} \leqslant \|x\|, \quad x \in E, \dim E = n.$$

Let $\tilde{R}$ be the one point compactification of the reals and let

$$\Pi = \prod_{f \in B(U^*)} \tilde{R}_f$$

be the product of $B(U^*)$ copies of $\tilde{R}$. For every finite-dimensional subspace E of X let $\pi_E \in \Pi$ be defined by $\pi_E(f) = \varphi_E f$. Since, by Tychonoff's theorem, Π is compact, the net $\{\pi_E\}$ (the spaces E are ordered by inclusion) has a subnet converging to a limit point π. Let

$$Z = \{f; f \in B(U^*), \pi(|f|^p) \text{ is finite (i.e. not } \infty)\}.$$

Then

(i) Z is a linear subspace and sublattice of $B(U^*)$ and, moreover, $f \in Z$, $g \in B(U^*)$, $|g| \leqslant |f| \Rightarrow g \in Z$.

(ii) $|||f||| = (\pi|f|^p)^{1/p}$ is a semi-norm on Z which has the property that

$$\min(|f(x^*)|, |g(x^*)|) = 0$$

for every $x^* \in U^* \Rightarrow |||f \pm g|||^p = |||f|||^p + |||g|||^p$.

(iii) For every $x \in X$, f_x belongs to Z and $\lambda^{-1}\|x\| \leqslant |||f_x||| \leqslant \|x\|$ (use (7.3)).

By (i) and (ii) and the characterization of $L_p(\mu)$-spaces given by Nakano ([41], cf. also [6]) the completion $\tilde{Z}$ of $Z/\{f; |||f||| = 0\}$ is isometric to an $L_p(\mu)$-space for some measure μ. The operator $\tilde{T}$ sending $x \in X$ into the class determined by f_x in $\tilde{Z}$ satisfies, by (iii),

$$\lambda^{-1}\|x\| \leqslant |||\tilde{T}x||| \leqslant \|x\|$$

and this concludes the proof.

Remark. Proposition 7.1 is essentially known. For $p = 2$ the argument of Lemma 3 of [34] provides a proof of this proposition (begin the transfinite induction one step earlier, i.e., from the finite-dimensional case instead of the separable case). An explicit statement of Proposition 7.1 for $p = 2$ is given in [25]. For a general p but $\lambda = 1$ the proposition is proved in [7]. The proof of [7] (cf. also [57]) can be modified so that it will hold for a general λ. Like the proof given here, the proofs in the papers mentioned above are essentially a combination of a compactness argument with an application of an isometric characterization of $L_p(\mu)$-spaces.

COROLLARY 1. *Let X be an $\mathscr{L}_{2,\lambda}$-space. Then X is isomorphic to a Hilbert space. Precisely, there exists a Hilbert space H such that $d(X, U) \leqslant \lambda$.*

COROLLARY 2. *Let X be a separable Banach space which satisfies the assumptions of Proposition 7.1. Then there is a subspace Y of $L_p(0,1)$ such that $d(X, Y) \leqslant \lambda$.*

Proof. By [9], Lemma 5, p. 168, every separable subspace of an $L_p(\mu)$-space, $1 \leqslant p < \infty$, is isometric to a subspace of a separable $L_p(\nu)$-space. Every separable $L_p(\nu)$-space is isometric to a subspace of $L_p(0,1)$ (use [21], Theorem C, p. 173). Corollary 2 immediately follows from these facts and Proposition 7.1.

COROLLARY 3. *Let X be a Banach space, let $1 \leqslant p < \infty$ and let $\lambda \geqslant 1$. Assume that for every $\varepsilon > 0$ there is a measure $\mu(\varepsilon)$ and a subspace $Y = Y(\varepsilon)$ of $L_p(\mu(\varepsilon))$ such that $d(X, Y) \leqslant \lambda + \varepsilon$. Then there is measure μ and a subspace Y of $L_p(\mu)$ such that $d(X, Y) \leqslant \lambda$.*

Proof. It follows from the assumption that for every $\varepsilon > 0$ and every finite-dimensional subspace E of X there is an operator $T_E: E \to l_p$ such that

$$\|T_E x\| \leqslant \|x\| \leqslant (\lambda + \varepsilon)\|T_E x\|, \quad x \in E.$$

We may now proceed exactly as in the proof of Proposition 7.1.

For $\mathscr{L}_p$-spaces X a stronger result than Proposition 7.1 can be obtained.

THEOREM 7.1. *Let $1 < p < \infty$ and let X be an $\mathscr{L}_p$-space. Then there is a measure μ and a complemented subspace Y of $L_p(\mu)$ which is isomorphic to X.*

Proof. Let X be an $\mathscr{L}_{p,\lambda}$-space. By Proposition 7.1, X is isomorphic to a subspace of an $L_p(\mu)$-space and hence, in particular, X is reflexive. We are going to show that by the construction described in the proof of Proposition 7.1 we get that X is isomorphic to a complemented subspace of the $L_p(\mu)$-space Z. Let U^*, $B(U^*)$ and f_x $(x \in X)$ have the same meaning as in the proof of Proposition 7.1. We consider now only those finite-dimensional subspaces E of X for which

$$d(E, l_p^n) \leqslant \lambda \quad \text{where} \quad n = \dim E. \tag{7.4}$$

By our assumption on X the set of those spaces E for which (7.4) holds is ordered by inclusion. For every such E let $T_E: E \to l_p^n$ satisfy

$$\lambda^{-1}\|x\| \leqslant \|T_E x\| \leqslant \|x\|, \quad x \in E,$$

and define the functional φ_E on $B(U^*)$ by

$$\varphi_E f = \sum_{i=1}^{n} f(T_E^* \xi_i),$$

where $\{\xi_i\}_{i=1}^n$ is the usual basis of $(l_p^n)^*$. By using in the proof of Proposition 7.1 only those spaces E for which (7.4) holds, we construct the spaces Z and $\tilde{Z}$. Thus Z consists of all $f \in B(U^*)$ for which

$$|||f|||^p = \lim_{\gamma} \varphi_{E_\gamma} |f|^p < \infty, \tag{7.5}$$

where $\{E_\gamma\}$ is a net of subspaces satisfying (7.4) which is directed by inclusion, and $\tilde{Z}$ is the completion of $Z/\{f;\ |||f||| = 0\}$.

For every E satisfying (7.4) let $P_E: B(U^*) \to E$ be defined by

$$P_E f = T_E^{-1}\Big(\sum_{i=1}^{n} f(T_E^* \xi_i)\eta_i\Big),$$

where $\{\eta_i\}_{i=1}^n$ is the usual basis in l_p^n. Then, for $x \in E$,

$$P_E f_x = T_E^{-1}\Big(\sum_{i=1}^{n} T_E^* \xi_i(x)\cdot\eta_i\Big) = T_E^{-1}\Big(\sum_{i=1}^{n} \xi_i(T_E x)\eta_i\Big) = T_E^{-1}(T_E x) = x. \tag{7.6}$$

Also, since $\|T_E^{-1}\| \leqslant \lambda$ and $\Big\|\sum_{i=1}^{n} f(T_E^* \xi_i)\eta_i\Big\|^p = \varphi_E |f|^p$,

$$\|P_E f\| \leqslant \lambda(\varphi_E |f|^p)^{1/p}, \quad f \in B(U^*). \tag{7.7}$$

Hence, by (7.5),

$$\overline{\lim_{\gamma}} \|P_{E_\gamma} f\| < \infty \quad \text{if} \quad f \in Z.$$

Therefore, since every bounded set in the reflexive space X is w-conditionally compact, we infer by Tichonoff's theorem that there is a subnet $\{E_{\gamma'}\}$ of $\{E_\gamma\}$ such that

$$Pf = \lim_{\gamma'} P_{E_{\gamma'}} f$$

exists in the w-topology for every $f \in Z$. Clearly, P is a linear map from Z into X. By (7.6) we infer that $Pf_x = x$ for every $x \in X$ and, by (7.5) and (7.7), $\|P\| \leqslant \lambda$. By passing from Z to $\tilde{Z}$ we get from P an operator $\tilde{P}: \tilde{Z} \to X$ such that $\|\tilde{P}\| \leqslant \lambda$ and $\tilde{P}\tilde{T}x = x$ for every $x \in X$. Here $\tilde{T}: X \to \tilde{Z}$ is the operator appearing at the end of the proof of Proposition 7.1, namely the operator mapping every x to the equivalence class of f_x in $\tilde{Z}$. Hence

$\tilde{T}\tilde{P}$ is a projection from the $L_p(\mu)$-space $\tilde{Z}$ onto its subspace $\tilde{T}X$ which is isomorphic to X. This completes the proof.

In the next section we shall show that Theorem 7.1 does not hold if $p = 1$ and it is very easily seen that it fails also if $p = \infty$. (A separable infinite-dimensional $C(K)$-space is not a $\mathscr{P}$-space and hence not complemented in an $L_\infty(\mu)$-space.)

Some information concerning $\mathscr{L}_p$-spaces for those values of p is contained in

Corollary 1. *Let $1 \leqslant p \leqslant \infty$ and let X be an $\mathscr{L}_p$-space. Then X is isomorphic to a complemented subspace of an $L_p(\mu)$-space for some measure μ if and only if X is complemented in X^{**}.*

Proof. For $1 < p < \infty$ the corollary is equivalent to Theorem 7.1. Assume now that $p = \infty$. Since an $L_\infty(\mu)$-space is a $\mathscr{P}_1$-space, it follows that every complemented subspace X of an $L_\infty(\mu)$-space is a $\mathscr{P}$-space and hence is complemented in X^{**}. Conversely, if X is an $\mathscr{L}_\infty$-space and is complemented in X^{**}, then by [35], p. 28, X is a $\mathscr{P}$-space and hence, in particular, is isometric to a complemented subspace of $l_\infty(\Gamma)$ for a suitable Γ.

We turn to the case $p = 1$. Since every $L_1(\mu)$-space is complemented in its second dual, it follows that every complemented subspace of an $L_1(\mu)$-space is complemented in its second dual (cf. [15], p. 101, or [35], p. 16). Conversely, assume that X is an $\mathscr{L}_1$-space and that there is a projection Q from X^{**} onto X. In the proof of Theorem 7.1 we used the fact that $1 < p < \infty$ only in the proof of the existence of the limit P of the mappings P_{E_γ}. We can avoid using the reflexivity of X if we embed X in X^{**} and use in X^{**} the w^*-topology. Then the proof of Theorem 7.1 for $p = 1$ (and this argument can be used also for $p = \infty$) will give an operator $\tilde{P}\colon \tilde{Z} \to X^{**}$ such that $\tilde{P}\tilde{T}x = x$ for $x \in X \subset X^{**}$. Hence $\tilde{T}Q\tilde{P}$ will be a projection from $\tilde{Z}$ onto $\tilde{T}X$.

Corollary 2. *Let X be an $\mathscr{L}_1$-space; then X^* is a $\mathscr{P}$-space.*

Proof. As observed in the proof of Corollary 1, there is an $L_1(\mu)$-space Z and operators $T\colon X \to Z$ and $P\colon Z \to X^{**}$ such that PT is the canonical embedding J_0 of X in X^{**}. Let J_1 be the canonical embedding of X^* in X^{***}, and consider the operators

$$X^* \xrightarrow{J_1} X^{***} \xrightarrow{P^*} Z^* \xrightarrow{T^*} X^*.$$

Then $T^*P^*J_1 = J_0^*J_1$ and $J_0^*J_1$ is, as well known and easily checked, the identity mapping of X^*. Hence X^* is isomorphic to the complemented subspace $P^*J_1X^*$ of Z^*. Since Z^* is a $\mathscr{P}_1$-space, the result follows.

Corollary 3. *Let X be a separable infinite-dimensional $\mathscr{L}_1$-space. Then X^* is isomorphic to l_∞.*

Proof. This follows from Corollary 2 and [44], Corollary 6, p. 222.

The next corollaries deal with spaces which are $\mathscr{L}_{p,1+\varepsilon}$-spaces for every $\varepsilon > 0$. For those spaces we can say much more than for general $\mathscr{L}_p$-spaces.

Corollary 4. *Let $1 < p < \infty$. A separable Banach space X is isometric to an $L_p(\mu)$-space for some measure μ if and only if X is an $\mathscr{L}_{p,1+\varepsilon}$-space for every $\varepsilon > 0$.*

Proof. We remarked already in Section 3 that an $L_p(\mu)$-space is an $\mathscr{L}_{p,1+\varepsilon}$-space for every $\varepsilon > 0$. Assume now that X is an $\mathscr{L}_{p,1+\varepsilon}$-space for every $\varepsilon > 0$. It follows from the proof of Theorem 7.1 that there is a measure μ and a subspace Y of $L_p(\mu)$ such that Y is isometric to X and there is a projection of norm 1 from $L_p(\mu)$ onto Y. Since X is separable we may assume that μ is a finite measure (use [9], Lemma 5, p. 168, and the fact that whenever an $L_p(\mu)$-space is separable μ is σ-finite and hence $L_p(\mu)$ is isometric to $L_p(\mu')$ for some finite measure μ'). By the results of Ando [1], Theorem 4, there is a measure ν such that Y (and hence X) is isometric to $L_p(\nu)$.

Remarks. (1) The assumption that X is separable can very probably be removed. Ando deals in [1] only with $L_p(\mu)$-spaces with μ finite. His Theorem 4 seems to be true even for general μ. However, the reduction of the general case to the case of finite μ is not straightforward and we did not work it out.

(2) M. Zippin in his Ph. D. thesis, which is being prepared at the Hebrew University, has proved the following result:

Let $1 \leqslant p < \infty$. A Banach space X is isometric to an $L_p(\mu)$-space for some measure μ if and only if there is a net $\{E_\gamma\}$ of finite-dimensional subspaces of X, directed by inclusion, such that $\bigcup_\gamma E_\gamma$ is dense in X and every E_γ is isometric to l_p^n with $n = \dim E_\gamma$. For $p > 1$ and X separable this is a weaker version of Corollary 4. For $p = 1$ Zippin's result is contained in

Corollary 5. *A Banach space X is isometric to an $L_1(\mu)$-space for some measure μ if and only if X is an $\mathscr{L}_{1,1+\varepsilon}$-space for every $\varepsilon > 0$.*

Proof. Let X be an $\mathscr{L}_{1,1+\varepsilon}$-space for every $\varepsilon > 0$. By the proof of Corollary 2 it easily follows that X^* is a $\mathscr{P}_1$-space. Hence, by a result of Grothendieck [16], X is isometric to an $L_1(\mu)$-space. This proves one direction of the assertion of Corollary 5. The other direction is trivial.

Let us mention that in [32] the analogue of Corollaries 4 and 5 for $p = \infty$ was obtained: A Banach space X is an $\mathscr{L}_{\infty,1+\varepsilon}$-space for every $\varepsilon > 0$ if and only if X^* is isometric to an $L_1(\mu)$-space.

Remark. There exist no infinite-dimensional $\mathscr{L}_{p,1}$-space if $1 \leqslant p < \infty, p \neq 2$. This immediately follows from Corollaries 4 and 5 and the fact that the space l_p is not an $\mathscr{L}_{p,1}$-space (if $p \neq 2$). There exist, however, infinite-dimensional $\mathscr{L}_{\infty,1}$-spaces. It was proved in [35], p. 100-101, that X is an $\mathscr{L}_{\infty,1}$-space if and only if X^* is an $L_1(\mu)$-space and the unit cell of every finite-dimensional subspace of X is a polyhedron. The simplest such space is c_0. More general examples are given in [35], p. 103.

The problem of a functional representation of general $\mathscr{L}_p$-spaces is still open. The next three propositions give some further information on $\mathscr{L}_p$-spaces.

PROPOSITION 7.2. *Let X be an $\mathscr{L}_{p,\lambda}$-space with $1 < p < \infty$ and let Y be a separable subspace of X. Then there exists a separable subspace Z of X containing Y such that Z is an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$ and such that there is a projection of norm 1 from X onto Z.*

Proof. We show first that there exists a separable subspace Z_1 of X containing Y which is an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$ (this part of the proof is valid also for $p = 1$ and ∞). Let $\{y_i\}_{i=1}^{\infty}$ be a dense sequence in Y. For every finite subset σ of the integers choose a finite-dimensional subspace E_σ of X such that $E_\sigma \supset \{y_i\}_{i\in\sigma}$ and $d(E_\sigma, l_p^n) \leqslant \lambda$, where $n = \dim E_\sigma$. Let Y_1 be the closed subspace of X spanned by $\bigcup_\sigma E_\sigma$. Cleary,I Y_1 is separable. Using Y_1, we construct next a subspace Y_2 of X in the same way as Y_1 was obtained from Y. Continuing inductively we get an increasing sequence $\{Y_n\}_{n=1}^{\infty}$ of separable closed subspaces of X. It is easily verified that $Z_1 = \overline{\bigcup_n Y_n}$ is an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$.

Since we assume that $1 < p < \infty$ we infer by Proposition 7.1 that X is reflexive. By the result of [37] it follows that there is a separable subspace Z_2 of X containing Z_1 such that there is a projection P_2 of norm 1 from X onto Z_2. Let next Z_3 be a separable subspace of X containing Z_2 which is an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$ and let $Z_4 \supset Z_3$ be a separable isubspace of X on which there is a projection P_4 with norm 1. Continuing enductively we get an increasing sequence $\{Z_n\}_{n=1}^{\infty}$ of separable subspaces f X such that Z_{2n+1} is an $\mathscr{L}_{p,\lambda+\varepsilon}$ space for every $\varepsilon > 0$ and every integer o and such that there is a projection P_{2n} of norm 1 from X onto Z_{2n} for nvery n. The space

$$Z = \overline{\bigcup_{n=1} Z_n}$$

has the properties required in the statement of the proposition (any limiting point P of the sequence $\{P_{2n}\}_{n=1}^{\infty}$ in the w-operator topology is a projection of norm 1 from X onto Z).

Remarks. Proposition 7.2 fails obviously to hold if $p = \infty$. It is very likely that it still holds if $p = 1$. If Y is not a separable subspace of X and $1 < p < \infty$, then the same proof as that of Proposition 7.2 shows that there is a subspace $Z \supset Y$ of X such that Z is an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$, Z has the same density character as Y and there is a projection of norm 1 from X onto Z.

PROPOSITION 7.3. *Let X be an infinite-dimensional $\mathscr{L}_p$-space with $1 \leqslant p < \infty$. Then X has a complemented subspace isomorphic to l_p.*

Proof. Assume first that $1 < p < \infty$. By Proposition 7.2 we can assume that X is separable. The desired result follows now from [27], Corollary 3, p. 168, and Theorem 7.1 (use the argument of the proof of Corollary 2 to Proposition 7.1).

Now let $p = 1$. By Proposition 7.1, X is isomorphic to a subspace Y of $L_1(\mu)$ for some measure μ. Since Y is not reflexive (this follows e.g. from Corollary 2 to Theorem 7.1), Y has a separable non-reflexive subspace Y_0. By [9], Lemma 5, p. 168, there is a separable subspace Z of $L_1(\mu)$ which contains Y_0 and which is isometric to $L_1(\nu)$ for some measure ν. From the construction of Z it follows that there is a projection of norm 1 from $L_1(\mu)$ onto Z (a conditional expectation operator; cf. [1] for details). By [27], Theorem 6, there is a subspace Y_1 of Y_0 which is isomorphic to l_1 and which is complemented in Z. Since Z is complemented in $L_1(\mu)$, Y_1 is also complemented in $L_1(\mu)$ and thus also in Y. This concludes the proof of the proposition.

PROPOSITION 7.4. *Let X be an $\mathscr{L}_\infty$-space. Then X^* is isomorphic to a complemented subspace of an $L_1(\mu)$-space for some measure μ.*

Proof. By [35], Theorems 2.1 and 3.3, X^{**} is a $\mathscr{P}$-space and is therefore isomorphic to a complemented subspace of a $C(K)$-space for some compact Hausdorff K. Hence X^{***} is isomorphic to a complemented subspace of $C(K)^*$ which is an $L_1(\mu)$-space. Since the canonical embedding of X^* in X^{***} is a complemented subspace of X^{***} (the projection being J^*, where $J: X \to X^{**}$ is the canonical embedding of X in X^{**}), the desired result follows.

We state now without proof the following result which was used already in Section 5. This result is contained implicitely in Levy [33] (cf. also Herz [22] and [7]), and in the context of Banach space theory it seems to appear first in Kadec [26].

THEOREM 7.2. *Let $1 \leqslant p \leqslant r \leqslant 2$. Then for every integer n, l_r^n is isometric to a subspace of $L_p(0,1)$.*

COROLLARY 1. *Let $1 \leqslant p \leqslant r \leqslant 2$. Then $L_r(0,1)$ is isometric to a subspace of $L_p(0,1)$.*

Proof. Use Corollary 2 to Proposition 7.1 and Theorem 7.2.

Remark. This corollary together with results of Banach and Mazur [2], Paley [43] and Kadec [26] solve the problem of linear dimension of $L_p(\mu)$-spaces (cf. Banach [2]).

COROLLARY 2. *Let $1 \leqslant p \leqslant r \leqslant 2$. Then every $\mathscr{L}_r$-space is isomorphic to a subspace of an $L_p(\mu)$-space for some measure μ.*

For $p = 2$, Theorem 7.2 and in fact a more general result can be proved very easily. We have

PROPOSITION 7.5. *Let $1 \leqslant p \leqslant \infty$ and let n be an integer. Then l_2^n is isometric to a subspace of $L_p(0,1)$.*

Proof. The case $p = \infty$ is trivial; so we consider only $p < \infty$. Let $S^n = \{x; x \in l_2^n, \|x\| = 1\}$ and let m be the normalized (i.e., $m(S^n) = 1$) rotation invariant measure on S^n. The integral $\int_{S^n} |(x,y)|^p dm(y)$ clearly depends only on $\|x\|$. Hence, if

$$c_{p,n} = \Big(\int_{S^n} |(x,y)|^p dm(y)\Big)^{1/p} \quad \text{for} \quad x \in S^n,$$

the map taking $x \in l_2^n$ into $f_x = c_{p,n}^{-1}(x,\cdot) \in L_p(m)$ is an isometry. Since $L_p(m)$ is isometric to $L_p(0,1)$ (cf. [21, p. 173]) the result follows.

Remark. Proposition 7.5 is a special case of a deep result of Dvoretzky [11].

COROLLARY 1. *Let $1 \leqslant p \leqslant \infty$. Then every Hilbert space is isometric to a subspace of $L_p(\mu)$ for a suitable measure μ. Every separable Hilbert space is isometric to a subspace of $L_p(0,1)$.*

From Theorem 7.2 and Proposition 7.5 it is easy to obtain some inequalities which resemble the inequalities of Section 2. These inequalities are probably useful though they are less deep than Theorem 2.1.

PROPOSITION 7.6. *Let $1 \leqslant p < \infty$ and let $p \leqslant r \leqslant 2$ if $p \leqslant 2$ or $r = 2$ if $p > 2$. Let $\{a_{i,j}\}_{i,j=1,\dots,N}$ and $\{b_i\}_{i=1}^N$ be real numbers such that*

$$\sum_{i=1}^N b_i \Big|\sum_{j=1}^N a_{i,j} t_j\Big|^p \geqslant 0 \quad \textit{for every real } \{t_j\}_{j=1}^N. \tag{7.8}$$

Then for every measure μ and every vector $\{x_j\}_{j=1}^N$ in $L_r(\mu)$

$$\sum_{i=1}^N b_i \Big\|\sum_{j=1}^N a_{i,j} x_j\Big\|^p \geqslant 0. \tag{7.9}$$

Proof. Assume that (7.8) holds and let ν be any measure on a measure space Ω. Let $\{f_j\}_{j=1}^N \in L_p(\Omega,\nu)$. Then for every $\omega \in \Omega$,

$$\sum_i b_i \Big|\sum_j a_{i,j} f_j(\omega)\Big|^p \geqslant 0.$$

By integrating with respect to ν we infer that

$$\sum_i b_i \Big\|\sum_j a_{i,j} f_j\Big\|^p \geqslant 0.$$

The desired result follows now from Theorem 7.2 (if $p \leqslant r \leqslant 2$) and Proposition 7.5 (if $r = 2$ and $1 \leqslant p < \infty$).

As an example of an inequality of the form (7.8) we take Hornich's inequality [24]. It is easily checked that for every real $\{t_i\}_{i=1}^n$, $\{s_j\}_{j=1}^m$ and u with

$$\sum_{i=1}^n t_i = \sum_{j=1}^m s_j$$

the following inequality is valid:

$$\sum_{i=1}^n (|t_i + u| - |t_i|) \leqslant \sum_{j=1}^m (|s_j + u| - |s_j|) + (n+m-2)|u|.$$

Hence, by Proposition 7.6, for every vector $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^m$ and z with

$$\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$$

in an $L_r(\mu)$-space with $1 \leqslant r \leqslant 2$ (and, in particular, in a Hilbert space)

$$\sum_{i=1}^n (\|x_i + z\| - \|x_i\|) \leqslant \sum_{j=1}^m (\|y_j + z\| - \|y_j\|) + (n+m-2)\|z\|.$$

We conclude this section by presenting a characterization of subspaces of $L_p(\mu)$-spaces which is related to Proposition 3.1.

THEOREM 7.3. *Let X be a Banach space and let $1 \leqslant p, \lambda \leqslant \infty$. Then there is a measure μ and a subspace Y of $L_p(\mu)$ with $d(X,Y) \leqslant \lambda$ if and only if whenever for every $x^* \in X^*$*

$$\sum_{i=1}^n |x^*(u_i)|^p \geqslant \sum_{j=1}^m |x^*(v_j)|^p, \qquad \{u_i\}_{i=1}^n, \{v_j\}_{j=1}^m \in X, \tag{7.10}$$

then

$$\lambda^p \sum_{i=1}^n \|u_i\|^p \geqslant \sum_{j=1}^m \|v_j\|^p. \tag{7.11}$$

Proof. Assume first that there is a measure μ and an operator $T: X \to L_p(\mu)$ with $\|x\| \leqslant \|Tx\| \leqslant \lambda\|x\|$ for every $x \in X$. Let $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ be vectors in X such that (7.10) holds and let B be the subspace of X which they generate. Let $\varepsilon > 0$. Since $L_p(\mu)$ is an $\mathscr{L}_{p,1+\varepsilon}$-space, there is a subspace $\tilde{B}$ of $L_p(\mu)$ containing TB such that $d(\tilde{B}, l_p^h) < 1+\varepsilon$ with $h = \dim \tilde{B} < \infty$. Hence there is an operator $\tilde{T}: B \to l_p^h$ with

$$\|x\| \leqslant \|\tilde{T}x\| \leqslant \lambda(1+\varepsilon)\|x\|, \qquad x \in B.$$

Let $\{\xi_k\}_{k=1}^h$ be the basis vectors of $(l_p^h)^*$. Then

$$\lambda^p(1+\varepsilon)^p\sum_{i=1}^n\|u_i\|^p\geqslant\sum_{i=1}^n\|\tilde{T}u_i\|^p=\sum_{i=1}^n\sum_{k=1}^h|\xi_k(\tilde{T}u_i)|^p$$

$$=\sum_{k=1}^h\sum_{i=1}^n|\tilde{T}^*\xi_k(u_i)|^p\geqslant\sum_{k=1}^h\sum_{j=1}^m|\tilde{T}^*\xi_k(v_j)|^p=\sum_{j=1}^m\|\tilde{T}v_j\|^p\geqslant\sum_{j=1}^m\|v_j\|^p.$$

Since $\varepsilon>0$ was arbitrary, (7.11) holds.

Assume conversely that (7.10) implies (7.11). By Proposition 7.1 we may assume without loss of generality that $\dim X<\infty$ and hence

$$S^*=\{x^*;\,x^*\in X^*,\ \|x^*\|=1\}$$

is a compact set. For every $x\in X$ let $f_x\in C(S^*)$ be defined by $f\ (x^*)=x^*(x)$. Let $K_0\subset C(S^*)$ be the convex hull of the set $\{f;\,f=|f_x|^p,\ x\in X,\ \|x\|=1\}$, and let

$$K_1=\bigcup_{\varrho>\lambda^p}(\varrho K_0-K_0).$$

K_1 is a convex set which is disjoint from the negative cone of $C(X^*)$. Indeed, if $g\in\varrho K_0-K_0$, $\varrho>\lambda^p$, then

$$g(x^*)=\lambda^p\sum_i\frac{\varrho}{\lambda^p}\alpha_i|x^*(u_i)|^p-\sum_j\beta_j|x^*(v_j)|^p,\qquad x^*\in X^*,$$

with $\alpha_i,\beta_j\geqslant0$, $\sum_i\alpha_i=\sum_j\beta_j=1$ and $\|u_i\|=\|v_j\|=1$ for every i and j. Since

$$\sum_j\|\beta_j^{1/p}v_j\|^p=1<\varrho/\lambda^p=\sum_i\|\varrho^{1/p}\alpha_i^{1/p}u_i/\lambda\|^p,$$

it follows from our assumption that for at least one $x^*\in S^*$ we have $g(x^*)\geqslant0$.

By the separation theorem and the Riesz representation theorem there is a positive measure μ on S^* such that for every $f,g\in K_0$ and every $\varrho>\lambda^p$,

$$\varrho\int fd\mu\geqslant\int gd\mu.$$

Hence for $x,y\in X$ with $\|x\|=\|y\|=1$

$$\lambda^p\int_{S^*}|x^*(x)|^pd\mu(x^*)\geqslant\int_{S^*}|x^*(y)|^pd\mu(x^*).\tag{7.12}$$

Let $\gamma^p=\inf\{\int|x^*(x)|^pd\mu(x^*);\ \|x\|=1\}$. The number γ is not 0. Indeed, if $\gamma=0$, then by (7.12)

$$\int|x^*(x)|^pd\mu(x^*)=0$$

for every $x\in X$ and this is impossible since

$$\sum_{h=1}^k|x^*(x_h)|^p>0$$

for every $x^*\in S^*$ if $\{x_h\}_{h=1}^k$ is any algebraic basis of X.

The operator $T\colon X\to L_p(\mu)$ defined by $Tx=\gamma^{-1}f_x$ satisfies $\|x\|\leqslant\|Tx\|\leqslant\lambda\|x\|$ for every $x\in X$ and this concludes the proof.

8. REMARKS, EXAMPLES AND OPEN PROBLEMS

This section contains some problems and a few results and examples which are related to the material of the preceding sections. Some of the problems were mentioned already in those sections.

We begin with examples.

EXAMPLE 8.1. *There exists an $\mathscr{L}_1$-space which is not isomorphic to a complemented subspace of any $L_1(\mu)$-space.*

Let $\{e_i\}_{i=1}^\infty$ be the usual basis of l_1 and let X be the subspace of l_1 spanned by the vectors

$$x_n=e_n-\tfrac{1}{2}(e_{2n}+e_{2n+1}),\qquad n=1,2,\ldots$$

This space was discussed in [36]. It was shown there that X is not isomorphic to a complemented subspace of an $L_1(\mu)$-space. The proof of this result in [36] was based on the fact that there is an operator T from l_1 onto $L_1(0,1)$ whose kernel is X. As observed in [36] it is easily seen that $\{x_i\}_{i=1}^\infty$ forms a basis of X. Let

$$B_n=\operatorname{span}\{x_i\}_{i=1}^n,\qquad n=1,2,\ldots$$

We shall show that $d(B_n,l_1^n)\leqslant2$ for every n and hence X is an $\mathscr{L}_{1,\lambda}$-space for every $\lambda>2$. It is easy to see that every B_n is spanned also by vectors $\{y_k^n\}_{k=1}^n$ of the form

$$y_k^n=e_k-\sum_{j=n+1}^{2n+1}\lambda_{j,k}^n\,e_j\quad\text{ with }\quad\lambda_{j,k}^n\geqslant0\ \text{ and }\ \sum_{j=n+1}^{2n+1}\lambda_{j,k}^n=1.$$

Hence for every n and every real $\{a_k\}_{k=1}^n$

$$\sum_{k=1}^n|a_k|\leqslant\Big\|\sum_{k=1}^na_ky_k^n\Big\|\leqslant2\sum_{k=1}^n|a_k|.$$

and this proves our assertion.

EXAMPLE 8.2. *Let $1<p\neq2<\infty$. Then the spaces l_p, $l_p\oplus l_2$, $(l_2\oplus l_2\oplus\ldots)_p$ and $L_p(0,1)$ are mutually non-isomorphic and all of them are $\mathscr{L}_p$-spaces. Hence $l_p\oplus l_2$ and $(l_2\oplus l_2\oplus\ldots)_p$ are examples of $\mathscr{L}_p$-spaces which are not isomorphic to $L_p(\mu)$-spaces.*

Proof. We show first that $l_p \oplus l_2$ is an $\mathscr{L}_p$-space. Let $\{g_i\}$ and $\{f_i\}$ denote the unit vector bases of l_p and l_2 respectively. For $1 \leqslant h < k < \infty$ let $G_{h,k}$ (resp. $F_{h,k}$) denote the spaces spanned $g_h, g_{h+1}, \ldots, g_{k-1}$ (resp. $f_h, f_{h+1}, \ldots, f_{k-1}$). Let B be a finite-dimensional subspace of $l_p \oplus l_2$. Without loss of generality we may assume that there are indices n and m such that $B \subset G_{1,n} \oplus F_{1,m}$. By using some properties of Rademacher functions it was shown in [44] (cf. the proof of Proposition 7 there) that in the space $G_{n,2^m+n}$, which is isometric to $l_p^{2^m}$, there is a subspace R_m such that $d(R_m, l_2^m) \leqslant a_p$ and a projection P_m from $G_{n,2^m+n}$ onto R_m with $\|P_m\| \leqslant b_p$, where a_p and b_p are constants depending only on p. Let $Y_m = \text{kernel } P_m$ and let $E = (G_{1,n} \oplus Y_m)_p \oplus F_{1,m}$. Clearly $E \supset G_{1,n} \oplus \oplus F_{1,m} \supset B$. Let $X_1 \overset{a}{\sim} X_2$ denote $d(X_1, X_2) \leqslant a$. Then, taking in account that $F_{1,m}$ is isometric to l_2^m and $G_{h,k}$ is isometric to l_p^{k-h}, we have

$$E = (G_{1,n} \oplus Y_m)_p \oplus F_{1,m} \overset{c_p}{\sim} G_{1,n} \oplus (Y_m \oplus F_{1,m})$$
$$\overset{d_p}{\sim} G_{1,n} \oplus (Y_m \oplus R_m) \overset{e_p}{\sim} (G_{1,n} \oplus G_{n,2^m+n})_p \overset{1}{\sim} l_p^{2^m+n}.$$

Hence $d(E, l_p^{2^m+n}) \leqslant c_p d_p e_p$, where c_p, d_p and e_p depend only on p (through a_p and b_p). Thus $l_2 \oplus l_p$ is an $\mathscr{L}_p$-space. Since the direct sum of $\mathscr{L}_{p,\lambda}$-spaces in the l_p-norm is an $\mathscr{L}_p$-space (acutally an $\mathscr{L}_{p,\lambda+\varepsilon}$-space for every $\varepsilon > 0$) and since

$$\begin{aligned}(l_2 \oplus l_2 \oplus l_2 \oplus \ldots)_p &\sim \big((l_2 \oplus R) \oplus (l_2 \oplus R) \oplus \ldots\big)_p \\ &\sim (l_2 \oplus l_2 \oplus \ldots)_p \oplus l_p \sim (l_2 \oplus l_2 \ldots)_p \oplus (l_p \oplus l_p \oplus \ldots)_p \\ &\sim \big((l_2 \oplus l_p) \oplus (l_2 \oplus l_p) \oplus \ldots\big)_p,\end{aligned}$$

we infer that $(l_2 \oplus l_2 \oplus \ldots)_p$ is also an $\mathscr{L}_p$-space (R denotes the 1-dimensional space and $X_1 \sim X_2$ denotes $d(X_1, X_2) < \infty$).

Clearly, neither $l_2 \oplus l_p$ nor $(l_2 \oplus l_2 \oplus \ldots)_p$ are isomorphic to l_p (because they contain l_2). If E is a subspace of $(l_2 \oplus l_2 \oplus \ldots)_p$ which is isomorphic to l_2, then, as easily seen, E is a complemented subspace and its complement is again isomorphic to $(l_2 \oplus l_2 \oplus \ldots)_p$. Hence $l_2 \oplus l_p$ is not isomorphic to $(l_2 \oplus l_2 \oplus \ldots)_p$. In order to show that both these spaces are not isomorphic to $L_p(0,1)$ we have just to remark that they do not contain subspaces isomorphic to l_r for $r \neq 2, p$ while $L_p(0,1)$ contains, if $p < 2$, subspaces isomorphic to l_r for all $p < r < 2$. (If $p > 2$, we pass to the dual spaces and thus come back to the case $p < 2$.)

Remark. Since $(l_2 \oplus l_2 \oplus \ldots)_p$ is an $\mathscr{L}_p$-space, it easily follows that $L_p(0,1; l_2)$ = the space of Bochner p-integrable functions on $(0,1)$ with values in l_2 is an $\mathscr{L}_p$-space. By Theorem 7.1, $L_p(0,1; l_2)$ is isomorphic to a complemented subspace of $L_p(0,1)$. Since clearly $L_p(0,1; l_2)$ contains a complemented subspace isomorphic to $L_p(0,1)$, it follows by using the decomposition method (cf. [44]) that $L_p(0,1)$ is isomorphic to $L_p(0,1; l_2)$. It can be shown that, for $2 < p < \infty$, $(l_p \oplus l_p \oplus \ldots)_2$ is not isomorphic to any subspace of $L_p(0,1)$. Hence, for $p \neq 2$, $(l_p \oplus l_p \oplus \ldots)_2$ is not an $\mathscr{L}_p$-space.

In Section 7 we gave for spaces which are $\mathscr{L}_{p,1+\varepsilon}$-spaces for every $\varepsilon > 0$ a functional representation. The preceding examples show that for general $\mathscr{L}_p$-spaces the situation is more complicated.

Problem 1. *Give a functional representation for general $\mathscr{L}_p$-spaces.*

This problem is quite general and vague. We formulate now some concrete problems which are essentially contained in Problem 1.

Problem 1a. *Is every $\mathscr{L}_\infty$-space isomorphic to a space $C(K)$ for a suitable compact Hausdorff space K?*

Problem 1b. *Let X be an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant \infty$. Is X^* and $\mathscr{L}_q$-space $(q^{-1}+p^{-1} = 1)$?*

Problem 1c. *Let X be an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant \infty$, and let Y be a complemented subspace of X. Is Y either an $\mathscr{L}_p$-space or (isomorphic to) a Hilbert space?*

Remark. If $p = 1$ or ∞ and Y infinite-dimensional, Y cannot be a Hilbert space.

In view of Proposition 7.3 the solution of Problem 1 for separable X will give important information in the general case. For separable X a more specific version of Problem 1 is:

Problem 1d. *Is every separable infinite-dimensional $\mathscr{L}_p$-space $(1 < p \neq 2 < \infty)$ isomorphic to one of the four spaces of Example 8.2?*

Another problem in the separable case is:

Problem 1e. *Let X be an infinite-dimensional subspace of l_p $(1 \leqslant p < \infty)$. Assume that X is isomorphic to a complemented subspace of $L_p(0,1)$. Is X isomorphic to l_p?*

We pass now to problems connected with p-absolutely summing operators.

Problem 2. *Let X and Y be infinite-dimensional Banach spaces such that every $T \in B(X, Y)$ is absolutely summing. Does it follow that X is an $\mathscr{L}_1$-space and Y is isomorphic to a Hilbert space?*

A partial answer to this problem is Theorem 4.2. We make now some further comments on this problem. We call a pair X, Y of Banach spaces *unconditionally trivial* (u. t. in symbols) if for every $T \in B(X, Y)$, $a_1(T) < \infty$. It is clear that if (X, Y) is u. t., then there is an $M < \infty$ such that $a_1(T) \leqslant M\|T\|$, and let us set $a_1(X, Y) = \inf M$.

Proposition 8.1. *Let X and Y be infinite-dimensional Banach spaces such that (X, Y) is u.t. Then*

1) *(X, l_2) is u.t.*
2) *For every unconditionally convergent series $\sum x_i$ in X, $\sum \|x_i\|^2 < \infty$.*
3) *Every operator from an $\mathscr{L}_\infty$-space into X is 2-absolutely summing.*

Proof. 1) This is an easy consequence of Dvoretzky's theorem on spherical sections [11].

2) Let $\sum x_i$ be an unconditionally convergent series in X. Then there is a constant ϱ such that

$$\sum_i |x^*(x_i)| \leqslant \varrho \|x^*\|$$

for every $x^* \epsilon X^*$. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis in l_2, and let $\{\lambda_i\}_{i=1}^\infty$ be a sequence of reals with $\sum \lambda_i^2 = 1$. Choose x_i^* in X^* with $\|x_i^*\| = 1$ and $x_i^*(x_i) = \|x_i\|$ for every i. Define $T \epsilon B(X, l_2)$ by

$$Tx = \sum_i \lambda_i x_i^*(x) e_i.$$

Clearly

$$\|T\| \leqslant \Big(\sum_i \lambda_i^2 \|x_i^*\|^2\Big)^{1/2} = 1 \quad \text{and} \quad \|Tx_i\| = \Big\|\sum_j \lambda_j x_j^*(x_i) e_j\Big\| \geqslant |\lambda_i| \, \|x_i\|.$$

Hence,

$$\sum_i |\lambda_i| \, \|x_i\| \leqslant \sum_i \|Tx_i\| \leqslant a_1(T) \sup_{\|x^*\| \leqslant 1} \sum_i |x^*(x_i)|$$
$$\leqslant \varrho a_1(T) \leqslant \varrho a_1(X, l_2) \|T\| \leqslant \varrho a_1(X, l_2).$$

Since this inequality holds whenever $\sum \lambda_i^2 = 1$, we infer that

$$\Big(\sum_i \|x_i\|^2\Big)^{1/2} \leqslant \varrho a_1(X, l_2) < \infty.$$

3) It is clearly enough to show that every $T \epsilon B(c_0, X)$ is 2-absolutely summing. Let $T \epsilon B(c_0, X)$ and let $x_n = Te_n$, where e_n is the n-th unit vector in c_0, $n = 1, 2, \ldots$ Let $\xi_i = \{\xi_i(n)\}_{n=1}^\infty$ be a sequence of elements in c_0 so that

$$\sup_n \Big(\sum_i |\xi_i(n)|^2\Big)^{1/2} = M < \infty.$$

We have to estimate

$$\Big(\sum_i \Big\|\sum_n \xi_i(n) x_n\Big\|^2\Big)^{1/2}.$$

Choose numbers λ_i and functionals x_i^* so that $\|x_i^*\| = 1$, $i = 1, 2, \ldots$, $\sum_i \lambda_i^2 = 1$ and

$$\Big(\sum_i \Big\|\sum_n \xi_i(n) x_n\Big\|^2\Big)^{1/2} = \sum_i \lambda_i x_i^* \Big(\sum_n \xi_i(n) x_n\Big).$$

Let $S: X \to l_2$ be defined by

$$Sx = \sum_i \lambda_i x_i^*(x) f_i,$$

where $\{f_i\}_{i=1}^\infty$ is an orthonormal basis of l_2. Then $\|S\| \leqslant 1$ and hence $a_1(S) \leqslant a_1(X, l_2) < \infty$ (by 1)). By the Schwartz inequality

$$\sum_i \lambda_i x_i^* \Big(\sum_n \xi_i(n) x_n\Big) = \sum_n \sum_i \lambda_i x_i^*(x_n) \xi_i(n)$$
$$\leqslant \sum_n \Big(\sum_i |\xi_i(n)|^2\Big)^{1/2} \Big(\sum_i |\lambda_i x_i^*(x_n)|^2\Big)^{1/2} \leqslant M \sum_n \|Sx_n\|$$
$$\leqslant a_1(S)\, M \sup_{\|x^*\|=1} \sum_n |x^*(x_n)| \leqslant a_1(X, l_2)\, M \|T\|$$

since

$$\sup_{\|x^*\|=1} \sum_n |x^*(x_n)| = \sup_{\|x^*\|=1} \sum_n |x^*(Te_n)| = \sup_{\|x^*\|=1} \|T^* x^*\| = \|T^*\| = \|T\|.$$

Hence

$$\Big(\sum_i \|T\xi_i\|^2\Big)^{1/2} = \Big(\sum_i \Big\|\sum_n \xi_i(n) x_i\Big\|^2\Big)^{1/2} \leqslant a_1(X, l_2) \|T\| \sup_{\|\eta^*\|=1} \Big(\sum_i |\eta^*(\xi_i)|^2\Big)^{1/2},$$

i.e. $a_2(T) \leqslant a_1(X, l_2) \|T\|$ and this concludes the proof.

Remark. The proof of part 3) can be considered also as a derivation of Theorem 4.3, for $p = 1$, from Theorem 4.1.

Problem 2 is closely connected to the following problem of Grothendieck ([15], Chap. II, p. 47):

Let X and Y be Banach spaces such that every $T \epsilon B(X, Y)$ is nuclear. Is either X or Y of a finite dimension?

Clearly a positive answer to problem 2 would imply a positive answer to Grothendieck's problem. By using the theorem of Dvoretzky [11] it is also easy to see that in order to answer the problem of Grothendieck it is enough to show that if X is infinite-dimensional and if (X, Y) is u.t., then Y is isomorphic to a Hilbert space.

Problem 3. *Let X and Y be infinite-dimensional Banach spaces such that every $T \epsilon B(X, Y)$ is p-absolutely summing for some fixed p, $1 < p < 2$. Does it follow that every $T \epsilon B(X, Y)$ is absolutely summing?*

By using the proof of Theorem 4.2 it can be shown that if every $T \epsilon B(X, Y)$ is p-absolutely summing ($p < 2$), then for every normalized unconditional basis $\{x_i\}_{i=1}^\infty$ in X or a complemented subspace of X there is a constant M such that

$$\Big\|\sum_{i=1}^\infty a_i x_i\Big\| \leqslant M \Big(\sum_i |a_i|^p\Big)^{1/p}$$

for every real $\{a_i\}_{i=1}^{\infty}$. Let us mention in this connection that by [44] there are for every $1 < p \leqslant r < 2$ a normalized unconditional basis $\{x_i\}_{i=1}^{\infty}$ in l_p (or $L_p(0,1)$) and a sequence of reals $\{a_i\}_{i=1}^{\infty}$ such that $\sum_i a_i x_i$ converges but $\sum_i |a_i|^r = \infty$.

PROBLEM 4. *Let $p > 2$. Is every operator from an $\mathscr{L}_\infty$-space to an $\mathscr{L}_p$-space p-absolutely summing?*

The notion of a p-absolutely summing operator can be generalized as follows (cf. [40]). Let $1 \leqslant r \leqslant p < \infty$, and let X and Y be Banach spaces. An operator $T \in B(X, Y)$ is said to be (p,r)-*absolutely summing* if there is a constant C such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant C \Big(\sup_{\|x^*\|=1} \sum_{i=1}^{n} |x^*(x_i)|^r\Big)^{1/r}, \quad \{x_i\}_{i=1}^{n} \subset X,\ n = 1, 2, \ldots$$

The (p,p)-absolutely summing operators coincide with the p-absolutely summing operators of Definition 3.2.

Let us observe that Theorem 4.3 can be completed by the following proposition:

PROPOSITION 8.2. *Let $2 < p < \infty$. Then every operator from an $\mathscr{L}_\infty$-space to an $\mathscr{L}_p$-space is $(p,2)$-absolutely summing.*

Proof. In the proof of Theorem 4.3 we used the fact that $p \leqslant 2$ only by passing from inequality (4.6) to inequality (4.7). Hence we can use here proof of Theorem 4.3 up to (4.6). Now, since $p > 2$,

$$\Big(\sum_k \Big(\sum_i \Big(\sum_j z_{i,j} a_{j,k}\Big)^2\Big)^{p/2}\Big)^{1/p} \geqslant \Big(\sum_k \sum_i \Big|\sum_j z_{i,j} a_{j,k}\Big|^p\Big)^{1/p}$$
$$= \Big(\sum_i \sum_k \Big|\sum_j z_{i,j} a_{j,k}\Big|^p\Big)^{1/p} = \Big(\sum_i \|T_0 z_i\|^p\Big)^{1/p}.$$

Thus, by (4.6),

$$\Big(\sum_i \|T_0 z_i\|^p\Big)^{1/p} \leqslant \lambda K_G \|T_0\| \leqslant \lambda K_G \|T\|.$$

Hence, as in the proof of Theorem 4.3, we infer that

$$\Big(\sum_i \|Tx_i\|^p\Big)^{1/p} \leqslant \varrho \lambda K_G \|T\|$$

which means that T is $(p,2)$-absolutely summing.

Recently Kwapień [31] obtained the following generalization of Theorem 4.1:

Every linear operator from an $\mathscr{L}_1$-space into an $\mathscr{L}_p$-space is $(a(p),1)$-absolutely summing where $a(p) = 2p/(3p-2)$ for $1 \leqslant p \leqslant 2$ and $a(p) = 2p/(p+2)$ for $2 < p \leqslant \infty$. We refer the reader to [31] for further information.

It follows from Proposition 3.1 that every (p,p)-absolutely summing operator is weakly compact (cf. [51]). This fact suggests the following problem:

PROBLEM 5. *For which values of p and r $(1 \leqslant r < p < \infty)$ is every (p,r)-absolutely summing operator weakly compact?*

By Orlicz's theorem (cf. [42]) every operator defined on an $\mathscr{L}_1$-space is $(2,1)$-absolutely summing. It is easily seen that a $(2,1)$-absolutely summing operator $T \in B(X, Y)$ is (p,r)-absolutely summing if $1 \leqslant r < 2$ and $p \geqslant 2r/(2-r)$. Indeed, let $\{x_i\}_{i=1}^{n} \subset X$ be such that

$$\Big(\sum_i |x^*(x_i)|^r\Big)^{1/r} \leqslant \|x^*\|$$

for every $x^* \in X^*$. Then

$$\sum_i |x^*(\lambda_i x_i)| \leqslant \|x^*\| \quad \text{whenever} \quad \sum_i |\lambda_i|^s \leqslant 1 \ \text{where} \ \frac{1}{r} + \frac{1}{s} = 1.$$

Since T is $(2,1)$-absolutely summing,

$$\Big(\sum_i \lambda_i^2 \|Tx_i\|^2\Big)^{1/2} \leqslant K$$

for some constant K depending only on T. This inequality holds whenever $\sum_i (\lambda_i^2)^{s/2} = 1$ and therefore

$$\sum \|Tx_i\|^q \leqslant K^2 \quad \text{where} \quad \frac{2}{q} + \frac{2}{s} = 1, \text{ i.e. } q = 2r/(2-r).$$

We have thus seen that if $1 \leqslant r < 2$ and $p \geqslant 2r/(2-r)$, then there are non-weakly compact (p,r)-absolutely summing operators. For other values of (p,r) we do not know the answer to problem 5. Let us remark that the argument above indicates that the most important case is that of $r = 1$ and $1 < p < 2$. By Theorem 8.1, given below, problem 5 reduces to the question whether the operator σ defined below is (p,r)-absolutely summing.

PROBLEM 6. *Let $\infty \geqslant p > s > r \geqslant 1$. Can every operator T from an $\mathscr{L}_p$-space to an $\mathscr{L}_r$-space be factored through an $\mathscr{L}_s$-space?*

The answer is yes, if $p \geqslant 2 \geqslant r$. Indeed, by Theorem 5.2 every such T can be factored through a Hilbert space H. Since H is isomorphic to a complemented subspace of $L_s(\mu)$ for some measure μ (this is well known see, e.g. [27]), it follows that T can be factored through $L_s(\mu)$.

Factorization theorems were obtained and applied by Grothendieck in many different situations (cf. [15] or [17] which is the basis of the present paper). It seems to us that there are many other areas in Banach space theory where factorization theorems can be obtained and used.

We present here one result in such a direction. Let $\sigma\colon l_1 \to l_\infty$ be the "sum operator", namely the operator mapping the sequence $\{a_i\}_{i=1}^{\infty}$ in l_1 into the sequence of its partial sums $\{\sum_{i=1}^{n} a_i\}_{n=1}^{\infty}$ in l_∞. The operator σ is obviously not weakly compact. We shall show now that it is a universal non-weakly compact operator in the sense that any non-weakly compact operator is a factor of σ.

THEOREM 8.1. *Let X and Y be Banach spaces and let $T \in B(X, Y)$. The operator T is not weakly compact if and only if there exist operators $S\colon l_1 \to X$ and $U\colon Y \to l_\infty$ such that $UTS = \sigma$.*

Proof. It is clear that if such U and S exist, then T is not weakly compact. To prove the converse, assume that T is not weakly compact and let $W = \{y;\, y = Tx;\, \|x\| \leqslant 1\}$. The subset W of Y is bounded and its closure is not weakly compact. Hence by [46] there are $\delta > 0$, $y^* \in Y^*$ and a basic sequence $\{w_n\}_{n=1}^{\infty}$ in W such that $y^*(w_n) \geqslant \delta$ for every n. (A sequence is called a *basic* one if it forms a basis in the subspace it spans.) Put $y_n = w_n/y^*(w_n)$, $n = 1, 2, \ldots$, and let $x_n \in X$ be such that $Tx_n = y_n$ and $\|x_n\| \leqslant \delta^{-1}$ (this is possible since $y_n \in \delta^{-1} W$ for every n). Define $S\colon l_1 \to X$ by

$$S(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n x_n \quad \text{for} \quad \{a_n\}_{n=1}^{\infty} \in l_1.$$

Clearly, S is an operator of norm $\leqslant \delta^{-1}$. Let E denote the closed linear subspace of Y spanned by the basic sequence $\{y_n\}_{n=1}^{\infty}$. Hence every $e \in E$ has a unique representation of the form

$$e = \sum_{n=1}^{\infty} y_n^*(e)\, y_n,$$

where $\{y_n^*\}_{n=1}^{\infty} \subset E^*$ and $y_n^*(y_m) = \delta_n^m$, and there is a ϱ such that for every e and n,

$$\Big\| \sum_{i=1}^{n} y_i^*(e)\, y_i \Big\| \leqslant \varrho \|e\|.$$

Since $y^*(y_n) = 1$, we have

$$\lim_n \sum_{i=1}^{n} y_i^*(e) = \lim_n y^*\Big(\sum_{i=1}^{n} y_i^*(e)\, y_i\Big) = y^*(e)$$

for e in E. Define $\tilde{U}\colon E \to c$ (c denotes the space of convergent sequences) by

$$\tilde{U} e = \Big\{\sum_{i=1}^{n} y_i^*(e)\Big\}_{n=1}^{\infty}.$$

$\tilde{U}$ is bounded since

$$\Big|\sum_{i=1}^{n} y_i^*(e)\Big| = \Big|y^*\Big(\sum_{i=1}^{n} y_i^*(e)\, y_i\Big)\Big| \leqslant \|y^*\| \Big\|\sum_{i=1}^{n} y_i^*(e)\, y_i\Big\| \leqslant \varrho \|y^*\|\, \|e\|.$$

Regarding c as a subspace of l_∞ and using the fact that l_∞ is a $\mathscr{P}_1$-space, we infer that $\tilde{U}$ can be extended to a bounded linear operator U from Y into l_∞. For $\{a_i\}_{i=1}^{\infty} \in l_1$

$$UTS(\{a_i\}_{i=1}^{\infty}) = UT\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \tilde{U}\Big(\sum_{i=1}^{\infty} a_i y_i\Big) = \Big\{\sum_{i=1}^{n} a_i\Big\}_{n=1}^{\infty}.$$

This concludes the proof.

Remark. If Y is separable, we can replace in the theorem the space l_∞ by c. Indeed, by Sobczyk's theorem [55], there is a projection P from the span of $c \cup UY$ onto c and so we could replace U by the operator $PU\colon Y \to c$.

We pass now to a "well known" problem which has been already raised by several authors in the last decade:

PROBLEM 7. *Does there exist a real-valued function $f(t)$ such that for every finite-dimensional Banach space X, $d(X, l_\infty^n) \leqslant f(p(X))$, where $p(X)$ is the projection constant of X?*

By Corollary 6 to Theorem 6.1, Problem 7 is equivalent to

PROBLEM 7a. *Does there exist a real-valued function $g(t)$ such that for every finite-dimensional Banach space X, $s(X) \leqslant g(p(X))$, where $s(X)$ (resp. $p(X)$) is the symmetry (resp. projection) constant of X?*

A more general problem than problem 7a is

PROBLEM 7b. *Does there exist a real-valued function $g(t, \lambda)$ such that whenever $Y \supset X$ are finite-dimensional Banach spaces for which there is a projection of norm $\leqslant \lambda$ from Y onto X, then $s(X) \leqslant g(s(Y), \lambda)$?*

An infinite-dimensional analogue of problem 7b is

PROBLEM 7c. *Let X be a complemented subspace of the Banach space Y. Assume that Y has an unconditional basis. Does it follow that also X has an unconditional basis?*

The example of [36] mentioned in the beginning of this section shows that the answer to Problem 7c is negative if we do not assume that X is complemented in Y.

Finally, let us introduce a notion which is suggested by Proposition 7.1. Let X be a Banach space. A Banach space Y is said to be of a *finite type not exceeding* X ($Y \prec X$) if for every finite-dimensional subspace B of Y and every $\varepsilon > 0$ there is a finite-dimensional subspace $\tilde{B}$ of X with $d(B, \tilde{B}) < 1 + \varepsilon$. A Banach space Y is said to be an *envelope* of a Banach space X if

a) $Y \prec X$.

b) Every Banach Z of density character not exceeding that of Y and satisfying $Z \prec X$ is isometric to a subspace of Y.

Corollary 2 to Proposition 7.1 shows that $L_p(0,1)$ is for $1 \leqslant p \leqslant \infty$ an envelope of l_p.

PROBLEM 8. *Does every separable Banach space have a separable envelope?*

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Some remarks on (p, q)-absolutely summing operators in l_p-spaces

by

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A linear operator $A\colon X \to Y$ (X, Y Banach spaces) is said to be (p, q)-*absolutely summing*, $1 \leqslant p, q < +\infty$, if there is a constant M such that for every finite set in X, $\{x_1, x_2, \ldots, x_n\}$, the inequality

$$\Big(\sum_{i=1}^{n} \|Ax_i\|^p\Big)^{1/p} \leqslant M \sup_{x^* \in K^*} \Big(\sum_{i=1}^{n} |x'(x_i)|^q\Big)^{1/q}$$

holds, where K^* is the unit ball of X^* — the dual of X.

The definition of a (p, q)-absolutely summing operator is due to Pełczyński and Mitjagin [7] and it generalizes earlier concepts of various authors. In [3] Grothendieck introduced "semi-intégrale à gauche" operators. They are exactly (1, 1)-absolutely summing operators. In [13] Saphar considered "Hilbert-Schmidt à gauche" operators in Banach spaces, which are (2, 2)-absolutely summing operators. In [10] Pietsch defined "absolut p-summierende Abbildungen", which are exactly (p, p)-absolutely summing according to our definition. In this paper we shall deal with (p, q)-absolutely summing operators in l_p-spaces. But all the results obtained here can be generalized to spaces of $\mathscr{L}_p$-type (for the definition see [8]). In the first part it is proved that every linear operator from l_1 to l_p is $\big(2p/(2p-|p-2|), 1\big)$-absolutely summing. In the second part we study (p, q)-absolutely summing operators in a Hilbert space.

0. Preliminaries. Let $(x_i)_{i\in I}$ be a finite family in a Banach space X, and K^* the unit ball of X^*. Let us put:

$$l^r(x_i, X) = \begin{cases} \Big(\sum_{i\in I} \|x_i\|^r\Big)^{1/r} & \text{if} \quad 1 \leqslant r < +\infty, \\ \sup_{i\in I} \|x_i\| & \text{if} \quad r = +\infty; \end{cases}$$

$$l^r[x_i, X] = \begin{cases} \sup_{x^*\in K^*} \Big(\sum_{i\in I} |x^*(x_i)|^r\Big)^{1/r} & \text{if} \quad 1 \leqslant r < +\infty, \\ \sup_{i\in I} \|x_i\| & \text{if} \quad r = +\infty. \end{cases}$$