The subspace M is doubly invariant. Indeed, if

$$m = \sum_{s\in S} T_s n_s \quad \text{with} \quad n_s \in N_0,$$

we infer by (3.6) that

$$T_\sigma m = \sum_{s\in S} T_{s\sigma} n_s \in M, \qquad T_\sigma^* m = \sum_{s_1=\sigma^{-1}s} T_{s_1} n_s \in M.$$

Since $M \cap N = N_0 \neq \{0\}$, we obtain $H = M$ and thus $N = N_0$ is a 1-dimensional space. This completes the proof of the theorem.

COROLLARY (Wermer). *An isometric non-unitary operator T on a Hilbert space H is unitarily equivalent to the translation operator on H^2 if and only if T has no proper doubly invariant subspaces.*

References

[1] C. Foiaş and I. Suciu, *Szegö measures and spectral theory in Hilbert spaces* Revue Roum. Math. Pures et Appl. 11 (1962), p. 147-159.

[2] P. R. Halmos, *Shifts on Hilbert spaces*, J. reine u. angew. Math., 1961.

[3] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables, II*, Acta Math. 106 (1961), p. 175-213.

[4] K. Hoffman, *Banach spaces of analytic functions*, New Jersey 1962.

[5] T. Ito, *On the commutative family of subnormal operators*, J. Faculty Sci., Hokkaido Univ., Ser. I. Math, 14 (1958), p. 1-15.

[6] W. Mlak, *Unitary dilation of contraction operators*, Rozprawy Matem. 46, Warszawa 1965.

[7] — *On semi-groups of contractions in Hilbert spaces*, Studia Math. 26 (1966), p. 263-272.

[8] B. Sz. Nagy et C. Foiaş, *Sur les contractions de l'espace de Hilbert IV*, Acta Sci. Math. 21 (1960), p. 251-259.

[9] I. Suciu, *Reprezentări spectral dilatabile ale unei algebre Banach comutative*, Studii şi cerc. mat. 16 (1964), p. 1211-1220.

[10] I. Wermer, *On restrictions of operators*, Proc. Amer. Math. Soc. 4 (1953), p. 860-865.

[11] H. Wold, *A study in the analysis of stationary time series*, Uppsala 1938

Reçu par la Rédaction le 30.7. 1967

STUDIA MATHEMATICA, T. XXX. (1968)

On multipliers preserving convergence of trigonometric series almost everywhere

by

MARY WEISS and ANTONI ZYGMUND (Chicago)

1. Consider a trigonometric series $\sum_{-\infty}^{+\infty} c_n e^{in\theta}$, which in the case $c_{-n} = \bar{c}_n$ can also be written in the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta) = \sum_0^\infty A_n(\theta),$$

say. Its conjugate is

$$\sum_{-\infty}^{+\infty}(-i\operatorname{sign} n)c_n e^{in\theta} = \sum_1^\infty (a_n \sin n\theta - b_n\cos n\theta) = \sum_0^\infty B_n(\theta),$$

say (with $B_0 = 0$).

One of the topics of the theory of trigonometric series that enjoyed popularity a few decades ago was the problem of the behavior (convergence or summability, at individual points or almost everywhere) of the series $\sum A_n(\theta)n^a$, $\sum B_n(\theta)n^a$, where a is a constant. The problem has obvious connections with differentiability or integrability (in general, of fractional order) of functions, and a was almost exclusively real. In this note we consider complex values of a, $a = \beta + i\gamma$, but in view of the fact that the case of real a has been exhaustively dealt with we limit ourselves to a purely imaginary, $a = i\gamma$, which shows some novel features. The problem we are discussing here arose out of some concrete applications but the latter are not considered here.

The main result of the paper is the following

THEOREM. *If the series $\sum_{n=0}^{\infty} A_n(\theta)$ is summable (C, k), $k > -1$, at each point of a set E of positive measure, then the series $\sum_0^\infty A_n(\theta)n^{i\gamma}$ is summable (C, k) almost everywhere in E. In particular, the convergence of $\sum A_n(\theta)$ in E implies the convergence of $\sum_0^\infty A_n(\theta)n^{i\gamma}$ almost everywhere in E.*

Here γ is a real number distinct from 0. Without loss of generality we may assume, wherever needed, that $a_0 = 0$, in which case successive termwise integrations of the series $\sum A_n(\theta)$ lead again to trigonometric series. We also assume that the a's and b's are real numbers.

2. The proof which follows systematically uses the notion of differentiability in the metric L^p, $1 \leqslant p < \infty$, and we recall the definition (for more details see [1]). We say that the function $f(\theta)$, defined almost everywhere in the neighborhood of the point θ_0, has an *m-th differential at* θ_0, *in* L^p, if there is a polynomial

$$P(t) = \sum_{j=0}^{m} a_j \frac{t^j}{j!}$$

of degree $\leqslant m$ such that

$$\Big(h^{-m} \int_{-h}^{h} |f(\theta_0+t) - P(t)|^p dt\Big)^{1/p} = o(h^m) \qquad (h \to 0). \tag{2.1}$$

The polynomial $P(t)$ is called the *m-th differential* (*in* L^p) *of* f *at* θ_0, and the number a_m is the *m-th derivative* (*in* L^p) *of* f *at* θ_0. The differentials in L^p have a number of properties missing in the classical case $p = \infty$ (when $f(\theta_0+t) = P(t) + o(t^m)$) and for this reason are both interesting and useful to consider.

The proof our theorem is based on a few lemmas which we now state.

LEMMA 1. *If a trigonometric series* $\sum A_n(\theta)$ *is summable* (C, k), $k = 0, 1, \ldots$, *to sum* $s(\theta)$ *in a set* E *of positive measure* (*thus, in particular,* $|a_n| + |b_n| = o(n^k)$), *then the function* $G(\theta)$ *obtained by integrating* $\sum A_n(\theta)$ *termwise* $k+1$ *times has almost everywhere in* E *a* $(k+1)$*-st derivative in the metric* L^p, $p < \infty$, *equal to* $s(\theta)$.

This lemma is known (see [4]) and we take it for granted here.

LEMMA 2. *Suppose that a trigonometric series* $\sum A_n(\theta)$ *is the Fourier series of a function* $F(\theta) \in L^p$, $1 \leqslant p < \infty$, *and that* F *has an m-th derivative in* L^p *at the point* θ_0 *equal to* s. *Then the series obtained by differentiating* $\sum A_n(\theta)$ *termwise* m *times is summable* $(C, m+2)$ *at* θ_0 *to sum* s.

This lemma holds even with $(C, m+2)$ replaced by $(C, m+\varepsilon)$, $\varepsilon > 0$, but the index of summability is of no importance and, in the form stated, the lemma is a simple corollary of known results.

For, in the first place, since differentiability in L^p clearly implies differentiability in L^{p_1} if $p_1 < p$, we may assume that $p = 1$. Thus we have (2.1) with f replaced by F and $p = 1$. Omitting the sign of absolute value we see that the indefinite integral G of F has an $(m+1)$-st derivative at θ_0 equal to s, in the classical sense. But then, by a very well known result (see [6_{II}], p. 60) the series obtained by differentiating the Fourier series of G termwise $m+1$ times — or what is the same thing differentiating the Fourier series of F termwise m times — is summable $(C, m+1+1) = (C, m+2)$ to s, and this is our lemma 2.

LEMMA 3. *If* $F(\theta) \sim \sum A_n(\theta)$, $F \in L^p$, $1 < p < \infty$, *then* $\sum A_n(\theta) n^{i\gamma}$ *is the Fourier series of a function* $\tilde{F}(\theta)$ *which is also in* L^p. *Moreover, if* F *has an m-th derivative in* L^p *at each point of a set* E *of positive measure, then* F *has an m-th derivative in* L^p *almost everywhere in* E.

The first part of this lemma is well known (see [6_{II}], p. 232, Example; another proof is contained implicitly in [2]). To the proof of the second part we return in the next section. Here we only observe that we require p to be strictly greater than 1.

LEMMA 4. *If a* (*numerical*) *series* $\sum\limits_0^\infty u_n$ *is summable* (C, k), $k > -1$, *and the series* $\sum\limits_0^\infty u_n n^{i\gamma}$ (γ *real*) *is Abel summable, then it is also summable* (C, k).

The proof of lemma 3 is given in §§ 3, 4 below. That of lemma 4 is briefly discussed in § 5. We shall now deduce our theorem from the lemmas above.

Suppose that $\sum A_n(x)$, with $a_0 = 0$, is summable (C, k), $k > -1$, at each point of a set E, $|E| > 0$; in particular, it is summable (C, k'), where k' is the least integer $\geqslant k$. By lemma 1, the sum $G(\theta)$ of the series obtained by integrating $\sum A_n(\theta)$ termwise $k'+1$ times has almost everywhere in E a $(k'+1)$-st derivative, in the metric L^p, $p < \infty$. Suppose, e.g. that $k'+1$ is even, so that

$$G(\theta) = (-1)^{\frac{1}{2}(k'+1)} \sum A_n(\theta) n^{k'+1}.$$

By lemma 3, the function

$$\tilde{G}(\theta) = (-1)^{\frac{1}{2}(k'+1)} \sum A_n(\theta) n^{k'+1} n^{i\gamma}$$

has a $(k'+1)$-st derivative in L^p, $1 < p < \infty$, at almost all points of E. By lemma 2, the last series differentiated termwise $k'+1$ times, that is the series $\sum A_n(\theta) n^{i\gamma}$, is summable $(C, k'+3)$ almost everywhere in E; in particular, it is Abel summable almost everywhere in E. Finally, by lemma 4, at each point where $\sum A_n(\theta) n^{i\gamma}$ is Abel summable, and so almost everywhere in E, it is summable (C, k). This completes the proof of the theorem provided we supply the proofs of lemmas 3 and 4.

3. In this and next sections we prove lemma 3, which is of independent interest, for general $1 < p < \infty$. It should however be observed that for the proof of theorem 1 we need only some fixed p, e.g. $p = 2$, in which case the fact that $\sum A_n(\theta) n^{i\gamma}$ is, like $\sum A_n(\theta)$, in L^2 is obvious.

The proof of lemma 3 resembles that of the fact that if a function has an m-th derivative in L^p, $1 < p < \infty$, at each point of a set E, then its Hilbert transform (conjugate function) has the same property almost everywhere in E (see [1], theorems 6, 7). The main idea is in both cases the same but unfortunately there is enough formal difference, of a not completely trivial nature, not to leave it to the reader to take care of the required changes. Thus we must go through certain details of computation.

Let C_n^α be the Cesàro numbers defined by the generating function

$$\sum_{n=0}^{\infty} C_n^\alpha z^n = (1-z)^{-\alpha-1}.$$

We have $C_n^\alpha \simeq n^\alpha/\Gamma(\alpha+1)$ $(\alpha \neq -1, -2, \ldots)$, $n \to \infty$, and in particular

$$C_n^{i\gamma} \simeq n^{i\gamma}/\Gamma(i\gamma+1).$$

We first prove lemma 3 with factors $n^{i\gamma}$ replaced by $C_n^{i\gamma}$ and we shall see later that this easily leads to lemma 3 as stated.

Suppose therefore that $F \sim \sum A_n(\theta) \in L^p$, $1 < p < \infty$, that F has an m-th derivative in L^p at each point of a set E, $|E| > 0$. It is known (see [1], theorems 9, 10) that given any $\varepsilon > 0$ we can find a closed set $P \subset E$, $|E-P| < \varepsilon$, and a decomposition

$$F = G+H,$$

where G and H are again periodic, G is in $C^{(m)}$ and $G^{(j)}(\theta)$, $j = 0, 1, \ldots, m$, coincides with the j-th derivative of F (in L^p); thus the derivatives of order $j = 0, 1, \ldots, m$ of H are all 0 on P. Moreover,

$$\Big(h^{-1}\int_{-h}^{h} |H(\theta+t)|^p dt\Big)^{1/p} \leqslant Mh^m \qquad (\theta \in P)$$

with M independent of θ and

$$(3.1)\qquad \Big(h^{-1}\int_{-h}^{h} |H(\theta+t)|^p dt\Big)^{1/p} = o(h^m) \qquad (h \to 0)$$

uniformly in $\theta \in P$. In particular, also

$$h^{-1}\int_{-h}^{h} |H(\theta+t)|\,dt = o(h^m) \qquad (h \to 0,\ \theta \in P).$$

Basic for the proof of theorem 1 is also the fact (see [1], theorem 10) that

$$(3.2)\qquad \int_{-\pi}^{\pi} \frac{H(t)}{|t-\theta|^{m+1}}\,dt < \infty$$

for almost all $\theta \in P$.

If we write

$$\tilde{F}(\theta) \sim \sum_{n=0}^{\infty} A_n(\theta) C_n^{i\gamma},$$

then $\tilde{F} = \tilde{G}+\tilde{H}$. Since the multipliers $C_n^{i\gamma}$ preserve the class L^p, $1 < p < \infty$ (see [6_{II}], p. 232, theorem 4.14), it is clear that $\tilde{G}(\theta)$ has almost everywhere an m-th derivative in L^p and it is enough to prove that $\tilde{H}(\theta)$ has an m-th derivative in L^p almost everywhere in P.

Let

$$K(r, t) = \frac{1}{2} + \sum_{n=1}^{\infty} C_n^{i\gamma} r^n \cos nt = \frac{1}{2}\left\{\frac{1}{(1-re^{it})^{1+i\gamma}} + \frac{1}{(1-re^{-it})^{1+i\gamma}} - 1\right\}.$$

Clearly, for almost all θ, $\tilde{H}(\theta)$ is the limit of the Abel means of its Fourier series, i.e.,

$$(3.3)\qquad \tilde{H}(\theta) = \lim_{r\to 1} \frac{1}{\pi}\int_{-\pi}^{\pi} H(\theta+t)K(r, t)\,dt$$

$$= \lim_{r\to 1} \frac{1}{\pi}\int_{-\pi}^{\pi} [H(\theta+t)-H(\theta)]K(r, t)\,dt + H(\theta).$$

On the other hand, it is well known (see [2]) that if $f(x) \in L^p(-\infty, +\infty)$, $1 \leqslant p < \infty$, then the expression

$$(3.4)\qquad g(x) = \int_0^1 \frac{f(x+t)-f(x)}{t^{1+i\gamma}}\,dt + \int_1^\infty \frac{f(x+t)}{t^{1+i\gamma}}\,dt = \lim_{\varepsilon\to 0}\int_\varepsilon^1 + \int_1^\infty$$

exists almost everywhere (clearly, it is only the existence of $\int_0^1$ that re-requires proof; the integral $\int_1^\infty$ converges absolutely and uniformly); moreover, if $1 < p < \infty$, we have

$$(3.5)\qquad \|g\|_p \leqslant A_p \|f\|_p,$$

where A_p depends on p only. All these facts remain essentially unchanged if in the definition of $g(x)$ instead of the decomposition $\int_0^1 + \int_1^\infty$ we use $\int_0^\omega + \int_\omega^\infty$, $0 < \omega < \infty$, the constant A_p in (3.5) remaining the same (this follows by a change of variable). Also, each integral $\int_0^\omega$ and $\int_\omega^\infty$ satisfies an inequality analogous to (3.5).

Using these facts it is easy to deduce from (3.3) that for periodic functions $H(\theta)$ that are merely integrable we have

$$\tilde{H}(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}[H(\theta+t)-H(\theta)]K(1,t)\,dt+H(\theta) \tag{3.6}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}[H(t)-H(\theta)]K(1,t-\theta)\,dt+H(\theta),$$

where

$$K(1,t) = \lim_{r\to 1}K(r,t) = \frac{1}{2}\left\{\frac{1}{(1-e^{it})^{1+i\gamma}}+\frac{1}{(1-e^{-it})^{1+i\gamma}}-1\right\},$$

and that if $H\in L^p$, $1<p<\infty$, then

$$\|\tilde{H}\|_p \leqslant A_p\|H\|_p,$$

where the norms are over the interval $(0,2\pi)$.

For example, in order to deduce (3.6) from (3.3) it is enough to verify that $|K(r,t)|\leqslant A/\delta$, where $\delta = 1-r$ and that $|K(r,t)-K(1,t)| \leqslant A\delta/t^2$ for $\delta\leqslant|t|\leqslant\pi$, so that the expression equal to $K(r,t)$ for $|t|\leqslant\delta$ and equal to $K(r,t)-K(1,t)$ for $\delta\leqslant|t|\leqslant\pi$ is majorized by a fixed multiple of the Poisson kernel.

We also add that since $K(t)$ is infinitely differentiable for $t\neq 0$, the second integral (3.6) shows that we do not affect the differentiability properties of H at the point θ if we modify H away from θ.

4. If $\varphi(z)$ denotes the function $\{(1-e^{iz})/iz\}^{-1-i\gamma}$, regular for $|z|<2\pi$, we have

$$K(1,t) = \frac{1}{2}\left\{\frac{\varphi(t)}{(it)^{1+i\gamma}}+\frac{\varphi(-t)}{(-it)^{1+i\gamma}}\right\},$$

and taking a sufficiently large number of terms of the Taylor series of φ we easily see that the problem reduces to showing that the functions

$$\int_{\theta}^{\pi}[H(t)-H(\theta)]\frac{(t-\theta)^k}{|t-\theta|^{1+i\gamma}}\,dt, \qquad \int_{-\pi}^{\theta}[H(t)-H(\theta)]\frac{(t-\theta)^k}{|t-\theta|^{1+i\gamma}}\,dt \tag{4.1}$$

have m-th derivatives in L^p at almost all points of P; the exponent k here takes a finite number of values $0,1,2,\ldots$ We may restrict ourselves to the first integral (4.1). We shall discuss only the case $k=0$ which is, in some sense, the most subtle (it corresponds to the constant term of the Taylor development of φ); for other values of k the proof is parallel.

Thus we will prove that the function

$$S(\theta) = \int_{\theta}^{\pi}[H(t)-H(\theta)]\frac{dt}{(t-\theta)^{1+i\gamma}} \tag{4.2}$$

has an m-th derivative in L^p at almost all points of P.

Let us consider any point $\theta_0\in P$ at which the integral (3.2) is finite. We shall prove that S has an m-th derivative in L^p at $\theta=\theta_0$. Assume, as we may, that $\theta_0=0$. We fix a small $h>0$ and we consider the function $S(\theta)$ in the interval $(-h,h)$. We write

$$S(\theta) = \int_{\theta}^{\theta+3h}\frac{H(t)-H(\theta)}{(t-\theta)^{1+i\gamma}}\,dt+\int_{\theta+3h}^{\pi}\frac{H(t)}{(t-\theta)^{1+i\gamma}}\,dt-H(\theta)\int_{\theta+3h}^{\pi}\frac{dt}{(t-\theta)^{1+i\gamma}}\,dt$$

$$= S_1+S_2+S_3,$$

say. Since, for $|\theta|\leqslant h$,

$$S_1(\theta) = \int_0^{3h}\{H(t+\theta)-H(\theta)\}t^{-1-i\gamma}\,dt = \int_0^{3h}[H_1(t+\theta)-H_1(\theta)]t^{-1-i\gamma}\,dt,$$

where $H_1(t)=H(t)$ for $|t|\leqslant 4h\leqslant 1$ and $H_1(t)=0$ elsewhere, an application of (3.5) (and the remarks that follow it) shows that

$$\Big(h^{-1}\int_{-h}^{h}|S_1|^p d\theta\Big)^{1/p} \leqslant \Big(h^{-1}\int_{-\infty}^{+\infty}|S_1|^p dt\Big)^{1/p} \leqslant A_p\Big(h^{-1}\int_{-\infty}^{+\infty}|H_1|^p d\theta\Big)^{1/p} \tag{4.3}$$

$$\leqslant A_p\Big(h^{-1}\int_{-4h}^{4h}|H|^p dt\Big)^{1/p} = o(h^m),$$

by (3.1). Also, since $|S_3(\theta)|\leqslant A_\gamma|H(\theta)|$,

$$\Big(h^{-1}\int_{-h}^{h}|S_3|^p d\theta\Big)^{1/p} = o(h^m). \tag{4.4}$$

Now observe that $2h\leqslant\theta+3h\leqslant 4h$, so that

$$S_2 = \int_{2h}^{\pi}\frac{H(t)}{(t-\theta)^{1+i\gamma}}\,dt-\int_{2h}^{\theta+3h}\frac{H(t)\,dt}{(t-\theta)^{1+i\gamma}} = S_{2,1}+S_{2,2}.$$

Clearly,

$$|S_{2,2}|\leqslant h^{-1}\int_0^{4h}|H(t)|\,dt = o(h^m). \tag{4.5}$$

On the other hand,

$$S_{2,1}(\theta) = \int_{2h}^{\pi} \frac{H(t)}{t^{1+i\gamma}(1-\theta/t)^{1+i\gamma}}\, dt \tag{4.6}$$

$$= \int_{2h}^{\pi} H(t) \frac{1}{t^{1+i\gamma}} \left\{ \sum_{j=0}^{m} \lambda_j \left(\frac{\theta}{t}\right)^j + O\left(\frac{\theta}{t}\right)^{m+1} \right\} dt$$

$$= \sum_{j=0}^{m} \lambda_j \theta^j \int_{2h}^{\pi} \frac{H(t)}{t^{j+1+i\gamma}}\, dt + O\left(|\theta|^{m+1} \int_{2h}^{\pi} \frac{|H(t)|}{t^{m+2}}\, dt \right).$$

Observe now that, by hypothesis, the integral $\int_0^{\pi} \frac{|H(t)|}{t^{m+1}}\, dt$ is finite, so that

$$\int_{2h}^{\pi} |H(t)|\, t^{-m-2}\, dt = o(h^{-1})$$

and the last term on the right of $o(h^m)$, also, if $j = 0, 1, \ldots, m$,

$$\int_{2h}^{\pi} \frac{H(t)}{t^{j+1+i\gamma}}\, dt = \int_0^{\pi} - \int_0^{2h} = \int_0^{\pi} + O(h^{m-j})$$

as easily seen by integration by parts and applying (3.1). Thus $S_{2,1}(\theta)$ is a fixed polynomial of degree $\leqslant m$ in θ plus an error term $o(h^m)$. Collecting results we obtain that $S(\theta)$ does indeed have an m-th derivative in L^p at $\theta = 0$.

We have thus proved lemma 3 with factors $n^{i\gamma}$ replaced by $C_n^{i\gamma}$. But the result holds also for the factors $C_n^{i\gamma-s}$, where s is any positive integer. For the new generating function is

$$\sum_{n=0}^{\infty} C_n^{i\gamma-s} z^n = \frac{(1-z)^s}{(1-z)^{i\gamma+1}}$$

and it is clear that the argument above holds for kernels $z^j/(1-z)^{i\gamma+1}$, $j = 0, 1, 2, \ldots$ Since the asymptotic formula

$$C_n^{i\gamma} = n^{i\gamma}\{\lambda_0 + \lambda_1 n^{-1} + \ldots + \lambda_s n^{-s} + O(n^{-s-1})\}$$

leads to

$$n^{i\gamma} = \mu_0 C_n^{i\gamma} + \mu_1 C_n^{i\gamma-1} + \ldots + \mu_s C_n^{i\gamma-s} + O(n^{-s-1});$$

taking s sufficiently large we are easily led to lemma 3 as stated.

5. It remains to consider lemma 4. We do not give a proof here since it is implicitly contained in paper [5] (Satz 2, p. 317). It is shown there that if the Cesàro means σ_n^k of a series $\sum_0^{\infty} u_n$ satisfy the condition $\sigma_n^k = o(\mu_n)$, where μ_n is a sequence monotonically tending to $+\infty$ and such that

$$\Delta^j \frac{1}{\mu_n} = O\left(\frac{1}{n^j \mu_n}\right) \qquad (j = 0, 1, \ldots, k'),$$

k' being the least integer $\geqslant k$, then $\sum u_n/\mu_n$ is either summable (C, k) or else is not Abel summable. Actually the proof without any change gives the following result: if $\{1/\mu_n\}$ is a bounded sequence satisfying $\Delta^j(1/\mu_n) = O(n^{-j})$ for $j = 0, 1, \ldots, k'$ and $\sum u_n$ is summable (C, k), then $\sum u_n/\mu_n$ is also summable (C, k), provided it is Abel summable.

The result clearly applies in the case when $\mu_n = n^{-i\gamma}$ $(n > 0)$.

Since the case $k = 0$ of theorem 1 is of special interest, it may be worth pointing out that in this case lemma 4 is immediate. For it is very well known that if a series $\sum_0^{\infty} v_\nu$ is Abel summable and

$$\sum_{\nu=0}^{n} \nu v_\nu = o(n), \tag{5.1}$$

then Σv_ν converges. But (5.1) is immediate, by summation by parts, if $v_\nu = u_\nu \nu^{i\gamma}$ and $\sum u_\nu$ converges.

6. The theorem of this paper has connection with some recent results of E. M. Stein. As he pointed out to us, the methods of his paper [3] give the following theorem:

If a function

$$F(z) = \sum_0^{\infty} c_n z^n$$

analytic in $|z| < 1$ has a non-tangential limit at each point of a set E situated on $|z| = 1$, then the function

$$G(z) = \sum_0^{\infty} c_n n^{i\gamma} z^n$$

has a non-tangential limit almost everywhere in E.

References

[1] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. 20 (1961), p. 171-225.

[2] B. Muckenhoupt, *On certain singular integrals*, Pacific J. Math. 10 (1961), p. 239-261.

[3] E. M. Stein, *Classes H^p, multiplicateurs et fonctions de Littlewood-Paley*, C. R. Acad. Sci. Paris 263 (1966), p. 716-719.

[4] Mary Weiss, *On symmetric derivatives in L^p*, Studia Math. 24 (1964), p. 89-100.

[5] A. Zygmund, *Über einige Sätze aus der Theorie der divergenten Reihen*, Bull. Intern. Acad. Pol. Sci. et Lettres, 1927, p. 309-331.

[6] — *Trigonometric series*, 2nd ed., Cambridge 1959, vol. I and II.

UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE
UNIVERSITY OF CHICAGO

Reçu par la Rédaction le 22. 8. 1967

STUDIA MATHEMATICA, T. XXX (1968)

Absolute continuity of vector-valued finitely additive set functions, I

by

W. ORLICZ (Poznań)

1. Throughout this paper we shall use the following notations and notions. X denotes a real Banach space and Ξ its conjugate space, ξ stands for a functional from Ξ. By Ξ_0 we denote a set of functionals from Ξ such that $\|\xi\| \leqslant 1$ and $\sup|\xi(x)| \geqslant c\|x\|$ for some $c > 0$, where the supremum is taken over all ξ in Ξ_0. The sets Ξ_0 are called *fundamental*. X is called *weakly sequentially compact* with respect to Ξ_0 if for every sequence ξ_n, where $\xi_n \epsilon \Xi_0$, there exists a subsequence $\xi_{n_i}(x)$ converging for any $x \epsilon X$. It is a simple matter to prove that a separable space X is weakly sequentially compact with respect to $\Xi_0 = \{\xi: \|\xi\| \leqslant 1\}$. There exist also non-separable spaces weakly sequentially compact with respect to some Ξ_0. As an example of such kind of spaces one can take the space M_0 of real-valued functions $f(\cdot)$, bounded on $\langle a, b\rangle$, having for any $t \epsilon (a, b)$ the right- and left-hand limit and the limits $f(a+), f(b-)$. Here the norm for $f \epsilon M_0$ is defined as $\sup\limits_{\langle a,b\rangle} |f(t)|$, and for Ξ_0 one can take the set of all functionals of the form

$$\xi(x) = \pm n \int\limits_u^{v+1/n} f(t)\,dt,$$

where $a \leqslant u, v+1/n \leqslant b, n = 1, 2, \ldots$ By E we will denote an abstract set of points (elements), the symbol $e_n\uparrow$ or $e_n\downarrow$ will stand for a sequence of sets e_n in E such that $e_1 \subset e_2 \subset \ldots$ or $e_1 \supset e_2 \supset \ldots$ respectively. $e_n\downarrow e$ or $e_n\uparrow e$ means $e_n\downarrow, e = \bigcap\limits_1^\infty e_n$, or $e_n\uparrow, e = \bigcup\limits_1^\infty e_n$ respectively.

Besides the notation given before we shall use throughout the paper the letter $\boldsymbol{E}$ to denote the class of zero-one sequences $\boldsymbol{\varepsilon} = \{\varepsilon_i\}$, that is to say with terms $\varepsilon_i = 1, 0$. If e is a set of points from E, then $\varepsilon_i a$ means a if $\varepsilon_i = 1$, and the empty set if $\varepsilon_i = 0$. For a sequence of sets a_i the symbol $a(\boldsymbol{\varepsilon})$ or $a^n(\boldsymbol{\varepsilon})$ denote the set $\bigcup\limits_1^\infty \varepsilon_i a_i$, or $\bigcup\limits_{i \geqslant n} \varepsilon_i a_i$, respectively. $\mathscr{E}, \mathscr{F}, \ldots$ always denotes a ring or an algebra of subsets from E. The class $\mathscr{E}$ is